# QUANTUM THEORY AND ENUMERATIVE PROBLEMS

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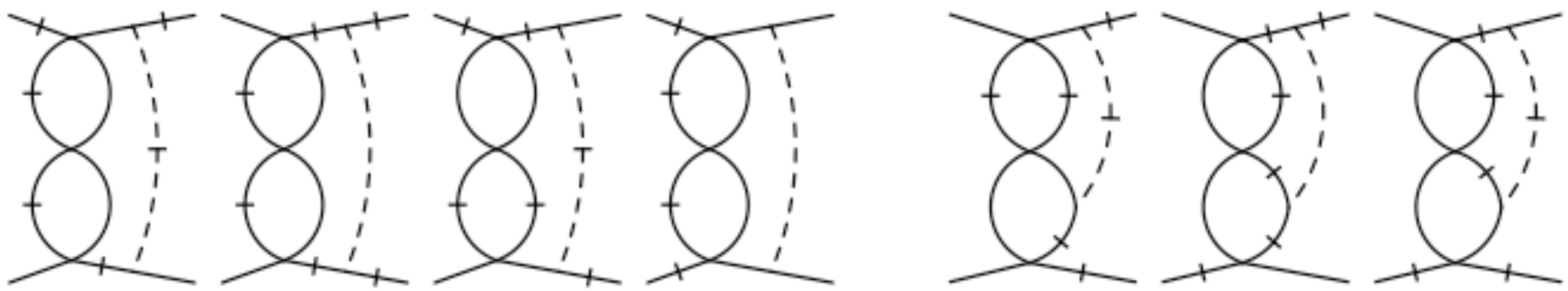
Why are quantum theorists interested in counting? What is the most important thing to count, for them?

This is a time-dependent question. For a quantum theorist in the 70's, the most likely answer would be

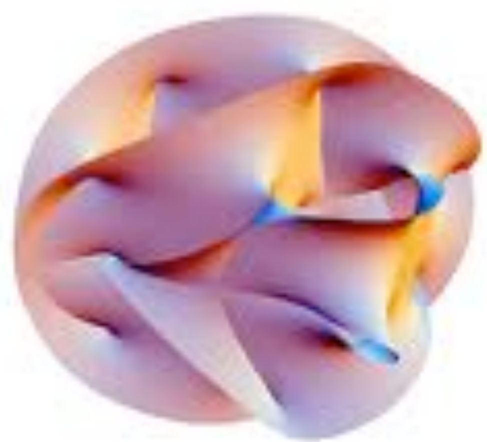
*graphs* or *diagrams*



This is because perturbative calculations in quantum theory can be typically organized in terms of the famous Feynman diagrams

However, a string theorist in 2010 might tell you that she is mostly interested in counting

*holomorphic curves* or *sheaves* in an algebraic variety



since some interesting quantities in string theory (including in some cases black hole entropies) can be computed by counting those objects

The asymptotic counting of graphs in quantum theories was addressed by physicists and mathematicians in the 70s, and surprisingly it turns out to be related to *instanton* effects -i.e. effects which are invisible with perturbative methods.

In this talk, I will start with the problem of counting graphs. But I'm interested in how this story generalizes to *string theory*, and so I will end up with the problem of counting curves, i.e. embedded strings in spacetime.

**Warning**: in this talk I will touch on *many* different subjects. The main point will be to show you that there are fruitful connections among them. But the counterpart is that I won't be able to tell you all the details -far from it. So please be patient!

