Dynamical Systems 2015
Exercises — 2 — linear ordinary differential equations

1) For this group of exercises, consider the matrices:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 3 & 7 \\ 0 & -5 \end{pmatrix} \qquad A_5 = \begin{pmatrix} 6 & -3 \\ 1 & -2 \end{pmatrix} \qquad A_6 = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

$$A_7 = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \qquad A_8 = \begin{pmatrix} -5 & 5 \\ -4 & -1 \end{pmatrix} \qquad A_9 = \begin{pmatrix} -5 & -2 \\ 2 & -1 \end{pmatrix}$$

$$A_{10} = \begin{pmatrix} 3 & -1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad A_{11} = \begin{pmatrix} -2 & 0 & -2 \\ 0 & -3 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad A_{12} = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 5 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

a) Which matrices A_j, $j = 1, \ldots, 12$ are similar to a real diagonal matrix? Which are similar to a complex diagonal matrix?

b) For which systems $\dot{x} = A_j \cdot x$, $j = 1, \ldots, 12$ the origin is Liapunov stable? For which the origin is asymptotically stable? For which there is more than one equilibrium? For which there are nonconstant periodic solutions?

c) Compute $e^{A_j t}$ for each one of the A_j.

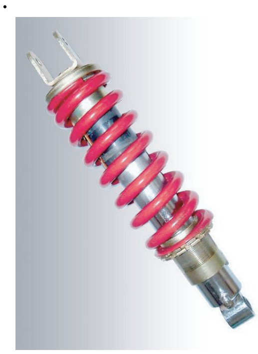
d) Solve the initial value problems below and plot the graph of the first coordinate of the solution:

$$\begin{cases} \dot{x} = A_1 \cdot x \\ x(0) = (1,2,3) \end{cases} \qquad \begin{cases} \dot{x} = A_2 \cdot x \\ x(0) = (1,0,0) \end{cases} \qquad \begin{cases} \dot{x} = A_2 \cdot x \\ x(2) = (1,0,1) \end{cases}$$

$$\begin{cases} \dot{x} = A_3 \cdot x \\ x(0) = (1,1,0) \end{cases} \qquad \begin{cases} \dot{x} = A_4 \cdot x \\ x(0) = (1,2) \end{cases} \qquad \begin{cases} \dot{x} = A_6 \cdot x \\ x(-3) = (1,-1) \end{cases}$$

e) Draw the phase portrait of $\dot{x} = A_j \cdot x$, $x \in \mathbf{R}^2$ for $j = 4, \ldots, 9$.

2) An object with mass $m > 0$ is placed on top of the suspension of a motorbike wheel. The suspension consists of a shock absorber with reaction constant $A \geq 0$ and a spring with constant $K \geq 0$. The position of the body satisfies the differential equation:



(1) $$\ddot{x} + A\dot{x} + \frac{K}{m}x = 0$$

that is equivalent to the first order differential equation in $\mathbf{R}^2$:

(2) $$\dot{x} = y \qquad \dot{y} = -\frac{K}{m}x - Ay$$

Find the solution $x(t)$ of (1) that satisfies the initial conditions $x(0) = x_0$, $\dot{x}(0) = y_0$ and with the values of x_0, A, K and m below. Plot the solution in the plane (t, x) and in the phase plane $(x, \dot{x})$.

a) $A = 2$, $K = 8$, $m = 2$, $x_0 = -1$, $y_0 = 0$.

b) $A = 2$, $K = 8$, $m = 2$, $x_0 = 0$, $y_0 = 1$.

c) $A = 0$, $K = 8$, $m = 8$, $x_0 = 1$, $y_0 = 0$.

d) $A = 1$, $K = 0$, $m = 8$, $x_0 = 0$, $y_0 = 1$.

e) $A = 1$, $K = 3$, $m = 1/2$, $x_0 = 1$, $y_0 = 1$.

f) Obtain the different phase portraits of (2) for $A \geq 0$, $K \geq 0$, $m > 0$. Represent in the (A, K)-plane the region corresponding to each type of phase portrait.

3) Two simple frictionless pendulums of length $\alpha > 0$ and supported at points at a distance $\ell > 0$ are connected by a spring of constant $k > 0$. Denoting by θ and φ the angles that each pendulum makes with the vertical, the equations of motion are:

$$(3) \begin{cases} \dfrac{d^2\theta}{dt^2} = \dfrac{-g\theta}{\alpha} - k\ell\dfrac{\theta - \varphi}{\alpha} \\[2ex] \dfrac{d^2\varphi}{dt^2} = \dfrac{-g\varphi}{\alpha} + k\ell\dfrac{\theta - \varphi}{\alpha} \end{cases}$$

where g is the acceleration of gravity.

a) The system (3) is equivalent to a first order linear differential equation $\dot{x} = Tx$ in $\mathbf{R}^4$, on the coordinates $x = (x_1, x_2, x_3, x_4) = (\theta, \dot{\theta}, \varphi, \dot{\varphi})$. Obtain the matrix T.

b) Show that the eigenvalues of T are $\pm i\sqrt{\dfrac{g}{\alpha}}$ and $\pm i\sqrt{\dfrac{g + 2k\ell}{\alpha}}$.

c) Find two planes P_1 and P_2 such that $T(P_1) \subset P_1$, $T(P_2) \subset P_2$ and $P_1 \oplus P_2 = \mathbf{R}^4$, $P_1 \cap P_2 = \{0\}$.

d) Draw the phase portraits of the restriction of $\dot{x} = Tx$ to each one of the planes P_1, P_2.

e) Discuss the stability of the equilibria of $\dot{x} = Tx$.

f) Describe the behaviour of the pendulums when $\theta(0) = \varphi(0) = 0$ and $\dot{\theta}(0) = \dot{\varphi}(0) = v > 0$ in the case α numerically equal to g and k numerically equal to g/ℓ. Plot the solutions you have found as curves in the plane (θ, φ). Repeat the exercise for α numerically equal to g and k numerically equal to $3g/\ell$ and to $4g/\ell$ — does it make any difference?