# Dynamical Systems 2015 — Exam

17th of December 2015 — time allowed 2h and 30 minutes

1) Let $h(x_1, x_2)$ be the diffeomorphism of $\mathbf{R}^2$ given by $h(x_1, x_2) = (x_1 + x_2 - 2, (2 - x_2)/3)$, with $h^{-1}(y_1, y_2) = (y_1 + 3y_2, -3y_2 + 2)$. Obtain the expression of $w(y_1, y_2) = (h_* v)(y_1, y_2)$ for the vector field $v(x_1, x_2) = (2 - x_1 - x_2 - (2 - x_2)^3, (2 - x_2)^3)$. Discuss the stability of the equilibria of $(\dot{y}_1, \dot{y}_2) = w(y_1, y_2)$ and plot its phase portrait. Do the same for $(\dot{x}_1, \dot{x}_2) = v(x_1, x_2)$.

2) Let $u : \mathbf{R} \longrightarrow \mathbf{R}$ be a non constant function of class $C^\infty$. Show that if there are two points $x_1 < x_2 \in \mathbf{R}$ such that $u(x_1) = u(x_2)$ and $u'(x_1) < 0$ then the equation $\ddot{x} = -u'(x)$ has a non constant periodic solution.

3) For the differential equation in $\mathbf{R}^2$ equivalent to $\ddot{x} = 8x - 4x^3$ describe $\omega(2, 0)$, $\alpha(-\sqrt{2}, 0)$ and $\alpha(1, 0)$.

4) Plot the possible phase portraits of $(\dot{x}, \dot{y}) = (y + a(2x - x^3), 8x - 4x^3)$ for small $a \geq 0$. *Hint: for $a = 0$ this is the equation in **??**).*

5) The Lie derivative of $f(x, y) = x^2 + y^2$ with respect to a certain $C^\infty$ vector field $v$ in $\mathbf{R}^2$, satisfies $L_v f(x, y) > 0$ if $|(x, y)| = 2$ and $L_v f(x, y) < 0$ if $|(x, y)| = 3$. Show that if $|v(x, y)| \neq 0$ at all points $(x, y)$ such that $2 \leq |(x, y)| \leq 3$, then the differential equation $(\dot{x}, \dot{y}) = v(x, y)$ has a non constant periodic solution.

6) Let $f : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ be the linear map represented by the matrix $A = \begin{pmatrix} 1/4 & 1/20 \\ 0 & 1/5 \end{pmatrix}$. Plot the trajectories of the points $(1, -1)$, $(1, 0)$ and $(1, 1)$ and identify the invariant subspaces $E^s$ and $E^u$.