Dynamical Systems 2015 — Exam

Licenciaturas em Matemática e Ciência de Computadores

17[th] of December 2015 — time allowed 2h and 30 minutes

1)

$$Dh(h^{-1}(y_1,y_2)) = \begin{pmatrix} 1 & 1 \\ 0 & -1/3 \end{pmatrix}$$

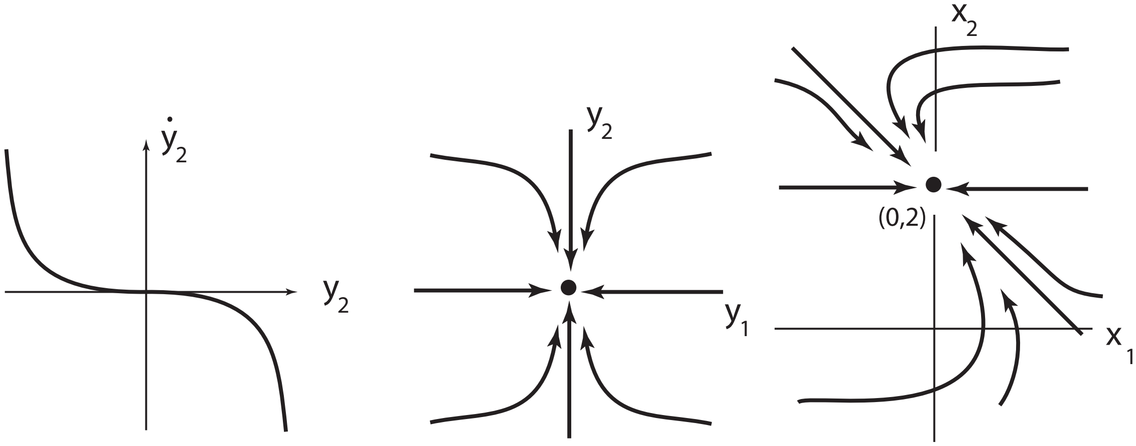
$$v(y_1+3y_2,-3y_2+2) = \big(2-(y_1+3y_2)-(-3y_2+2)-(-3y_2)^3,(-3y_2)^3\big) = \big(-y_1-27y_2^3,27y_2^3\big)$$

$$w(y_1,y_2) = (h_*v)(y_1,y_2) = \big(-y_1,-9y_2^3\big)$$

The only equilibrium of $(\dot{y}_1,\dot{y}_2) = w(y_1,y_2)$ is the origin. Let $f(y_1,y_2) = y_1^2+y_2^2$, which is positive definite at the origin. Since $L_w f(y_1,y_2) = -2y1^2 - 18y_2^4 \leq 0$, with $L_w f(y_1,y_2) < 0$ for $(y_1,y_2) \neq (0,0)$, then, by Liapunov's theorem, the origin is an asymptotically stable equilibrium of $(\dot{y}_1,\dot{y}_2) = w(y_1,y_2)$.

The only equilibrium of $(\dot{x}_1,\dot{x}_2) = v(x_1,x_2)$ is (0,2), asymptotically stable because the diffeomorphism h^{-1} maps solutions of $(\dot{y}_1,\dot{y}_2) = w(y_1,y_2)$ into solutions of $(\dot{x}_1,\dot{x}_2) = v(x_1,x_2)$.

Phase portraits:
To get the phase portrait of $(\dot{y}_1,\dot{y}_2) = w(y_1,y_2)$, we plot $\dot{y}_2$ as a function of y_2 (figure on the left) and this gives the behaviour on the y_2 axis. For the global phase portrait note that the flow has the form $\varphi(t,y_{10},y_{20}) = (\mathrm{e}^{-t}y_{10},y_2(t))$ where $y_2(t)$ is a solution of $\dot{y}_2 = -9y_2^3$, with $y_2(0) = y_{20}$, and that $\mathrm{e}^{-t}y_{10}$ tends to 0 as $t \to \infty$ much faster than $y_2(t)$.
For the other phase portrait, note that h^{-1} maps the phase portrait of $(\dot{y}_1,\dot{y}_2) = w(y_1,y_2)$ into the phase portrait of $(\dot{x}_1,\dot{x}_2) = v(x_1,x_2)$.



2) Solutions $x(t)$ of $\ddot{x} = -u'(x)$ are the first coordinates of solutions of $(\dot{x},\dot{y}) = (y,-u'(x))$, for which there is a first integral $f(x,y) = u(x)+y^2/2$. If there is a closed non trivial level curve of $f(x,y)$ not containing any equilibria of $(\dot{x},\dot{y}) = (y,-u'(x))$, then $\ddot{x} = -u'(x)$ has a non constant periodic solution. Note that all equilibria of $(\dot{x},\dot{y}) = (y,-u'(x))$ are of the form $(x_*,0)$ where $u'(x_*) = 0$. Since $u'(x_1) < 0$, there are infinitely many closed level curves of $f(x,y)$ through points $(x_0,0)$ whith $x_1 < x_0 < x_2$ and not all of them cross the $y = 0$ axis at critical points of u.

3) For $u(x) = -4x^2 + x^4$, the function $f(x, y) = u(x) + y^2/2$ is a first integral for this differential equation. See the phase portrait in 4) below.