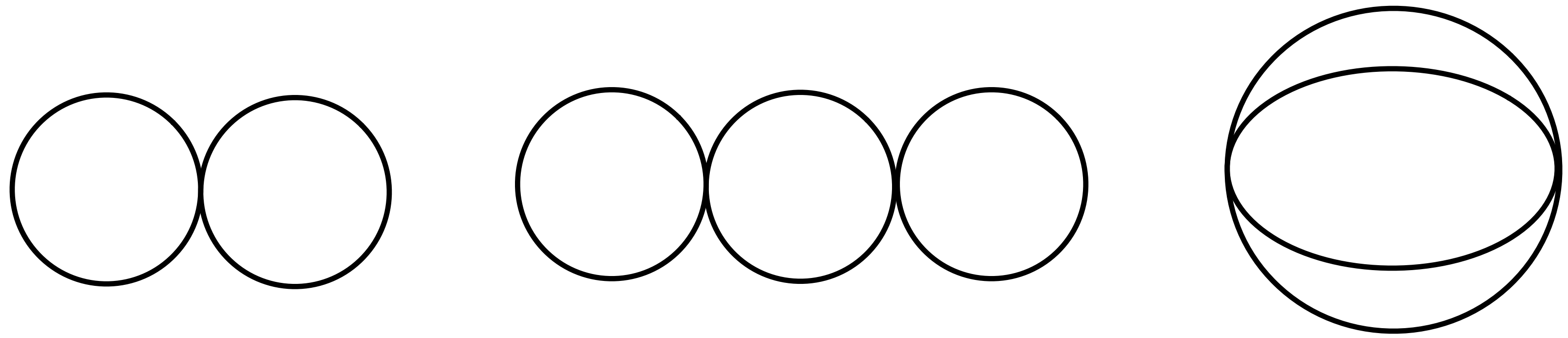
A very simple example: the quartic anharmonic oscillator. Given the potential

$$V(x) = \frac{1}{2}x^2 + \frac{g}{4}x^4$$

you want to calculate the ground state energy as a perturbative series in g (a standard problem in undergraduate QM)

$$E(g) = \frac{1}{2} + \sum_{n\geq 1} a_n g^n$$

It turns out that, in order to calculate $a_n$, you have to draw first all possible *connected* diagrams with *n* quartic vertices and no legs, and count them taking into account their automorphism group



$|\mathrm{Aut}(\Gamma)|$ $8$ $16$ $48$

This is not the whole story, since each diagram has to be multiplied by a so-called *Feynman integral* $I_\Gamma$ . One obtains

$$a_n = \sum_{\Gamma_n} \frac{I_{\Gamma_n}}{|\mathrm{Aut}(\Gamma_n)|}$$

The number of *disconnected* graphs of order *n* is easy to calculate

$$\frac{(4n-1)!!}{(4!)^n n!}$$

This grows *factorially* with *n*: $$\frac{(4n-1)!!}{(4!)^n n!} \sim n!$$

A simple analysis shows that restricting oneself to connected diagrams does not change this asymptotic growth. The Feynman integrals only grow exponentially, and we conclude that the $a_n$ grow factorially, too

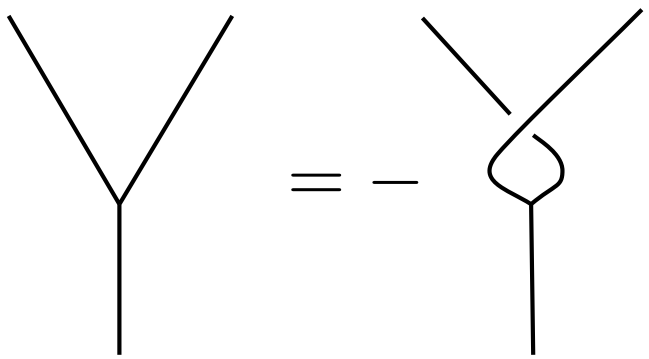
Therefore, the series for *E(g)* has *zero radius of convergence*. It is believed that generic perturbative series in QFT are divergent (more precisely, they are *asymptotic*), and much effort has been devoted to making sense of them

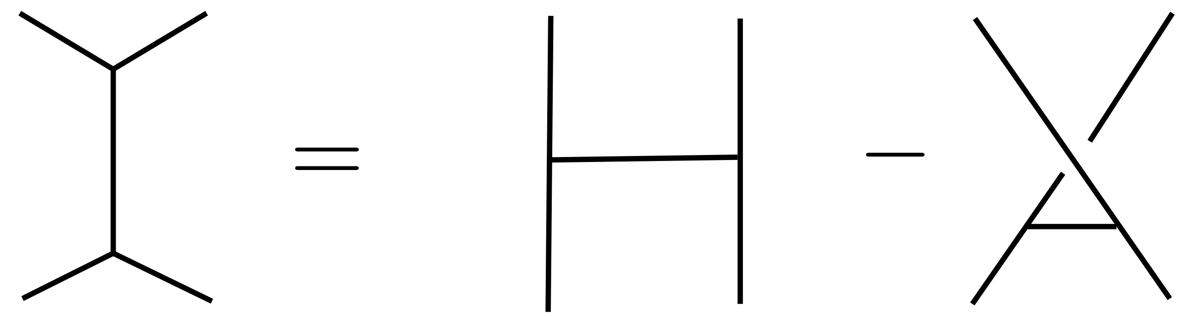
# Quantum topology and Feynman diagrams

The above example was in Quantum Mechanics. What happens in QFT? Fortunately for us, there is a QFT which has been reformulated mathematically in a very precise way. This is *Chern-Simons theory*, introduced by Witten in 1989 to understand knot invariants.

