Dynamical Systems 2015

Exercises — 1 — dynamical systems in 1-dimensional domains

1. Population models

1) Consider the Malthus law for the dimension $x(t)$ of a population on the instant $t \in \mathbf{R}$:

$$\dot{x} = \alpha x \tag{1}$$

with $\alpha \in \mathbf{R}$ constant ($\alpha =$ growth rate= birth rate-mortality rate).

a) Find the equilibrium points of (1) if $\alpha \neq 0$.

b) Obtain the explicit form of the solution of (1) that satisfies the initial condition $x(0) = x_0 \geq 0$.

c) For $\alpha = -1$, plot the solutions of (1) that satisfy $x(0) = 1$, $x(0) = 2$ and $x(0) = 0$. Do the same for $\alpha = 2$.

d) Draw the phase portraits of (1) for $\alpha > 0$ and for $\alpha < 0$.

2) Another model for the dimension $x(t)$ of a population on the instant $t \in \mathbf{R}$ is given by the Velhurst equation:

$$\dot{x} = ax(1 - bx) \qquad a, b > 0 \tag{2}$$

a) Find the equilibrium points of (2).

b) Draw the phase portrait of (2).

c) Plot the solutions of (2) that satisfy $x(0) = 1/b$, $x(0) = (1/2)b$, $x(0) = 2b$, $x(0) = -1$ and $x(0) = 0$. You do not need to find the solutions to do this.

d) Check that the general solution of (2) that satisfies $x(0) = x_0$ is given by

$$x(t) = \frac{x_0}{bx_0 + (1 - bx_0)\mathrm{e}^{-at}}$$

Does this information help you to plot $x(t)$?

2. Difference equations

3) Study of the family of functions

$$L_\mu : [0,1] \longrightarrow [0,1] \qquad L_\mu(x) = \mu x(1 - x) \qquad 0 < \mu \leq 2 \tag{3}$$

Check the following:

a) The maximum value of L_μ is $\mu/4$, achieved at $x = 1/2$, hence if $0 < \mu \leq 2$ then L_μ maps $[0,1]$ into itself .

b) The point $x = 0$ is always a fixed point of L_μ.

c) Compute the first few iterates in the orbits of $x_0 = 1/4$ and of $x_0 = 3/4$ for $L_{1/2}$ and plot $L_{1/2}$ and $f(x) = x$ on the same graph.

d) If $0 < \mu \leq 1$ then the only fixed point of L_μ in $[0,1]$ is $x = 0$. In this case if $0 < x_0 < 1$ then $L_\mu(x_0) < x_0$, hence the orbit of x_0 accumulates on the fixed point $x = 0$. What is the orbit of $x = 1$ in this case?

e) Compute the first few iterates in the orbits of $x_0 = 1/4$, $x_0 = 2/3$ and of $x_0 = 3/4$ for $L_{3/2}$ and plot $L_{3/2}$ and $f(x) = x$ on the same graph.

f) If $1 < \mu \leq 2$ then $x = 0$ and $x = 1 - 1/\mu$ are the only fixed points of L_μ and $1 - 1/\mu \in [0, 1]$. In this case, if $0 < x_0 < 1 - 1/\mu$ then $x_0 < L_\mu(x_0) < 1 - 1/\mu$. If $1 - 1/\mu < x_0 < 1/\mu$ then $1 - 1/\mu < L_\mu(x_0) < x_0$. If $1/\mu < x_0 < 1$ then $0 < L_\mu(x_0) < 1 - 1/\mu$.

g) Compute the orbits of $x_0 = 1/\mu$, and $x_0 = 1$ under L_μ when $1 < \mu \leq 2$.

h) Use the information in f) and g) above to plot the different types of orbits for L_μ when $1 < \mu \leq 2$.