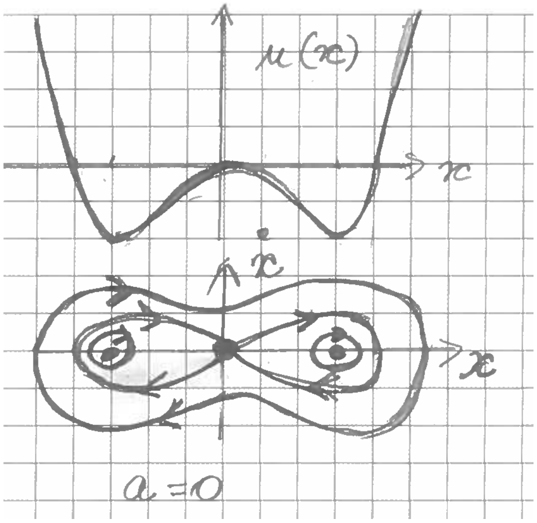
Since $u(2) = u(0) = 0$ then the points $(2, 0)$ and (0,0) lie on the same level curve of $f(x, y)$. The only equilibrium on this level curve is (0,0), hence $\omega(2, 0) = \{(0, 0)\}$.

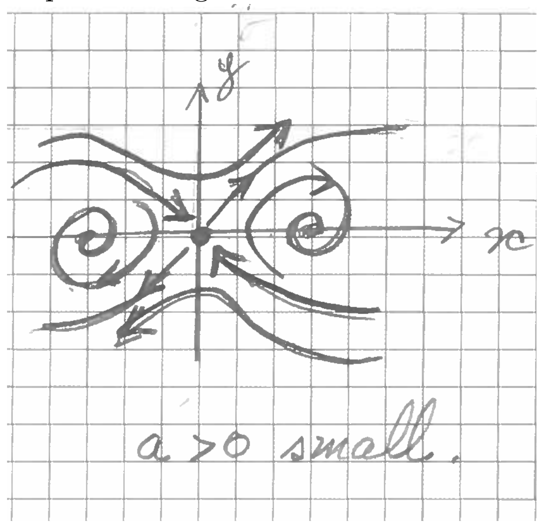
The point $(-\sqrt{2}, 0)$ is an equilibrium, hence $\alpha(-\sqrt{2}, 0) = \{(-\sqrt{2}, 0)\}$.

There are no equilibria on the closed level curve of $f(x, y)$ containing (1,0), hence $\alpha(1, 0)$ is the closed curve $\{(x, y);\ -4x^2 + x^4 + y^2/2 = -3\}$.

4) For $a = 0$ this is a conservative system with one degree of freedom, as observed in 3). It equilibria are $(0, 0)$ (a saddle) and $(\pm\sqrt{2}, 0)$ (centres).

For $0 < a < 2$ the equilibria are $(0, 0)$ (a saddle) and $(\pm\sqrt{2}, 0)$ (stable foci). If there are no periodic solutions except for the constant ones, the phase portrait is given below.





5) First note that the hypothesis implies that the ring $R = \{(x, y) \in \mathbf{R}^2 :\ 2 \leq |(x, y)| \leq 3\}$ is positively invariant, i.e., if $(x(t), y(t))$ is a solution of $(\dot{x}, \dot{y}) = v(x, y)$ with $(x(0), y(0)) \in R$ then $(x(t), y(t)) \in R$ for all $t > 0$. This is because $f(x(t), y(t))$ increases with t when $|(x, y)| = 2$ and $f(x(t), y(t))$ decreases with t when $|(x, y)| = 3$, hence all solutions starting at the boundary of R go inside R in positive time.

Since R is a positively invariant compact set, then all trajectories starting at R have a non-empty ω-limit set contained in R. Since by hypothesis there are no equilibria in R, by the Poincaré-Bendixson Theorem, the ω-limit set is a non-trivial closed orbit, corresponding to a non constant periodic solution.

6) The eigenvalues of the triangular matrix A are 1/4 and 1/5. Since both eigenvalues have norm less than 1, then A is a contraction, $E^s = \mathbf{R}^2$ and $E^u = \{(0, 0)\}$.

(1,0) is an eigenvector of A corresponding to the eigenvalue $1/4$, hence $A^n(1, 0)^T = 1/4^n(1, 0)^T$. The trajectory of $(1, 0)$ lies on the segment with endpoints (0,0) and (1,0).

(1,-1) is an eigenvector of A corresponding to the eigenvalue 1/5, hence $A^n(1,-1)^T = 1/5^n(1,-1)^T$. The trajectory of $(1,-1)$ lies on the segment with endpoints (0,0) and (1,-1).

(1,1)=(2,0)-(1,-1), hence $A^n(1,1)^T = 1/4^n(2,0)^T - 1/5^n(1,-1)^T$. The trajectory of $(1,1)$ does **not** lie on the segment with endpoints (0,0) and (1,1), it is contained in a curve that, at the origin, is tangent to the horizontal axis.

Trajectories:
of $(1,1)$ blue +
of $(1,0)$ red x
of $(1,-1)$ green dot.