The reformulation we are interested in is the so-called *LMO invariant of homology 3-spheres*, which is based on the diagrams one finds in Chern-Simons quantum perturbation theory

We first introduce the relevant diagrams. These are connected, trivalent graphs modulo the so-called AS and IHX relations:



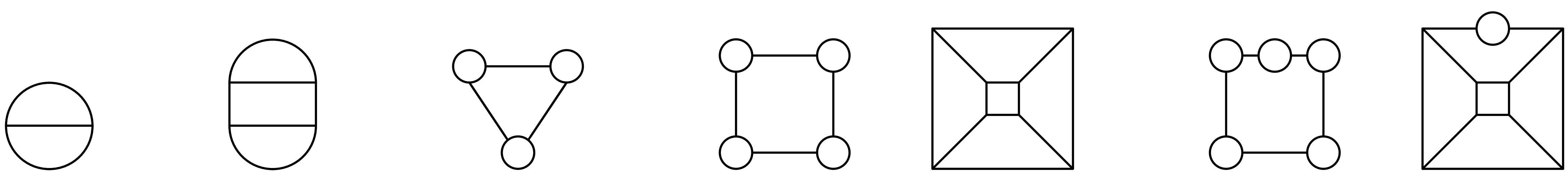


finite dimensional!

graded space

$$\mathcal{A}^{\mathrm{conn}}(\emptyset) = \oplus_{n=1}^{\infty}\mathcal{A}_n^{\mathrm{conn}}(\emptyset)$$

degree=number of vertices/2



1 2 3 4 5

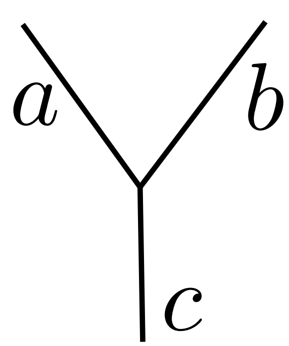
Given a rational homology sphere *M*, the LMO invariant produces an element in this space of graphs

$$F_{\mathrm{LMO}}(M) = \sum_{\Gamma} c_{\Gamma}(M)\Gamma \in \mathcal{A}^{\mathrm{conn}}(\emptyset)$$

*Each* coefficient is a topological invariant of *M*; it can be expressed in terms of *configuration integrals* over *M*

$c_{\ominus}(M)$ = Casson-Walker invariant of *M*

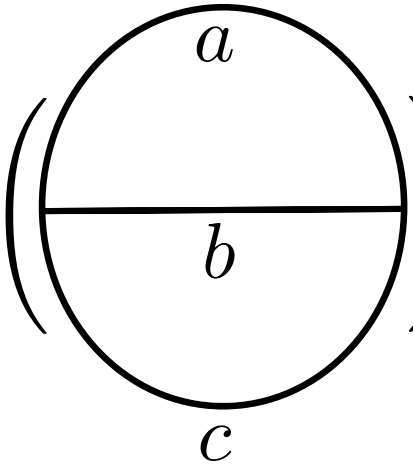
Given now a *Lie algebra* **g** with structure constants $f_{abc}$ ,we can associate a *number* to each graph by the substitution rule

$W_{\mathbf{g}}\Big($  $\Big) = f_{abc}$



This is called a *weight system*

For example,

$W_{\mathbf{g}}\Big($ 

$\Big) = \sum_{a,b,c} f_{abc} f_{abc}$

Therefore, LMO invariant of *M* + Lie algebra **g** gives a *formal power series*

$$F(M, \mathbf{g}, \hbar) = \sum_{\Gamma} c_{\Gamma}(M) W_{\mathbf{g}}(\Gamma) \hbar^{n(\Gamma)}$$

grading

This is conjecturally the perturbative quantum series that is obtained from *Chern-Simons gauge theory*

$$S = \frac{1}{\hbar} \int_M \mathrm{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

around *A=0*

**g**-valued connection

$$F(M,\mathbf{g},\hbar) = \sum_{\Gamma} c_{\Gamma}(M) W_{\mathbf{g}}(\Gamma) \hbar^{n(\Gamma)}$$

free energy

Feynman integral

group factor

perturbation parameter, or coupling constant

What is the nature of this series?

