XIX FALL WORKSHOP "GEOMETRY AND PHYSICS"

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BI-PARACONTACT STRUCTURES AND LEGENDRE FOLIATIONS

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Contact manifolds

A **contact manifold** is a smooth manifold M endowed with a 1-form η such that

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- M is necessarily of odd dimension, $2n+1$
- There exists a unique vector field ξ, called **Reeb vector field**, such that

$$\eta(\xi) = 1 \quad \text{and} \quad d\eta(\xi,\cdot) = 0.$$

Contact manifolds

$\mathcal{D} := \ker(\eta)$ is called the **contact distribution**.

Foliation theory $\Longrightarrow$ $\mathcal{D}$ is integrable if and only if $\eta \wedge d\eta = 0$.

Thus the geometric interpretation of

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The maximal dimension of an integrable subbundle of $\mathcal{D}$ is n.

Legendre foliations

Definition

Let (M^{2n+1}, η) be a contact manifold. A foliation $\mathcal{F}$ of M^{2n+1} is called **Legendre foliation** if

- $\dim(\mathcal{F}) = n$
- $T(\mathcal{F})$ is a subbundle of the contact distribution $\mathcal{D}$

Legendre foliations

More in general, a **Legendre distribution** is an *n*-dimensional sub-bundle L of the contact distribution such that

$$d\eta(X,X') = 0$$

for all $X,X' \in \Gamma(L)$. Note that this condition holds if L is involutive, since

$$2d\eta(X,X') = X(\eta(X')) - X'(\eta(X)) - \eta([X,X']).$$

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Almost contact and paracontact structures

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- An **almost contact structure** on (M,η) is given by a tensor field φ such that

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- An **almost paracontact structure** on (M,η) is given by a tensor field φ such that
  - $\varphi^2 = I - \eta\otimes\xi$
  - φ induces an *almost paracomplex structure* on the contact distribution (i.e. the ±1 eigendistributions of $\varphi|_D$ have equal dimension n).

Almost contact and paracontact structures

The geometry of an almost contact / paracontact manifold $(M^{2n+1},\varphi,\xi,\eta)$ is related to almost complex / paracomplex structures by considering

$$M^{2n+1} \times \mathbf{R}$$

and

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X \mp f\xi, \eta(X)\frac{d}{dt}\right)$$

for all $X \in \Gamma(TM^{2n+1})$ and $f \in C^\infty(M^{2n+1} \times \mathbf{R})$.

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- $J^2 = -I$ / $J^2 = I$, respectively.
- When

  $$[J,J] \equiv 0,$$

  the almost contact/paracontact manifold M^{2n+1} is called **normal**.

Almost contact and paracontact structures

It suffices to evaluate $[J,J]$ in the couples of vector fields of type $((X,0),(Y,0))$ and $((X,0),(0,d/dt))$, for any $X,Y \in \Gamma(TM^{2n+1})$:

$$[J,J]((X,0),(Y,0))$$

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where

$$N^{(1)}(X,Y) := [\varphi,\varphi](X,Y) + 2d\eta(X,Y)\xi$$

$$N^{(2)}(X,Y) := (\mathcal{L}_{\varphi X}\eta)(Y) - (\mathcal{L}_{\varphi Y}\eta)(X)$$

$$N^{(3)}(X) := (\mathcal{L}_{\xi}\varphi)X$$

$$N^{(4)}(X) := (\mathcal{L}_{\xi}\eta)X.$$

Almost contact and paracontact structures

Then

$$(\varphi, \xi, \eta) \text{ is normal}$$

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$$N^{(1)} = 0$$

Bi-paracontact structures

Definition

An **almost bi-paracontact structure** on a contact manifold (M,η) is given by the triplet $(\varphi_1,\varphi_2,\varphi_3)$, where $\varphi_1,\varphi_2,\varphi_3$ are tensor fields of type (1,1) on M such that

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For each α =1,2 set

$$\mathcal{D}_\alpha^+ := \{X\in T(M) \mid \varphi_\alpha X = X\} \qquad \mathcal{D}_\alpha^- := \{X\in T(M) \mid \varphi_\alpha X = -X\}$$

Bi-paracontact structures

Proposition

The canonical distributions $\mathcal{D}_1{}^{+}$, $\mathcal{D}_1{}^{-}$, $\mathcal{D}_2{}^{+}$, $\mathcal{D}_2{}^{-}$ satisfy the following properties:

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1. $\mathcal{D}_1{}^{\pm} = \{X + \varphi_3 X \mid X \in \mathcal{D}_2{}^{\pm}\}$, $\mathcal{D}_2{}^{\pm} = \{Y + \varphi_3 Y \mid Y \in \mathcal{D}_1{}^{\mp}\}$.

