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1 Introduction to dynamical systems

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Deterministic time evolutions, and signals or time series derived from these, are well known from physics, biology, chemistry, economics, etc. The theory of dynamical systems provides an encompassing framework for this. We here restrict to systems with a finite dimensional state space, or as it is said in classical mechanics: with 'finitely many degrees of freedom'.

Below, with help of several examples, we first develop a general definition of 'dynamical system', where continuous- resp. discrete-time systems are just special cases, linked together by the Poincaré mapping and the suspension. Second we distinguish between various types of dynamical behaviour: steady state or equilibrium behaviour, periodic behaviour, etc. Also, as a prelude to later chapters, we briefly meet more complicated behaviour, like quasi-periodic and chaotic dynamics.

While introducing the corresponding language, we present a problem setting. Roughly speaking our interest is with the asymptotic dynamics for infinite time. To fix thoughts, one may think of evolutions in systems with dissipation, that in the state space have settled down on an attractor. One important feature is the geometry of these attractors. Another point is the dependence of the asymptotic dynamics on the initial value of the evolution. As we shall see, certain types of asymptotic behaviour can be more or less typical for the system at hand, than others. Finally, it will be of importance how this whole structure depends on perturbations of the system. We'll call a dynamical property *persistent* if it is unchanged under small enough perturbations of the system. Here the notion of persistent can be seen as opposed to pathological.

As said before, many of these subjects will be explored in later chapters of this volume. Also, for background reading we refer to e.g. Arnold [1], Broer and Takens [4] and Ruelle [8,9].

1.1 What is a dynamical system?

To fix thoughts we start presenting three examples of a dynamical system.

1.1.1 Example: Ordinary differential equations Consider the system of autonomous, ordinary differential equations

$$\dot{x} = F(x), \tag{1.1}$$

$x \in \mathbb{R}^n$, with $F\colon \mathbb{R}^n \to \mathbb{R}^n$ a smooth function. As is known from e.g. Arnold [1] or Hirsch and Smale [6] the system (1.1) has a (global) solution

$$\phi\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n,$$

determined by

$$\phi(x,0) = x \quad \text{and} \quad \frac{\partial}{\partial t}\phi(x,t) = F(\phi(x,t)).$$

For each $x \in \mathbb{R}^n$ fixed, the solution of (1.1) with initial value x is given by $t \mapsto \phi(x,t)$. These solutions only need to exist for $|t|$ small: it may occur that for some $t_0 \in \mathbb{R}$ we have that $\phi(x,t_0) = \infty$, i.e. that the solution disappears to infinity in finite time. Therefore, the solution ϕ in general is only defined on a neighbourhood of $\mathbb{R}^n \times \{0\}$ in $\mathbb{R}^n \times \mathbb{R}$. Moreover ϕ is smooth both in t and in x: the smooth solution also depends smooth on the initial condition. For t fixed, with $|t|$ sufficiently small, we also may consider the map

$$\phi^t\colon \mathbb{R}^n \to \mathbb{R}^n, \quad \phi^t(x) = \phi(x,t),$$

possibly only locally defined. The map ϕ^t is called the *flow over time* t of the system (1.1). As far as defined, it has the *group property*:

$$\phi^0 = \mathrm{Id} \quad \text{and} \quad \phi^s \circ \phi^t = \phi^{s+t}. \tag{1.2}$$

Notice that ϕ^t is smoothly invertible: $(\phi^t)^{-1} = \phi^{-t}$. Such a smooth map with a smooth inverse is generally called a *diffeomorphism*.

Example: The mathematical pendulum As an explicit example we consider the planar mathematical pendulum, with equation of motion

$$\ddot{x} = -\omega^2 \sin x,$$

$x \in \mathbb{R}$, where $\omega > 0$ is some constant. If we write $y := \dot{x}$ we obtain a system

$$\dot{x} = y,$$
$$\dot{y} = -\omega^2 \sin x,$$

$(x,y) \in \mathbb{R}^2$, of the form (1.1). The (x,y)-plane is usually called the *phase plane*. In this plane we sketched the images of some solution curves $t \mapsto \phi^t(x,y)$, so obtaining a picture called *phase portrait*, cf. Figure 1.1. The arrows indicate the direction of the time evolution. It is not hard to see that in this case the solutions are defined for all t. Also it is easily seen to what kind of motion of the pendulum each of the sketched curves does correspond.

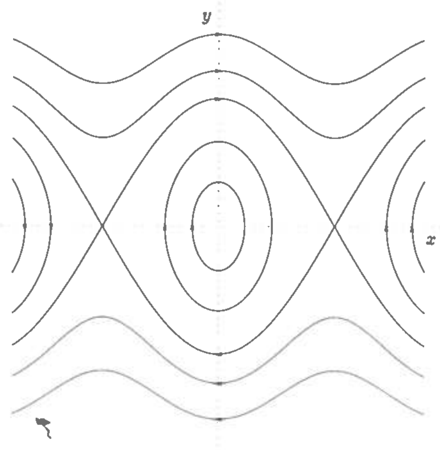


Figure 1.1 Phase portrait of the planar mathematical pendulum.

In this example we worked with continuous time. The following two examples will have discrete time.

1.1.2 Example: Endomorphisms In stead of giving a general definition of endomorphism, we illustrate our ideas with help of two concrete cases.