$|c_{\Gamma}(M)| \leq C_M^n$ for all graphs of degree *n* [Garoufalidis-Le]

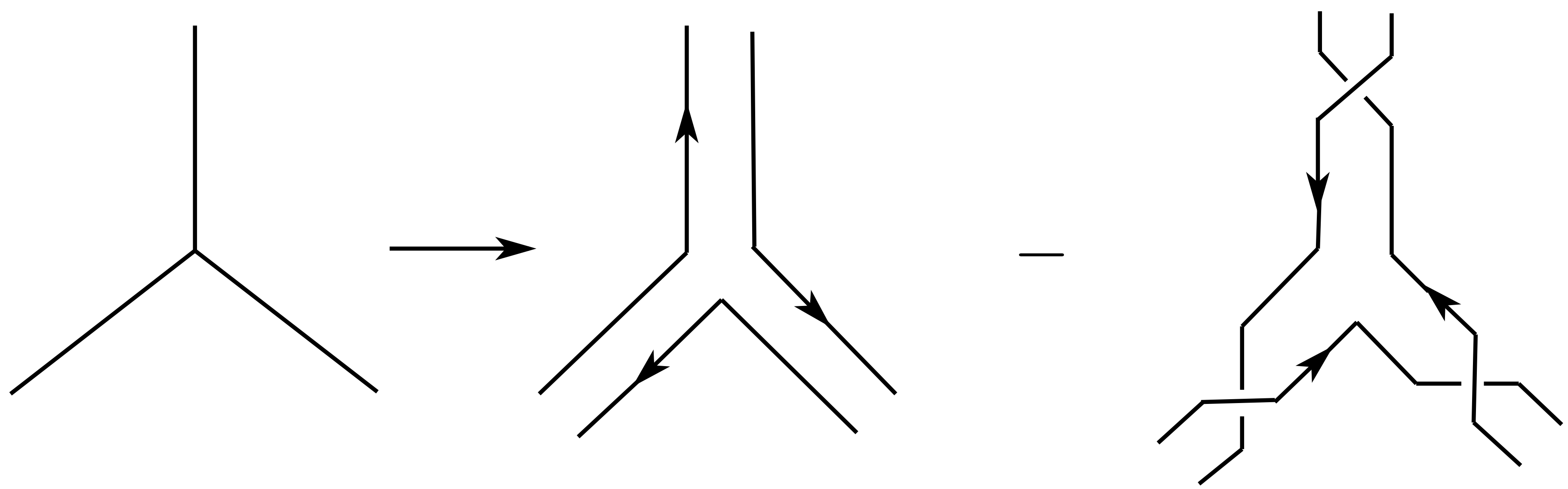
$$\dim \mathcal{A}_n^{\mathrm{conn}}(\emptyset) \sim n!$$

Therefore, just like in the Quantum Mechanics problem above, *the series diverges for all choices of gauge groups, due to the factorial growth in the number of diagrams*

Note: in a realistic QFT things are even worst, since Feynman integrals also grow factorially (renormalons)