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2. For any $\alpha \neq \beta$,

$$T(M) = \mathcal{D}_\alpha{}^+ \oplus \mathcal{D}_\alpha{}^- \oplus \mathbf{R}\xi = \mathcal{D}_\alpha{}^{\pm} \oplus \mathcal{D}_\beta{}^{\pm} \oplus \mathbf{R}\xi.$$

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3. $\dim(\mathcal{D}_1{}^{+}) = \dim(\mathcal{D}_1{}^{-}) = \dim(\mathcal{D}_2{}^{+}) = \dim(\mathcal{D}_2{}^{-}) = n$.

   It follows that φ_1 and φ_2 are almost paracontact structures.

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Are $\mathcal{D}_\alpha{}^{+}$ and $\mathcal{D}_\alpha{}^{-}$ Legendre distributions?

Bi-paracontact structures

Theorem

Let $(\varphi_1,\varphi_2,\varphi_3)$ be an almost bi-paracontact structure on (M,η). Then, for each $\alpha=1,2$,

$\mathcal{D}_\alpha^+$ and $\mathcal{D}_\alpha^-$ are *Legendre distributions*

$$\Updownarrow$$

$d\eta$ is of type (1,1) with respect to φ_α

$$\Updownarrow$$

$$N_\alpha^{(2)} = 0$$

where

$$N_\alpha^{(2)}(X,Y) := (\mathcal{L}_{\varphi_\alpha X}\eta)(Y) - (\mathcal{L}_{\varphi_\alpha Y}\eta)(X).$$

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$N_\alpha^{(1)}$ vanishes on the contact distribution

where

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Definition

When, for all $\alpha=1,2$, one among the above two conditions holds, the almost bi-paracontact structure is called **integrable**.

Bi-paracontact structures

Theorem

Let $(\varphi_1,\varphi_2,\varphi_3)$ be an almost bi-paracontact structure on (M,η). Then, for each $\alpha=1,2$,

$\mathcal{D}_\alpha^+$ and $\mathcal{D}_\alpha^-$ are *flat* (= ξ is basic) *Legendre foliations*

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When, for all $\alpha = 1,2$, one among the above two conditions holds, the almost bi-paracontact structure is called **normal**.

- $(\varphi_1,\varphi_2,\varphi_3)$ is normal $\Leftrightarrow$ (φ_1,ξ,η), (φ_2,ξ,η) are normal as almost paracontact struct. and (φ_3,ξ,η) is normal as almost contact struct.

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normality $\Rightarrow$ integrability

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Even dimension

almost
bi-paracontact
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Bi-paracontact structures

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Obata connection
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ii) $\nabla^1 \varphi_1 = 0$, $\nabla^1 \varphi_2 = \eta \otimes (2h_2 - h_1 \varphi_3 + \varphi_3 h_1)$, $\nabla^1 \varphi_3 = \eta \otimes (2h_3 - h_1 \varphi_2 + \varphi_2 h_1)$

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iii) $T^\alpha(\varphi_\alpha X,Y) - T^\alpha(X,\varphi_\alpha Y) = 2\big(d\eta(\varphi_\alpha X,Y) - d\eta(X,\varphi_\alpha Y)\big)\,\xi + \eta(Y)h_\alpha X + \eta(X)h_\alpha Y$

where

$$h_\alpha := \frac{1}{2}\mathcal{L}_\xi \varphi_\alpha$$

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ii) $\nabla^c \varphi_\alpha = \frac{2}{3} \eta \otimes h_\alpha$ for each α=1,2,3