The Newton algorithm Let $p = p(x)$ be a real polynomial with degree different from 1. Consider the associated map

$$\phi: x \mapsto x - (p(x)/p'(x)).$$

Apparently ϕ is a rational map, therefore having an extension as a complex analytic, and therefore smooth, map from the Riemann sphere $\mathbb{C} \cup \{\infty\}$ to itself. The real part of this sphere is the circle $\mathbb{R} \cup \{\infty\}$.

Now let $x_0, x_1, x_2, \ldots$ be a sequence of points with $x_{j+1} = \phi(x_j)$, for all $j \geq 0$. If this sequence converges to a value $x_\infty \neq \infty$, then $\phi(x_\infty) = x_\infty$ and $p(x_\infty) = 0$. This directly follows from the definition. The method to find zeroes of p based on this method is named after Newton, hence the name *Newton algorithm* for our map ϕ.

So we are lead to consider iterates $\{\phi^j\}_{j\geq 0}$ of ϕ. This set of iterates forms a halfgroup, i.e. here the property (1.2) holds for all $s, t \in \mathbb{Z}$ with $s, t \geq 0$.

Notice that in general ϕ is not invertible; such smooth maps usually are called *endomorphisms*. If ϕ is not injective this means that for some points x_0 the preimage $\phi^{-1}(x_0)$ consists of more than one point. Interpreting j as 'time', we conclude that the 'past' of x_0 can not be reconstructed. Also notice that this problem does not occur in Example 1.1.

The Logistic Map Another example of an endomorphism is the so-called Logistic Map

$$\phi: [0,1] \to [0,1], \quad \phi(x) := 4x(1-x).$$

Again we can consider the associated halfgroup of iterates $\{\phi^j\}_{j\geq 0}$. In this case there exists an interesting interpretation in terms of population dynamics. If $x_0, x_1, x_2, \ldots$ is a sequence of points with $x_{j+1} = \phi(x_j)$ for all $j \in \mathbb{Z}, j \geq 0$, then x_j can be thought of as the (scaled) size of a generation of one-day-flies, on the day numbered j. In this way j really signifies time. It will be clear that also in this case the past in general can not be reconstructed.

Remarks

(i) A different form of the Logistic Map is

$$\psi: [-1,1] \to [-1,1], \quad \psi(x) := 1 - 2x^2;$$

(ii) One also frequently considers the quadratic families

$$\phi_\mu(x) = \mu x(1-x) \quad \text{or} \quad \psi_\mu(x) = 1 - \mu x^2,$$

where μ is a real parameter. In fact, for population dynamical purposes it is more realistic to restrict to ϕ_μ or ψ_μ with $\mu \leq 4$.

1.1.3 Example: Diffeomorphisms As said before, a *diffeomorphism* is an endomorphism with a smooth inverse. It may be clear from the above that in the case of a diffeomorphism the past always can be reconstructed. Related to this is the fact that for a diffeomorphism ϕ the iterates $\{\phi^j\}_{j=-\infty}^{+\infty}$ form a group, again compare (1.2).

We now present two constructions that link diffeomorphisms to the autonomous systems of ordinary differential equations of Example 1.1.

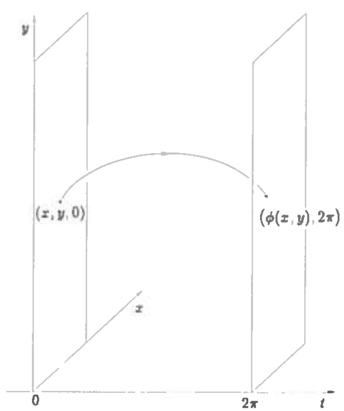


Figure 1.2 Poincaré- or period mapping of the periodically forced pendulum.

The Poincaré map Again, instead of a general treatment we give an example, for instance compare [2]. Consider the periodically forced pendulum with equation of motion

$$\ddot{x} = -\omega^2 \sin x + \varepsilon \cos t,$$

$x, t \in \mathbb{R}$, where ε and ω are positive constants. In the (generalized) phase space with coordinates x, y and t this defines an autonomous system

$$\dot{x} = y,$$
$$\dot{y} = -\omega^2 \sin x + \varepsilon \cos t,$$
$$\dot{t} = 1,$$

of ordinary differential equations of the form (1.1). Let $\psi = \psi^\tau(x, y, t)$ be the associated (global) solution or flow. Because of the periodicity in t one here may consider the *Poincaré- or period mapping*

$$\phi\colon \mathbb{R}^2 \to \mathbb{R}^2,$$

defined by the relation

$$\psi^{2\pi}(x, y, 0) = (\phi(x, y), 2\pi). \tag{1.3}$$

Geometrically think of following the solution curve $\tau \mapsto \psi^\tau(x, y, 0)$ from the initial value $(x, y, 0)$, which lies in the plane $t = 0$, till it hits the plane $t = 2\pi$ in the point $(\phi(x, y), 2\pi)$. It is not difficult to see that ϕ is a diffeomorphism. The behaviour of ϕ, and in particular of the group $\{\phi^j\}_{j=-\infty}^{+\infty}$, gives a lot of information on the behaviour of the system of differential equations. For example, if (x_0, y_0) is a periodic point of ϕ of period $n \geq 1$, i.e. if

$$\phi^n(x_0, y_0) = (x_0, y_0),$$

then the solution $\tau \mapsto \phi^\tau(x_0, y_0, 0)$ is periodic with period $2\pi n$.