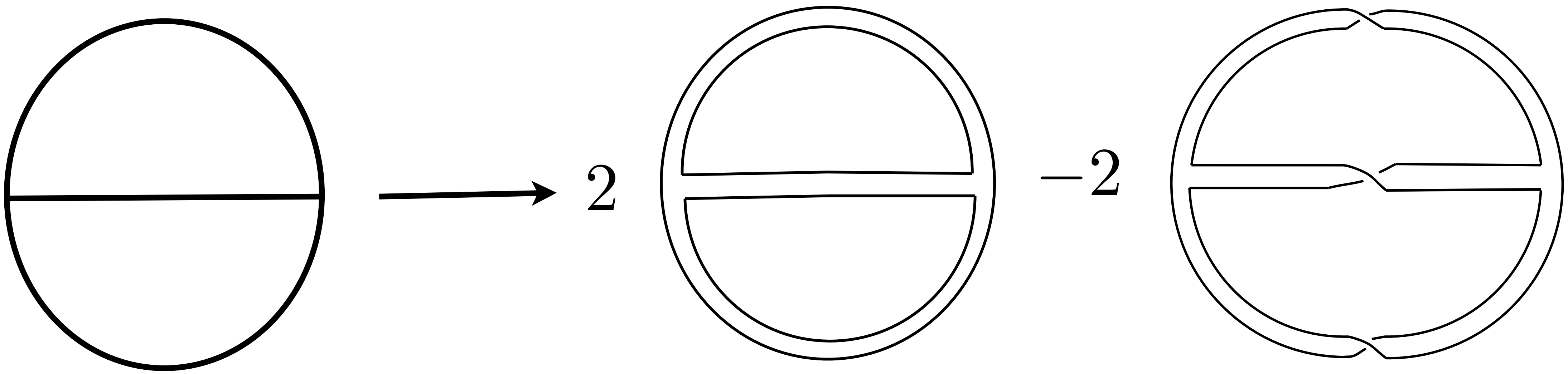
# Taming the graphs

One way to tame this divergence is to change the space of graphs we want to consider. To do this in a systematic way, one notices with ‘t Hooft that the weight system associated to *u(N)* can be interpreted in terms of *thickening the graphs*



This gives, for each graph, a formal linear combination of *fatgraphs*, which are Riemann surfaces with boundaries. In the combinatorics literature, these are called *maps*

# Riemann surfaces from perturbation theory



$$\Gamma \to \sum_{g,h} p_{g,h}(\Gamma)\,\Gamma_{g,h}$$

map of genus *g* with
*h* boundaries

$$W_{u(N)}(\Gamma) = \sum_{g,h} p_{g,h}(\Gamma) N^h \qquad n = 2g - 2 + h$$

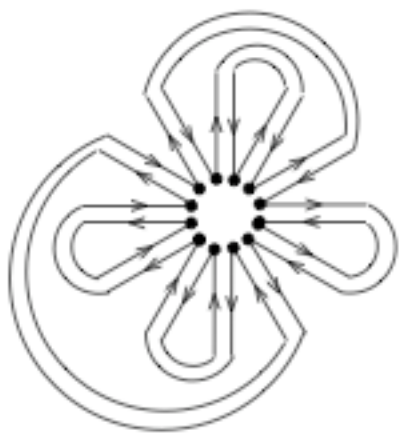
For example, $W_{u(N)}(\theta) = 2N^3 - 2N$

# Counting maps

However, for each *fixed* genus, the number of maps grows only *exponentially* with the number of holes (or equivalently, the degree). This can be already seen by counting flowers with *n* petals:

thin petals

$$(2n-1)!! \sim n!$$



planar, thick petals

$$\text{Catalan number } C_n \sim 4^n$$

Number of cubic maps of degree *n* on a surface of genus *g*

$$T_g(n) \sim t_g n^{5(g-1)/2} c^n$$

[Bender-Canfield]

Using this result we deduce that

$$F_g(t) = \sum_{\Gamma_{g,h}} c_\Gamma(M) p_{g,h}(\Gamma) t^h$$

*are analytic* in a *common* neighbourhood of *t=0* [Garoufalidis-Le-M.M.]

So in CS theory with *U(N)* gauge group we can repackage the divergent series into an infinite sequence of analytic functions!

$$F(M, u(N), \hbar) = \sum_{g=0}^{\infty} F_g(t) \hbar^{2g-2} \qquad t = \hbar N$$