Bi-paracontact structures

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Any almost bi-paracontact manifold admits a unique connection ∇^c (called the **canonical connection**) such that

i) $\nabla^c \xi = 0$

ii) $\nabla^c \varphi_\alpha = \frac{2}{3}\eta \otimes h_\alpha$ for each $\alpha=1,2,3$

iii) $$T^c = d\eta - \frac{1}{3}\big(d\eta(\varphi_1\cdot,\varphi_1\cdot) + d\eta(\varphi_2\cdot,\varphi_2\cdot) - d\eta(\varphi_3\cdot,\varphi_3\cdot)\big) - \frac{1}{6}\big(N^{(1)}_{\varphi_1} + N^{(1)}_{\varphi_2} - N^{(1)}_{\varphi_3}\big).$$

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$$- \frac{1}{6}\left(N^{(1)}_{\varphi_1} + N^{(1)}_{\varphi_2} - N^{(1)}_{\varphi_3}\right).$$

Furthermore, ∇^c is explicitly given by

$$\nabla^c = \frac{1}{3}\left(\nabla^1 + \nabla^2 + \nabla^3\right).$$

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2) $T^c = 2d\eta \otimes \xi$

3) $R^c(\varphi_1 \cdot, \varphi_1 \cdot) = R^c(\varphi_2 \cdot, \varphi_2 \cdot) = -R^c(\varphi_3 \cdot, \varphi_3 \cdot) = -R^c$

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4) The Ricci tensor is given by

$$\mathrm{Ric}^c(X,Y) = -\frac{1}{2}\mathrm{trace}(R^c(X,Y))$$

Hence, Ric^c is skew-symmetric and

$$\mathrm{Ric}^c(\varphi_1 \cdot, \varphi_1 \cdot) = \mathrm{Ric}^c(\varphi_2 \cdot, \varphi_2 \cdot) = -\,\mathrm{Ric}^c(\varphi_3 \cdot, \varphi_3 \cdot) = \mathrm{Ric}^c$$

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7) The 1-dimensional foliation defined by the Reeb vector field ξ is **transversely para-hypercomplex**.

∇^c is (locally) projectable to the Obata connection on the leaf space.

Examples

The model in the normal case

Consider $\mathbf{R}^{2n+1}$ with coordinates $(x_1,\ldots,x_n,y_1,\ldots,y_n,z)$ and contact form

$$\eta := dz - \sum_{i=1}^{n} y_i dx_i \,.$$

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Set, for each $i \in \{1,\ldots,n\}$

$$X_i = \frac{\partial}{\partial y_i}, \quad Y_i = \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial z}.$$

Define

$$\begin{array}{lll} \varphi_1 X_i := X_i & \varphi_1 Y_i := -Y_i & \varphi_1 \xi := 0 \\ \varphi_2 X_i := -Y_i & \varphi_2 Y_i := -X_i & \varphi_2 \xi := 0 \\ \varphi_3 X_i := Y_i & \varphi_3 Y_i := -X_i & \varphi_3 \xi := 0 \end{array}$$

Then $(\varphi_1,\varphi_2,\varphi_3)$ is a normal almost bi-paracontact structure on $\mathbf{R}^{2n+1}$.

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Then $(\varphi_1,\varphi_2,\varphi_3)$ is a normal almost bi-paracontact structure on $\mathbf{R}^{2n+1}$.

The canonical distributions $\mathcal{D}_1^{+}$, $\mathcal{D}_1^{-}$, $\mathcal{D}_2^{+}$, $\mathcal{D}_2^{-}$ are given by

$$\mathcal{D}_1^{+} = \mathrm{span}\{X_1,\dots,X_n\}, \quad \mathcal{D}_1^{-} = \mathrm{span}\{Y_1,\dots,Y_n\}$$

$$\mathcal{D}_2^{+} = \mathrm{span}\{X_1-Y_1,\dots,X_n-Y_n\}, \quad \mathcal{D}_2^{-} = \mathrm{span}\{X_1+Y_1,\dots,X_n+Y_n\}$$

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Every contact Riemannian manifold endowed with a Legendre distribution admits an almost bi-paracontact structure.