Remarks

(i) The time periodicity of the system is better expressed if for the (generalized) phase space we take $\mathbb{R}^2 \times \mathbb{R}/(2\pi\mathbb{Z})$ with coordinates x, y and $t(\text{mod}2\pi)$. The time coordinate t then varies over the circle $\mathbb{R}/(2\pi\mathbb{Z})$ and our Poincaré mapping ϕ exactly is the *return mapping* of the plane $t = 0\,(\text{mod}2\pi)$.
Notice that in this setting the periodic solutions of the system of differential equations are closed curves. Also notice that an invariant circle for the return map ϕ corresponds to an invariant 2-torus for the system of differential equations;

(ii) In this particular example there is no friction, which by Stokes' Theorem, e.g. see Chapter 9 of this volume, implies that ϕ is area preserving. However, if we consider the damped pendulum with periodic forcing

$$\ddot{x} = -\omega^2 \sin x - c\dot{x} + \varepsilon \cos t$$

($c > 0$), the Poincaré mapping is defined analogously, though no longer area preserving.

With help of hyperplanes that are transversal to the solution curves of systems of differential equations, Poincaré mappings can be defined in great generality. On the other hand every diffeomorphism can be regarded as the Poincaré mapping of some system of differential equations, as we shall see now.

The suspension Suppose we are given a diffeomorphism $\psi\colon N \to N$, where N is some (smooth) manifold. A manifold M is wanted with on it a vector field, such that ψ is a corresponding Poincaré mapping. Here a vector field is meant to be a globally defined system of ordinary differential equations. In fact we shall find a vector field such that for the associated (global) solution or flow the relation (1.3) holds. For this purpose first consider the manifold-with-boundary $N \times [0, 2\pi]$, where the last coordinate wil be called t. On $N \times [0, 2\pi]$ consider the vector field $\partial/\partial t$, in any set of local coordinates $(x_1, x_2, \ldots, x_n)$ on N given by

$$\dot{x}_j = 0 \quad (1 \leq j \leq n)$$
$$\dot{t} = 1.$$

The Poincaré mapping $N \times \{0\} \to N \times \{2\pi\}$ of this vector field only is the Identity Map. However, next let us use the diffeomorphism ψ to glue the boundaries $N \times \{0\}$ and $N \times \{2\pi\}$, namely by identifying $(x, 2\pi) \sim (\psi(x), 0)$, for all $x \in N$. This leaves us with a smooth manifold M, on which $\partial/\partial t$ induces a smooth vector field, the so-called *suspension* of ψ. One directly proves that the global solution flow of this vector field is given by

$$\phi^\tau(x, t) = \big(\psi^k(x), (t + \tau)\,(\text{mod}2\pi)\big),$$

$x \in N,\ t \in \mathbb{R}$, where k is the number of integer multiples of 2π between t and $t + \tau$. From this it follows that the relation (1.3) holds, with ϕ and ψ having their role interchanged. Then, indeed ψ is the Poincaré mapping of the constructed vector field, as required.

We conclude this section with a general definition of dynamical system. This definition includes the above examples.

1.1.4 Definition of dynamical system *A dynamical system is a triple (M, T, ϕ), consisting of a state space or phase space M, a time set T, and an evolution operator ϕ: $M \times T \to M$, such that*

(i) *$T \subseteq \mathbb{R}$ is an additive halfgroup, i.e. that $0 \in T$ while for all $t, s \in T$ also $t + s \in T$;*

(ii) *For all $x \in M$ and $t, s \in T$ we have*

$$\phi(x, 0) = x \quad \text{and} \quad \phi(\phi(x,t), s) = \phi(x, s+t).$$

As before we write $\phi^t(x) := \phi(x,t)$, $(x,t) \in M \times T$. We then see that $\{\phi^t\}_{t\in T}$ is a halfgroup with respect to composition, compare (1.2). The elements of M represent de possible *states* of the 'system'. The state that we get from $x \in M$ by evolution over the time $t \in T$ then is $\phi^t(x)$. Therefore the set $\{\phi^t(x)\}_{t\in T}$ is named the *evolution* of x .

As announced, the above three examples now are special cases, if only we forget about the problems of the domain of definition of ϕ in Example 1.1.1. The respective time sets are $T = \mathbb{R}$, $T = \mathbb{Z}_+ := \{j \in \mathbb{Z} | j \geq 0\}$ and $T = \mathbb{Z}$. Observe that in the case of Example 1.1.1 the evolution of a point x exactly is the solution curve with initial value x.

1.2 Setting of the problem

As said briefly before, we are interested in the asymptotic behaviour of evolutions as $|t| \to \infty$; for simplicity we mainly restrict to the case where $t \to \infty$. So let a dynamical system (M, T, ϕ) be given. Moreover let $\{\phi^t(x_0)\}_{t\in T}$ be the evolution with initial value x_0. To begin with we discuss some simple types of this.

1.2.1 Definitions of some simple types of evolution We say that the evolution is *stationary* if $\phi^t(x_0) = x_0$ for all $t \in T$. The point x_0 then is called an *equilibrium-* or *restpoint*. For a dynamical system as in Example 1.1.1, generated by the differential equation $\dot{x} = F(x)$, this means that $F(x_0) = 0$. For the endomorphisms ϕ of the Examples 1.1.2, 3 it means that $\phi(x_0) = x_0$: x_0 is a *fixed point* or *fixpoint* of ϕ.