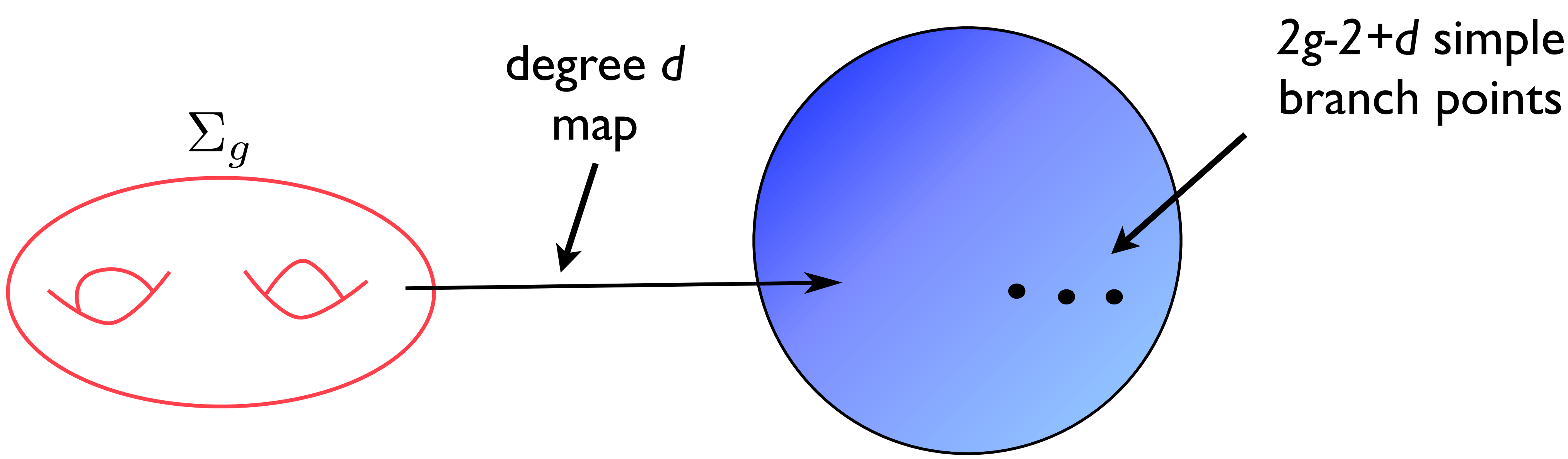
This repackaging is called the *1/N* expansion. In many cases it tames the divergent series of quantum theory, provided there is an underlying *U(N)* symmetry. It can be also done in our Quantum Mechanics problem.

# String theory

In this talk, a string theory will be regarded as *a theory of maps from a Riemann surface to a "target manifold"*

A *counting problem in string theory* is just a procedure to count these maps (like Gromov-Witten theory, cf. Candelas' course)

Example: maybe the simplest string theory (in the above sense) is *Hurwitz theory*, where we count branched coverings from a Riemann surface to a target space, which we will take to be the two-sphere (we restrict to simple branch points).



# String theory and large N dualities

One of the most amazing recent ideas in theoretical physics is the conjecture that *quantum gauge theories in the $1/N$ expansion are equivalent to string theories*

This idea has been tested in detail in many examples. Here we are interested in extracting some simple enumerative consequences. The analyticity of the fixed genus amplitudes in the $1/N$ expansion suggests in particular the following

*Conjecture*: the number of maps from a *fixed genus* Riemann surface to a target manifold grows at most *exponentially* (with the degree)

# Testing the conjecture

Hurwitz theory: count branched coverings with *Hurwitz numbers*

Natural generating functions

$$F_g(t) = \sum_{d\geq 1} \frac{H_{g,d}}{(2g-2+d)!} t^d$$

*analytic* in a common neigbourhood of *t=0*

Kontsevich: count the number $N_d$ of genus zero curves of degree *d* in $\mathbb{P}^2$, passing through *3d-1* points

$$F_0(t) = \sum_{d\geq 1} \frac{N_d}{(3d-1)!} t^d$$

*analytic* at *t=0*

[di Francesco-Itzykson]

Candelas *et al*: Gromov-Witten invariants of the quintic at genus zero grow only exponentially with the degree

# The price to pay

Let us consider the generating functions $F_g(t)$ at a fixed value of *t* inside the common neighbourhood of convergence. What is their asymptotic behavior with *g*?

we expect $$F_g(t) \sim (2g)!$$ [Shenker]

In the LMO case, this follows from the asymptotics of the $t_g$ in the formula of Bender-Canfield for $T_g(n)$

$$T_g(n) \sim t_g n^{5(g-1)/2} c^n \qquad t_g \sim g^{-g/2}$$

[Goulden-Jackson, Bender-Gao-Richmond, Garoufalidis-Le-M.M.]