Therefore we have many examples of almost bi-paracontact manifolds:

- *The cotangent sphere bundle of a Riemannian manifold*
- *Every contact Riemannian manifold such that ξ is an Anosov flow (e.g. the tangent sphere bundle of a negatively curved manifold)*
- *Any contact metric (κ,μ)-space*

Contact metric (κ,μ)-spaces

Definition

A **contact metric (κ,μ)-space** is a contact Riemannian manifold (M,φ,ξ,η,g) such that

$$R^g(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \qquad (*)$$

for some $\kappa,\mu \in \mathbf{R}$, where

$$h := \frac{1}{2}\mathcal{L}_\xi\varphi.$$

$(*)$ is called "(κ,μ)-nullity condition". If in $(*)$ $\kappa = 1$, then $h \equiv 0$ and the manifold is Sasakian.

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D.E. Blair, T. Koufogiorgos, B.J. Papantoniou, *Contact metric manifolds satisfying a nullity condition*, Israel J. Math. 1995

Motivations

1. The (κ,μ)-nullity condition determines the curvature completely.

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4. In the non-Sasakian case ($\kappa \neq 1$) the contact metric structure is not "projectable". Thus **contact metric (κ,μ)-spaces have no analogues in even dimension**.

5. A classification is known

   E. Boeckx, *A full classification of contact (κ,μ)-spaces*, Illinois J. Math. (2000)

Contact metric (κ,μ)-spaces

Theorem (Blair et alt. 1995; C.M. 2010)
Let (M,φ,ξ,η,g) be a contact (κ,μ)-manifold. Then $\kappa \leq 1$.

- If $\kappa = 1$ (M,φ,ξ,η,g) is Sasakian.
- If $\kappa < 1$ the operators h and φh admit 3 eigenvalues 0, λ, $-\lambda$, where $\lambda = \sqrt{1-\kappa}$.

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  The corresponding eigendistributions are Legendre foliations of M and satisfies

$$
\begin{aligned}
T(M) &= \mathcal{D}_h(\lambda) \oplus \mathcal{D}_h(-\lambda) \oplus \mathbf{R}\xi \qquad \text{(orthogonal)} \\
&= \mathcal{D}_h(\lambda) \oplus \mathcal{D}_{\varphi h}(\lambda) \oplus \mathbf{R}\xi \\
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Contact metric (κ,μ)-spaces

Corollary

Every (non-Sasakian) contact metric (κ,μ)-space admits a canonical almost bi-paracontact structure $(\varphi_1,\varphi_2,\varphi_3)$ given by

$$\varphi_1 := \frac{1}{\sqrt{1-\kappa}}\varphi h, \quad \varphi_2 := \frac{1}{\sqrt{1-\kappa}}h, \quad \varphi_3 := \varphi.$$

Such almost bi-paracontact structure is *integrable*, *non-normal*.

Contact metric (κ,μ)-spaces

Theorem
The canonical connection ∇^c is a **contact connection** (i.e. $\nabla^c \eta = \nabla^c d\eta = 0$), and satisfies

$$\nabla^c \varphi_1 = -\frac{2}{3}\left(1-\frac{\mu}{2}\right)\eta \otimes \varphi_2$$

$$\nabla^c \varphi_2 = \frac{2}{3}\left(1-\frac{\mu}{2}\right)\eta \otimes \varphi_1 + \frac{2}{3}\sqrt{1-\kappa}\,\eta \otimes \varphi_3$$

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Conversely …

Contact metric (κ,μ)-spaces

Theorem

Let (φ_1,φ_2,φ_3) be an *integrable* almost bi-paracontact structure on a contact manifold (M,η) such that $\nabla^c\eta = \nabla^c d\eta = 0$ and

$$\begin{aligned} \nabla^c\varphi_1 &= \qquad\qquad - a\eta\otimes\varphi_2 \\ \nabla^c\varphi_2 &= a\eta\otimes\varphi_1 \qquad\qquad + b\eta\otimes\varphi_3 \\ \nabla^c\varphi_3 &= \qquad\qquad b\eta\otimes\varphi_2 \end{aligned}$$

for some a, b > 0.

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for some a, $b > 0$.

Set

$$g_1 := d\eta(\cdot,\varphi_1\cdot) + \eta\otimes\eta$$

$$g_2 := d\eta(\cdot,\varphi_2\cdot) + \eta\otimes\eta$$

$$g_3 := -d\eta(\cdot,\varphi_3\cdot) + \eta\otimes\eta$$

and assume that g_3 is positive definite.