Another simple kind of evolution is the *periodic* evolution. In that case we assume the evolution to be non-stationary, while for a certain $p \in T$ with $p > 0$ we have $\phi^p(x_0) = x_0$. By the group property this immediately implies that $\phi^{t+p}(x_0) = \phi^t(x_0)$ for all $t \in T$. In Example 1.1.1 this means that the curve $t \mapsto \phi^t(x_0)$ is a closed curve, with period p. In Examples 1.1.2, 1.1.3 it means that $\phi^p(x_0) = x_0$ for $p > 1$: x_0 then is a *periodic point* of ϕ, with period p.

We further distinguish so-called *asymptotically stationary* and *asymptotically periodic* evolutions: these are evolutions $\{\phi^t(y)\}_{t\in T}$, with initial value y, such that $\phi^t(y)$ converges to such a stationary resp. periodic evolution, as $t \to \infty$.

For a fixed stationary resp. periodic evolution $\{\phi^t(x)\}_{t\in T}$ the points $y \in M$ that are asymptotic to it for $t \to \infty$, form the *stable set* of $\{\phi^t(x)\}_{t\in T}$. Under very general conditions this stable set is an immerged submanifold of M, e.g. compare [5,7]. In particular this holds when the stable set is open in M; in this case we call the stable manifold the *basin of attraction* of $\{\phi^t(x)\}_{t\in T}$.

1.2.2 Example: The hyperbolic point attractor As in the Examples 1.1.2, 1.1.3, let ϕ: $M \to M$ be an endomorphism with a fixpoint x_0, so with $\phi(x_0) = x_0$. Suppose that for all eigenvalues λ of the derivative $D_{x_0}\phi$ we have $|\lambda| < 1$. It then can be shown, e.g. see [1,6], that the stationary evolution $\{\phi^t(x_0)\}_{t\in T}$ has a basin of attraction with x_0 as an interior point. For all initial values y in this basin we have $\lim_{t\to\infty} \phi^t(x_0) = x_0$. The point x_0 is called a *hyperbolic point attractor*.

Remarks

(i) The word 'hyperbolic' refers to the fact that no eigenvalue of $D_{x_0}\phi$ is on the complex unit circle. Since then they are all inside the unit disk, the attracting character of x_0 already can be seen from this linear part;

(ii) A similar concept exists for the continuous time systems $\dot{x} = F(x)$ of Example 1.1.1. Then we have $F(x_0) = 0$, while the eigenvalues λ of $D_{x_0}F$ have to satisfy $\text{Re}(\lambda) < 0$;

(iii) Analogously *hyperbolic periodic attractors* are defined.

To develop our ideas further we next give two explicit examples.

1.2.3 Example: The pendulum, with and without damping

The planar mathematical pendulum with linear damping has the equation of motion

$$\ddot{x} = -\omega^2 \sin x - c\dot{x},$$

here $c \geq 0$ is the damping coefficient. In the phase plane this gives the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\omega^2 \sin x - cy. \end{aligned}$$

We observe that by periodicity in the first variable x, a more appropriate phase space would be the cylinder $\mathbb{R}/(2\pi\mathbb{Z}) \times \mathbb{R}$, see above for a similar remark. A phase portrait for the case $c = 0$ was already given in Example 1.1.1, for $c > 0$ a phase portrait looks as in Figure 1.3.

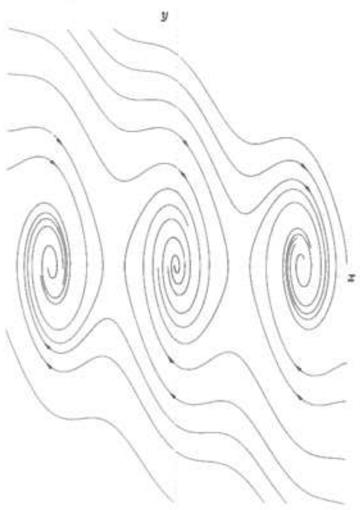


Figure 1.3 Phase portrait of the damped pendulum.

First let us discuss the case $c = 0$. From Example 1.1.1 we see that in the case $c = 0$ almost all evolutions are periodic. Furthermore there are two stationary and two asymptotically stationary evolutions.

Notice that these asymptotically stationary evolutions lie on one-dimensional stable manifolds: two curves in the phase plane. These

curves separate two types of motion, can you describe those? This is why such stable and unstable manifolds are called *separatrices*.

Also observe that neither the stationary, nor the periodic evolutions have a basin of attraction. The reason is that for $c = 0$ our vector field has divergence zero. This implies that the corresponding flow ϕ^t is area preserving, from which it follows that there can be no attractors at all. Compare Chapter 9 of this volume.

In the case $c > 0$ this is different. Now the point $(x, y) = (0, 0)$ is a hyperbolic point attractor, with in its basin of attraction almost the whole phase plane. Also the remaining evolutions are (asymptotically) stationary, as it is easily shown. What is the role of these separatrices?

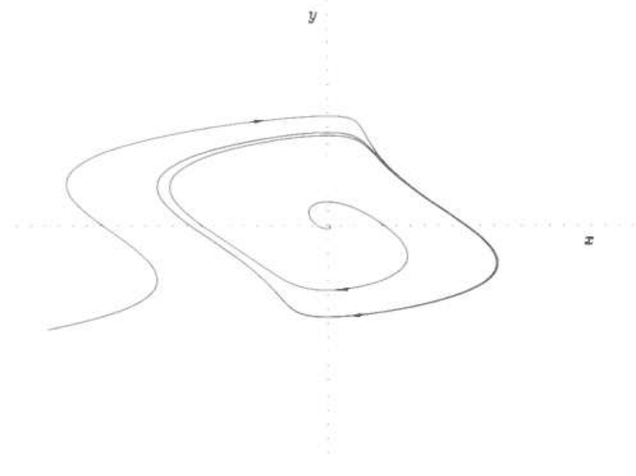


Figure 1.4 Phase portrait of the Van der Pol oscillator.