See also [Garoufalidis-M.M.] for an extension to non-orientable maps

# Asymptotic counting

The factorial asymptotics we have found in these examples can be refined as follows:

Quantum Mechanics, Chern-Simons/LMO $\quad a_n \sim n!(A_{\mathrm{inst}})^{-n}$

1/N expansion, string theory $\quad F_g(t) \sim (2g)!(A(t))^{-2g}$

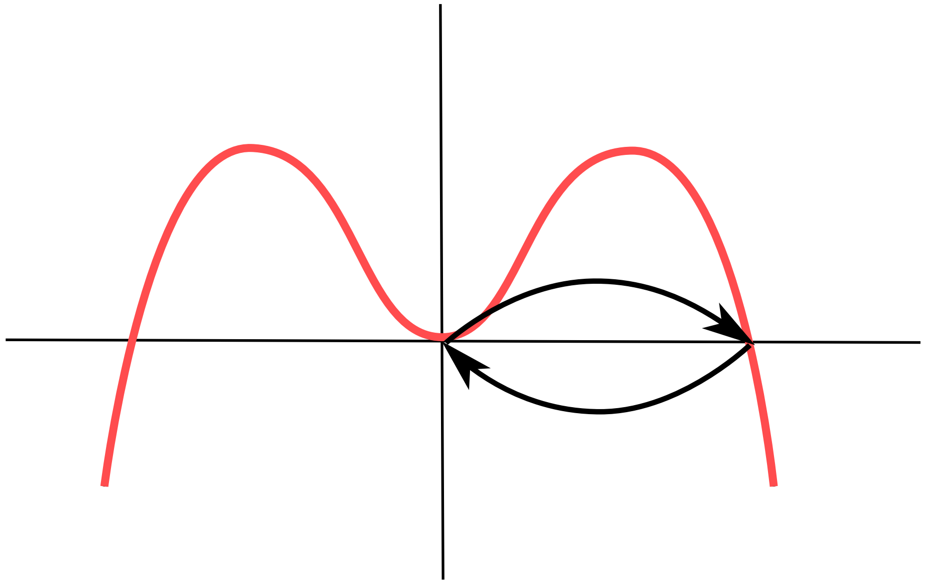
Is it possible to calculate $A_{\mathrm{inst}}$ , *A(t)* in the above formula? Why should be interested in this calculation?

The reason is a beautiful result going back to the 70's. In physics parlance it says that *the large order behavior of the perturbative series is controlled by instantons*. In this context, these are "exponentially small corrections" to the perturbative result.

# Instantons in Quantum Mechanics

This was in fact discovered by Bender and Wu in the example of the quartic potential that we discussed before [Note: this is a different Bender from the one involved in counting maps!]

In order to find the instantons here, we have to consider the *inverted* quartic potential $g \to -g$. There is now tunneling, and the energy acquires an imaginary part which is non-analytic in g



$$E(g) = \sum_{n \geq 0} a_n (-1)^n g^n + \mathcal{O}(\mathrm{e}^{-A_{\mathrm{inst}}/g})$$

where $A_{\mathrm{inst}}$ is the action of the instanton mediating the decay

$$a_n \sim n!(-1)^n (A_{\mathrm{inst}})^{-n}$$

[rigorous proof by Harrell-Simon]

# Instantons in ODEs

Simplest case of this idea: ODEs with irregular singular points

Euler $$y'(z) + A\,y(z) + \frac{1}{z} = 0 \qquad z = \infty$$

formal power series solution $$\phi(z) = \sum_{n\geq 0} \frac{A^{-n-1}n!}{z^{n+1}} \qquad a_n \sim A^{-n}n!$$

full solution ("trans-series") $$y(z) = \phi(z) + C\mathrm{e}^{-Az}$$

instanton

perturbative series

The asymptotics of the coefficients in the formal solution is governed by the small exponential, which is "non-perturbative" (meaning, non analytic at $z = \infty$ )

Much developed in the *theory of resurgence* of Écalle

# Instantons in Chern-Simons theory

The formal power series obtained from the LMO invariant by using weight systems has an asymptotics of the form

$$F(M, \mathbf{g}, \hbar) = \sum_n a_n \hbar^n \qquad a_n \sim n! (A_{\text{inst}})^{-n}$$

What is $A_{\text{inst}}$ here? This series describes the perturbative expansion of CS theory around the *trivial* *A=0* gauge connection. On general 3-manifolds, there are also *non-trivial* flat connections. These are the instantons of CS theory. We should then expect

$$A_{\text{inst}} = S_{\text{CS}}(A_*)$$

a non-trivial flat connection on *M*

[cf. recent work of Garoufalidis]

# Instantons in string theory

Finding the appropriate instanton configuration in string theories is less trivial. In fact, part of the motivation to introduce D-branes in string theory was precisely the search for stringy instantons!