Contact metric (κ,μ)-spaces

Then

- (φ_1,ξ,η,g_1), (φ_2,ξ,η,g_2) are **paracontact metric structures**
- (φ_3,ξ,η,g_3) is a **contact metric structure**

Moreover, the Levi Civita connections of g_1, g_2, g_3 **satisfy a (κ_α,μ_α)-nullity condition**, where

$$\kappa_1 = 9/4\, a^2 - 1, \qquad \mu_1 = 2 - 3b$$

$$\kappa_2 = 9/4\, (a^2 - b^2), \qquad \mu_2 = 2$$

$$\kappa_3 = 1 - 9/4\, b^2, \qquad \mu_3 = 2 + 3a$$

Contact metric (κ,μ)-spaces

Let us recall that Boeckx introduced the number

$$I_M = \frac{1 - \mu/2}{\sqrt{1 - \kappa}}$$

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i) If $|I_M|>1$

$$\varphi'_1 := \frac{1}{\sqrt{(1-\mu/2)^2 - 1 + \kappa}}\left(I_M \varphi h + \sqrt{1-\kappa}\varphi\right), \quad \varphi'_2 := \frac{1}{\sqrt{1-\kappa}} h, \quad \varphi'_3 := \varphi'_1 \varphi'_2$$

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ii) If $|I_M|<1$

$$\varphi'_1 := \frac{1}{\sqrt{1-\kappa}}h, \quad \varphi'_2 := \frac{1}{\sqrt{1-\kappa-(1-\mu/2)^2}}\left(I_M h+\sqrt{1-\kappa}\varphi h\right), \quad \varphi'_3 := \varphi'_1\varphi'_2$$

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Then one proves that

$$g'_3 := -d\eta(\cdot,\varphi'_3\cdot) + \eta\otimes\eta$$

is a Riemannian metric, which is compatible with the almost contact structure.

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Corollary
Any contact metric (κ,μ)-space such that $I_M \neq \pm 1$ admits a compatible Sasakian structure.
Consequently, if M is compact, the Betti numbers b_i, $1 \leq i \leq 2n$ are even.

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1. B. Cappelletti Montano, *Bi-Legendrian structures and paracontact geometry*, Int. J. Geom. Meth. Mod. Phys. 2009
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OBRIGADO !

Let (M,φ,ξ,η,g) be a contact metric manifold endowed with a Legendre distribution L.

Define $Q{:=}\varphi(L)$, Legendre distribution. Set

$$\psi\big|_L = I, \quad \psi\big|_Q = -I, \quad \psi\xi = 0.$$

Then

$$\varphi_1 := \varphi\psi, \quad \varphi_2 := \psi, \quad \varphi_3 := \varphi$$

is an almost bi-paracontact structure.

Every contact manifold (M,η) endowed with a Legendre distribution admits infinitely many bi-Legendrian structures.

Infact let L be a Legendre distribution, (φ,ξ,η,g) be an associated contact Riemannian structure and $Q=\varphi L$. Let ψ be the paracontact structure associated with (L,Q).

For any α,β such that $\alpha^2+\beta^2=1$ we set

$$\psi_{\alpha,\beta} := \alpha\psi + \beta\varphi\psi.$$

Then

- $\psi_{\alpha,\beta}{}^2 = I - \eta\otimes\xi$
- The eingendistributions $L^+_{\alpha,\beta}$ and $L^-_{\alpha,\beta}$ of $\psi_{\alpha,\beta}$ corresponding to 1 and -1 defines transversal Legendre distributions.

Thus $\left(L^+_{\alpha,\beta}, L^-_{\alpha,\beta}\right)$ is an almost bi-Legendrian structure on (M,η).

Example

M - smooth manifold, $\dim(M)=n$. $T^*(M)$ - the cotangent bundle of M.
$\omega = y_i dx^i$ - the Liouville form on $T^*(M)$, (x^i) being coordinates on M, (y_i) fiber coordinates.
Suppose there exists a function F: $T^*(M) \to [0,+\infty[$ such that

$$F(tv) = tF(v), \quad \text{for all } t \geq 0,\ v \in T^*(M).$$

The unit cotangent bundle with respect to F is the $(2n+1)$-dimensional hypersurface of $T^*(M)$

$$S_F^{\ *}(M) = \{v \in T^*(M) \mid F(v) = 1\}.$$

Typical case: (M,g) a Riemannian manifold and take $F = \|\cdot\|$. $S_F^{\ *}(M)$ is denoted also $T_1^{\ *}(M)$ and called *cotangent sphere bundle* of (M,g).