1.2.4 Example: The differential equation of Van der Pol

Consider the following second order equation

$$\ddot{x} = -x - \dot{x}(x^2 - 1),$$

which is the equation of motion of an oscillator with non-linear damping, negative for $(x, \dot{x})$ near $(0, 0)$. Such oscillators frequently occur in electronics; this particular example goes with the name of Van der Pol.

In the phase plane we get the system

$$\dot{x} = y,$$
$$\dot{y} = -x - y(x^2 - 1),$$

with a phase portrait as depicted in Figure 1.4. It appears that here we deal with a hyperbolic periodic attractor, with $\mathbb{R}^2 \setminus \{0\}$ as a basin of attraction. Moreover the point 0 is a repelling equilibrium, sometimes called (point) source. Generally speaking we call a dynamical system with continuous time an *oscillator* if it has one periodic attractor with almost the whole state space in its basin of attraction. In such a case, for almost all initial values, the evolution is asymptotic to this periodic one.

1.2.5 Provisory setting of the problem Inspired by the above examples we come to a provisory setting of the problem, which is to give a description of the asymptotic behaviour of dynamical systems in dependence of the initial value, where attention has to be paid to continuity resp. discontinuity in this dependence.

In order to have wider interpretation of the term 'asymptotic behaviour' we now first give a more general definition of the concept of attractor.

1.2.6 Definition of attractor *We say that $A \subseteq M$ is an attractor of the dynamical system (M, T, ϕ) if the following three properties hold:*

(i) *A is compact;*

(ii) *$\phi^t(A) = A$ for all $t \in T$, which means that A is invariant under the evolutions of the system;*

(iii) *There exists a neighbourhood $\mathcal{U}$ of A in M such that for all $y \in \mathcal{U}$ the point $\phi^t(y)$ converges to A as $t \to \infty$. In somewhat other words: $A = \bigcap_{t>0} \phi^t(\mathcal{U})$.*

Remark In the literature, e.g. compare [4,8,9], the freshly defined object often is called 'attracting set'. For an attracting set to be an attractor then an extra requirement is the so-called *transitivity*: for a certain $a \in A$ the evolution $\{\phi^t(a)\}_{t \in T}$ has to be *dense* in the set A. Transitivity prevents an attractor to fall apart into different components.

It may be clear that the hyperbolic point- and periodic attractors fall under this definition. Next let us discuss some further examples.

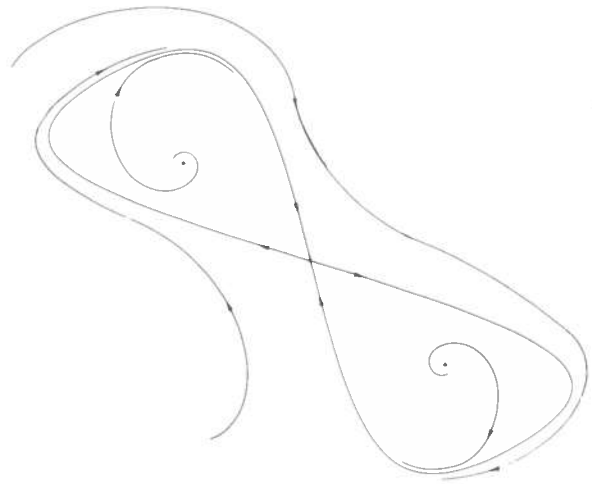


Figure 1.5 A universal attractor.

1.2.7 Examples: Some other attractors

The universal attractor On $\mathbb{R}^n$ consider a dynamical system and suppose that there exists a compact subset C of $\mathbb{R}^n$, such that every evolution from a certain time on remains in C forever. (We sometimes abbreviate this by saying that ∞ is a source.) The universal attractor A of such is system then is defined as the largest, compact, invariant set. One can show that $A = \bigcap_{t>0} \phi^t(C)$.

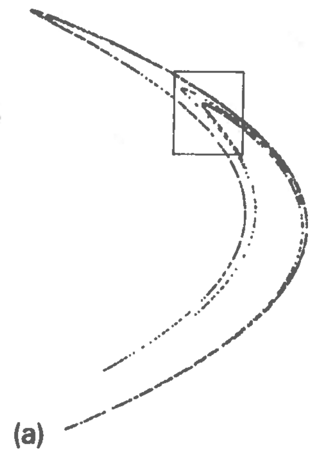
It is easy to verify that the universal attractor is an attractor in the sense of Definition 1.2.6. Figure 1.5, depicting a system with continuous time in $\mathbb{R}^2$, moreover shows that a universal attractor does not have to be transitive.