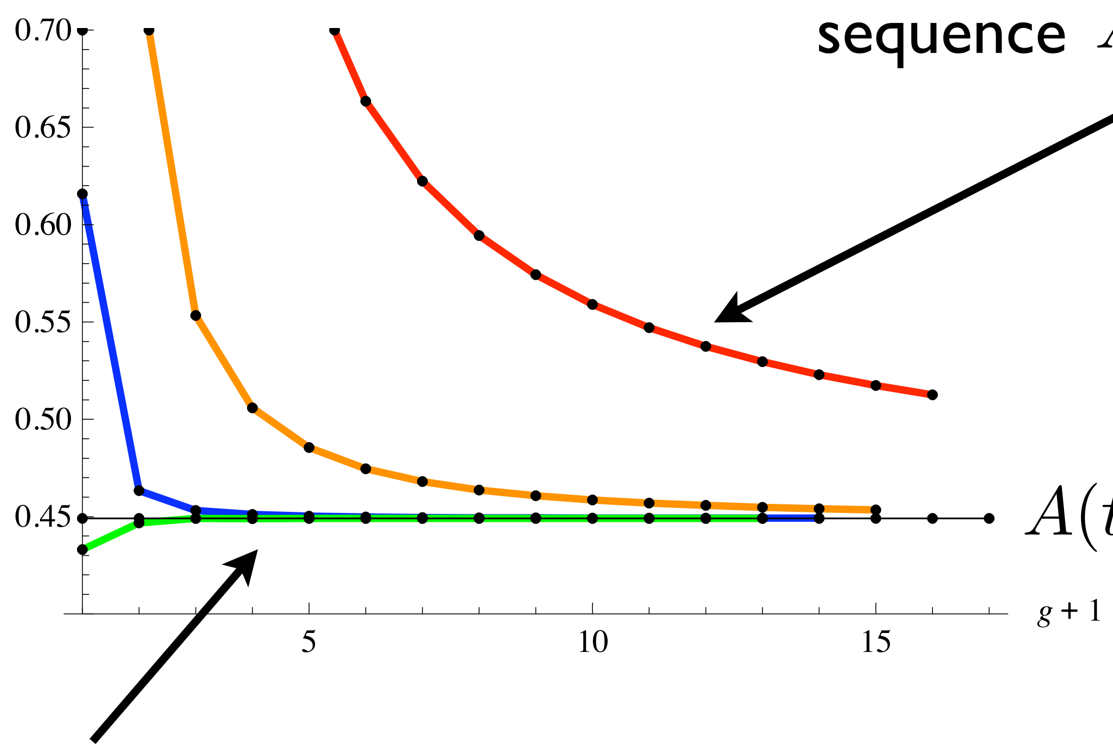
In the case of Hurwitz theory, one can identify these instantons indirectly by using the correspondence with a quantum gauge theory in the $1/N$ expansion, or the "trans-series" approach. This is because the total generating function

$$F(\hbar, t) = \sum_{g=0}^{\infty} F_g(t)\hbar^{2g-2}$$

satisfies a difference equation -namely, the Toda equation

[Pandharipande]

In particular, *A(t)* can be calculated explicitly (but a complicated formula!), and it can be checked that it governs the asymptotics of Hurwitz generating functions at large genus




[M.M.-Schiappa-Weiss, M.M.]

Understanding this in more complicated string theories is still an open problem

# Conclusions

In this talk I have reviewed a nice story linking perturbative series and the counting of diagrams in QM and QFT on one side, to instanton configurations on the other side.

In some cases, the perturbative series can be reformulated in the so-called $1/N$ expansion. We find in this way analytic functions related to the counting of fatgraphs. The resulting objects are closely related to string theory and to the counting of curves in varieties.

In these string theories there are more elusive instantons which also control the “perturbative” genus expansion.

Perhaps the most important lesson is that the generating functionals that Gromov-Witten theorists like so much are *just* the perturbative sector of a much bigger theory involving non-trivial instanton sectors.