Then $(S^*_F(M),\eta)$ is a contact manifold, where $\eta = i^*(\omega)$ and the foliation $\mathcal{F}_F$ by fibers of the projection map π: $S_F^{\ *}(M) \to M$ is a Legendre foliation of $(S_F^{\ *}(M),\eta)$.

Theorem (Pang)

Any Legendre foliation $\mathcal{F}$ on (M,η) is locally equivalent with one of the form $\mathcal{F}_F$. Moreover, the equivalence is global provided that the leaves of $\mathcal{F}$ are compact and simply connected; in this case F defines a Finsler metric on M.

A Finsler metric is a function $F\colon T^*M \to [0,+\infty[$ such that

- F is positively homogeneous: $F(tv) = tF(v)$ for any $t > 0$
- $F(v_1+v_2) \leq F(v_1) + F(v_2)$.

The triangle inequality is equivalent to the convexity of the function F^2, and this is again equivalent to the condition that $(F^2)_{ij}$ is positive definite on T^*M.

Theorem

Let (M,η) *be a contact manifold. Then there exist*

- *a Riemannian metric* g *such that*

$$g(X,\xi) = \eta(X),$$

  i.e. $g(\xi,\xi) = 1$ and $\xi \perp \mathcal{D}$

- *a tensor field* φ *of type* $(1,1)$ *such that*

$$\varphi^2 = -I + \eta \otimes \xi, \qquad g(X,\varphi Y) = d\eta(X,Y).$$

The geometric structure (φ,ξ,η,g) is called contact Riemannian (or metric) structure.

Example (Anosov flows)
Let (M,φ,ξ,η,g) be a contact Riemannian manifold for which ξ is Anosov, i.e. there exist subbundles E^s and E^u (*stable* and *unstable subbundles*), invariant along the flow $\{\omega_t\}$ of ξ, such that

$$T(M) = E^s \oplus E^u \oplus \mathbf{R}\xi$$

and

$$\|\omega_{t*}v\| \le a\exp(-ct)\|v\| \quad \text{for all } t\geq 0 \text{ and } v\in E^s{}_x$$
$$\|\omega_{t*}w\| \le a\exp(ct)\|w\| \quad \text{for all } t\leq 0 \text{ and } w\in E^u{}_x,$$

where $a,c>0$ are constants independent of $x\in M$ and $v\in E^s{}_x$, $w\in E^u{}_x$.

Then (E^s,E^u) is a flat bi-Legendrian structure on M.

- The most notable example of contact manifold for which ξ is Anosov is the tangent sphere bundle of a negatively curved manifold.

Let π': $TM \to M$ be the tangent bundle of (M,g). The tangent space to TM at a point (x,u) splits into the direct sum of the vertical subspace $VTM_{(x,u)} = \ker(\pi'_*|_{(x,u)})$ and the horizontal subspace $HTM_{(x,u)}$ with respect to the Levi Civita connection ∇^g on M:

$$T_{(x,u)}TM = VTM_{(x,u)} \oplus HTM_{(x,u)}.$$

If $X \in \Gamma(TM)$, X^h and X^v are the horizontal and vertical lift of X on TM.

The Sasaki metric g_S on TM is defined by

$$g_S(X^h,Y^h) = g_S(X^v,Y^v) = g(X,Y)\circ\pi', \quad g_S(X^h,X^v) = 0.$$

TM admits an almost complex structure J defined by

$$JX^h = X^v, \quad JX^v = -X^h.$$

The tangent sphere bundle π: $T_1M \to M$ is the hypersurface of TM defined by

$$T_1M = \{(x,u)\in TM \mid g_x(u,u)=1\}.$$

The geodesic flow of (M,g) is the horizontal vector field of TM defined by

$$\xi'_u = -JN = u^i\left(\frac{\partial}{\partial x^i}\right)^h$$

where $(x,u)\in TM$ and $N = u^i\,\dfrac{\partial}{\partial u^i}$. If $(x,z)\in T_1M$ then ξ'_z is tangent to T_1M. Let η' be the 1-form on T_1M dual to ξ' with respect to g_S and φ' be the (1,1) tensor field given by

$$\varphi' X = JX - \eta'(X)N.$$

Then

$$(\varphi',\, 2\xi',\, \tfrac{1}{2}\eta',\, \tfrac{1}{4}g_S)$$

is the standard contact metric structure on T_1M.