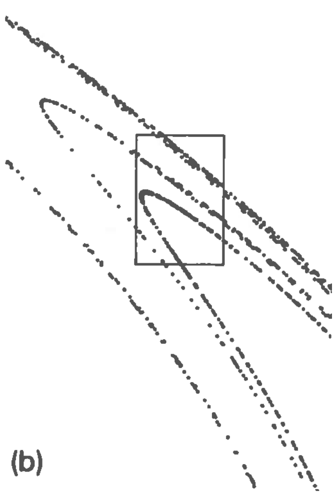
Remark In the sequel, especialy the Chapters 3 and 5, many considerations assume the state space to be a compact manifold. Most of these are equally valid on $\mathbb{R}^n$, provided the existence of a universal attractor and provided the restriction to the case where $t > 0$.

The Hénon attractor The Hénon map is a diffeomorphism

$$\phi\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto (1 + y - ax^2, bx),$$

where $a > 0$ and $b > 0$ are constants. Following the recipe of the Examples 1.1.2, 1.1.3 the map ϕ gives us a dynamical system with time set $T = \mathbb{Z}$.





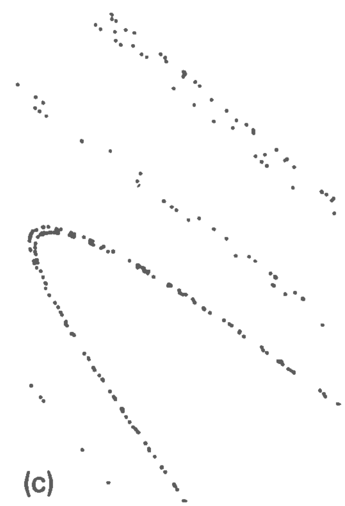


Figure 1.6a, b, c The Hénon attractor and some enlargements.

We take $a = 1.4$ and $b = 0.3$ and in Figure 1.6a have a computer sketch for us the evolution with initial value $(x, y) = (0, 0)$. Apart from the first iterates $(x, y) = (0, 0)$, $(x, y) = (1, 0)$ and $(x, y) = (-0.4, 0.3)$ we find a complicated picture that looks much like a strongly oscillating curve. This impression is more or less confirmed by the Figures 1.6b and 1.6c showing an enlargement of the rectangle indicated in the preceeding picture. Experimentally it appears that, apart from the first few iterates, whatever the initial value near $(x, y) = (0, 0)$, always this same figure emerges. Let H be the name of this, experimentally defined, figure: H is called the *Hénon attractor*. It is, however, not yet known whether or not H satisfies Definition 1.2.6, although recently partial results have been obtained in this direction.

From the above it will be clear that an attractor can be larger and geometrically more complicated than a point on a circle. An important geometric characteristic of an attractor is its *dimension*, which may be integer valued or not. In the latter case the attractor is a *fractal*. A numerical estimation of the fractal dimension (i.e. the *limit capacity*) of the Hénon attractor is 1.2... These attractors often are given the adjective *strange*.

Another point is the dynamics within such an attractor which may be more or less orderly. The criterion for disorderly or *chaotic* dynamics is the so-called ***sensitive dependence on initial values***. This roughly means that evolutions with nearby initial values in the attractor are driven apart as $t \to \infty$. Also here characteristics occur like ***entropy***, and ***Lyapunov exponents*** which are a measure for the amount of chaos. As is to be expected, stationary and periodic attractors are not chaotic, while usually fractal attractors do have chaotic dynamics. On the other hand also important examples exist of smooth attractors, so with integer dimension, which have chaotic dynamics. An example of this is the so-called Doubling or Baker Transformation on the circle, given by $\phi: \mathbb{R}/(2\pi\mathbb{Z}) \to \mathbb{R}/(2\pi\mathbb{Z})$, $x \mapsto 2x(\mathrm{mod} 2\pi)$. For further reading on this subject we refer to the Chapters 5 and 6 of this volume, resp. by Takens and Van Strien. Other references are, for instance, [3,4,8,9].

Next let us commit some phenomenology: we take a relatively simple dynamical system, viz. a damped pendulum with periodic excitation, and have the computer sketch projections of various evolutions. We shall try to determine the type of dynamics involved.

The interpretation of numerical or experimental data, necessarily having a finite amount of precision and duration over a finite time interval, always involves uncertainties. First of all it can never be decided whether or not the signal is periodic with a period that is larger than the time interval at hand. For example, the Hénon attractor of Example 1.2.7 could very well be a periodic attractor of large period.

Nevertheless here we may be guided by the theory. If on theoretical grounds we know that a certain type of evolution occurs for a large set of initial values, then this may support our guess.

1.2.8 Example: Some evolutions of periodically excited pendulum

We consider the damped, periodically forced pendulum, with the following equation of motion:

$$\ddot{x} = -\omega^2 \sin x - c\dot{x} + \varepsilon \cos t,$$

compare previous examples. To begin with we work in the (generalized) phase space $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ with coordinates $x, y := \dot{x}$ and t, where we obtain the system form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\omega^2 \sin x - cy + \varepsilon \cos t, \\ \dot{t} &= 1. \end{aligned}$$

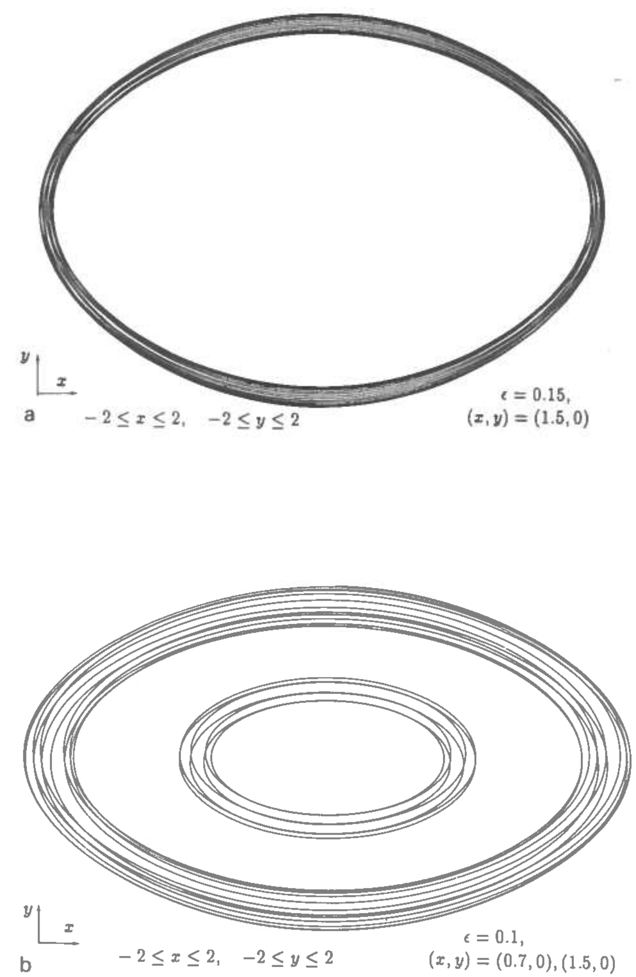


Figure 1.7a, b Projected evolutions of the undamped pendulum.

From now on we fix $\omega = 1.43$ and, as before, distinguish between the cases $c = 0$ and $c > 0$. In all pictures we specify the constants ε, resp. c and ε, as well as the initial values (x, y).

As we saw earlier, the case $c = 0$ has divergence zero, which excludes the existence of attractors. To fix thoughts, in Figure 1.7a, b and c we presented the projections of several evolutions on the (x, y)-plane or the (x, t)-plane.

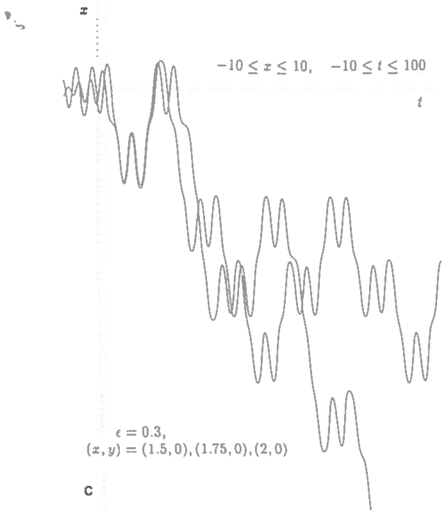


Figure 1.7c Projected evolutions of the undamped pendulum.

Interpretation: From the theory we can derive that for small ε there are a lot of so-called quasi-periodic evolutions. This follows from the KAM-Theorem, compare this volume, Chapters 4 and 9. In fact, if the initial value is chosen at random, the probability that it lies on a quasi-periodic evolution is positive; it is even close to 1 for small ε. In the phase space $\mathbb{R}^2 \times \mathbb{R}/(2\pi\mathbb{Z})$ with coordinates (x, y) and $t(\bmod 2\pi)$, such a quasi-periodic evolution takes place on an invariant 2-torus. Up to a smooth transformation it is generated by a system

$$\dot{\varphi}_1 = \omega_1, \quad \dot{\varphi}_2 = \omega_2$$

$\varphi_1, \varphi_2 \in \mathbb{R}/(2\pi\mathbb{Z})$, of differential equations on the standard 2-torus $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, where the frequency ratio ω_1/ω_2 is irrational. The

corresponding evolution curves are screw-lines each of which densely fills the torus, again compare Chapter 4. Now the curves in the Figures 1.7a and b very much look like projections of such quasi-periodic evolutions, so this would be our guess in these cases. When ε grows larger, the probability of hitting a quasi-periodic solution decreases. In Figure 1.6c we see how close to initial values with a, probably, quasi-periodic evolution we find initial values with an evolution that looks more complicated: more or less oscillatory behaviour is alternated with behaviour where the pendulum sweeps a certain number of times around its point of suspension. One might think that there is a sensitive dependence on initial value, which also here would imply chaos.

Next we come to the case $c > 0$. In order to express all the periodicities of our system properly we also take $x(\bmod 2\pi)$, hence considering the phase space $\mathbb{R}/(2\pi\mathbb{Z}) \times \mathbb{R} \times \mathbb{R}/(2\pi\mathbb{Z})$, with coordinates (x, y, t). From the positive damping we conclude two important dynamical features. The first is the existence of a universal attractor, compare Example 1.2.7. The second consequence is the non-existence of invariant 2-tori: in the three-dimensional phase space they would enclose some invariant volume, which can be shown to be impossible in the case $c > 0$.

Interpretation: The Figures 1.8a and b seem to show an asymptotically periodic evolution with a long 'initial piece'.