D-homotethic deformations

Let (M,φ,ξ,η,g) be a contact metric manifold. By a D_a-homothetic deformation we mean the change of structure tensor of the form

$$\bar{\eta} = a\eta, \quad \bar{\xi} = \frac{1}{a}\xi, \quad \bar{\varphi} = \varphi, \quad \bar{g} = ag + a(a-1)\eta \otimes \eta$$

where $a > 0$. Then

- $(M,\bar{\varphi},\bar{\xi},\bar{\eta},\bar{g})$ is also a contact metric manifold.
- A (κ,μ)-manifold is transformed in a $(\bar{\kappa},\bar{\mu})$-manifold where

$$\bar{\kappa} = \frac{\kappa + a^2 - 1}{a^2}, \quad \bar{\mu} = \frac{\mu + 2a - 2}{a}.$$

Let $(M,\mathcal{F})$ be a foliated space and ∇ be a linear connection on M.

∇ is $\mathcal{F}$-projectable if it projects to connections of the local slice space of F.
The conditions for this are:

a. if σ is a projectable cross section of $N(\mathcal{F}):=TM/T\mathcal{F}$, then $\nabla_X\sigma = 0$ for all $X\in\Gamma(T\mathcal{F})$,
b. if σ is a projectable cross section of $N(\mathcal{F})$ and X is a basic vector field, then $\nabla_X\sigma$ is a projectable cross section of $N(\mathcal{F})$.

Here by "projectable cross section" we mean that the vector field that represents σ is basic: $\sigma = [Y]_{\mathcal{F}}$, $Y\in\Gamma(TM)$, $[Y_1]=[Y_2] \Leftrightarrow Y_1-Y_2 \in\Gamma(T\mathcal{F})$.

- A *3-web* is a triple of foliations $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ on M^{2n} such that for any $\alpha, \beta \in \{1,2,3\}$, $\alpha \neq \beta$, $T(M) = T(\mathcal{F}_\alpha) \oplus T(\mathcal{F}_\beta)$.

- A *para-hypercomplex structure* on M^{2n} is given by two anti-commuting product structures I, J and a complex structure K such that $IJ = K$. Then M^{2n} admits a unique connection, called *Obata connection*, defined as the unique torsion-free connection preserving the para-hypercomplex structure.

Let N be an n-dimensional manifold and $T^{\mathbf{C}}N$ its complexifed tangent bundle. Let H be a complex subbundle of complex dimension l.

A ***CR*-manifold** of real dimension n and CR-dimension l is a pair (N,H) such that

- $H_p \cap \overline{H}_p = \{0\}$
- H is involutive.

Then there exists an unique subbundle D of TN such that $D^{\mathbf{C}} = H \oplus \overline{H}$ and an unique bundle map $J: D \to D$ such that $J^2 = -I$ and

$$H = \{X - \mathbf{i}JX \mid X \in D\}.$$

Since $\varphi^2 = -I + \eta \otimes \xi$ and $\varphi\xi = 0$, the eigenvalues of φ are 0, $\pm\mathbf{i}$. Thus

$$\mathcal{D}^{\mathbf{C}} = \mathcal{D}' \oplus \mathcal{D}''$$

where

$$\mathcal{D}' = \{X - \mathbf{i}\varphi X \mid X \in \mathcal{D}\}$$
$$\mathcal{D}'' = \{X + \mathbf{i}\varphi X \mid X \in \mathcal{D}\}$$

Theorem (Tanno)

$(M^{2n+1}, \mathcal{D}')$ *is (strongly pseudo-convex) CR-manifold* if and only if

$$(\nabla^g{}_X\varphi)Y = g(X+hX,Y)\xi - \eta(Y)(X+hX).$$