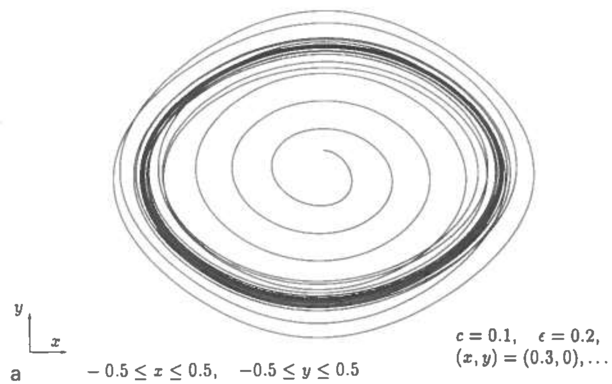


Figure 1.8a Projected evolutions of the damped pendulum.

In Figure 1.8c we find a behaviour similar to that of Figure 1.7c, of course, apart from the quasi-periodic evolutions. Also here our guess points into the direction of chaos.

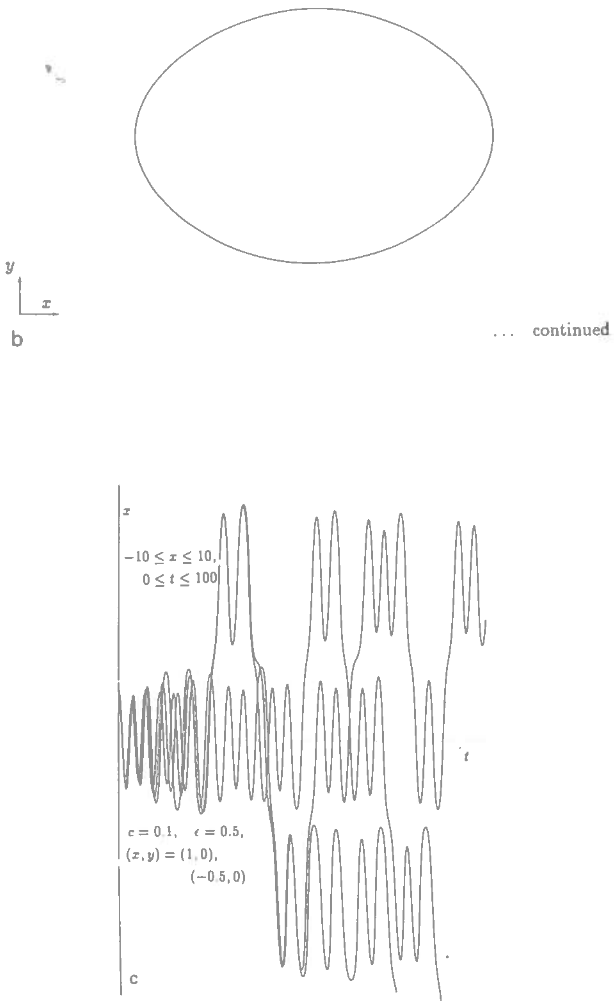


Figure 1.8b, c Projected evolutions of the damped pendulum.

A question is whether the (universal) attractor is a strange attractor or not: iteration of the Poincaré mapping seems to give a picture similar to the Hénon attractor, see Figure 1.6. We note that the present example is a model for the so-called Josephson junction from the theory of superconductivity. For more details on this and on other examples also compare [4].

Since our provisory problem setting 1.2.5, we saw several examples of asymptotic behaviour as $t \to \infty$. Let us briefly summarize some aspects regarding the dependence on the initial value. Here, however, we do not attempt to be complete. In simple cases, like the autonomous pendulum, we saw how the type of this asymptotic dynamics depends on the initial value: where it is continuous and where not. Also we can distinguish between more and less typical behaviour: e.g., for the undamped pendulum the typical dynamics is periodic, while for the damped pendulum the typical dynamics is asymptotically stationary to the lower equilibrium. All other dynamical behaviour is exceptional, at least from the view point of choosing initial values at random. In the case of the undamped pendulum with periodic excitation we met quasi-periodic dynamics, being rather typical, since it occurs with positive probability when the initial value is chosen randomly. However, as we shall see in Chapter 4, here the behaviour usually is not continuous under variation of the initial value. Both here and in the version with damping it seems that chaotic asymptotics also is rather typical.

We now conclude this section with an extension of the problem setting.

1.2.9 Setting of the problem Continuing the line of thought of this chapter, we are interested in what is the typical and a-typical asymptotic dynamics of a given system. The extension is, that we also wish to know how this behaviour changes under perturbations of the whole system. To this end we need a suitable *topology* on the space of dynamical systems under consideration. If, under small perturbation, this behaviour does not change much we speak of *persistent* behaviour. In the next chapter of this volume we shall define such a topology and give a further discussion on persistence of dynamical properties.

Sometimes the behaviour is not persistent, but we can introduce a parameter μ, such that the non-persistent behaviour occurs for $\mu = 0$, while both for $\mu < 0$ and for $\mu > 0$ there is persistence. In important cases we so obtain an *unfolding* that, regarded as a 1-parameter family of dynamical systems, has persistent dynamical behaviour. Taking this point of view, we end up in the realm of bifurcation theory. Chapter 3 of this volume, by Takens, treats the bifurcation theory of 'simple' attractors.

Remark In the above we several times met a structural difference between systems with, and systems without damping. We shall illustrate this on the autonomous case.

The undamped pendulum has typically periodic behaviour and this property appears to be persistent in the universe of undamped autonomous systems. However if we perturb in a wider universe by introducing a small amount of damping, we see that all periodicity disappears: so periodic behaviour is not at all persistent for such perturbations.

The message of this is that, in general, the persistence of dynamical properties may depend on the structures that have to be preserved and on the corresponding restrictions in the universe of perturbations.

1.3 References

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