Dynamical Systems 2015

Exercises — 3 — Lie derivatives, changes of coordinates, conservative systems

1) Show that $f : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ given by $f(x, y) = (x + y, x - y)$ is a diffeomorphism and compute $f_*v(x, y)$ for the vector field in the plane given by $v(x, y) = (x - y^2, x^2 + y)$.

2) a) Check that the expression $h(x, y) = (2 + 2x + (y - 2)\mathrm{e}^x, 2 + (y - 2)\mathrm{e}^x)$ defines a local diffeomorphism near the origin in $\mathbf{R}^2$.

b) Compute $h_*v(h(x, y))$, $w(h(x, y))$, $h_*w(h(x, y))$ and $v(h(x, y))$ for the vector fields

$$v(x, y) = (y, y - 2) \qquad w(x, y) = (1, 0)$$

and check that $h_*w(h(x, y)) = v(h(x, y))$ (hence $h_*w((x, y)) = v((x, y))$).

3) For the differential equation on $\mathbf{R}^2$:

$$(3) \begin{cases} \dot{x} = y + x(1 - x^2 - y^2) \\ \dot{y} = -x + y(1 - x^2 - y^2) \end{cases}$$

a) Show that the change of coordinates $x = r\cos\theta$, $y = r\sin\theta$ transforms the equation:

$$(3.a) \begin{cases} \dot{r} = r(1 - r^2) \\ \dot{\theta} = -1 \end{cases}$$

into the equation (3).

b) Sketch the phase portraits of (3.a) and of (3).

4) Compute f_*v for $f(x, y) = (x - y, y)$ and $v(x, y) = (x + y, 2y)$.

5) Rectify the flow of the vector field in $\mathbf{R}^2$ given by $v(x, y) = (\mathrm{e}^y, 1)$.

6) Sketch the phase portrait of $\ddot{x} = -\dfrac{dU}{dx}$ with $U(x) = (x^4 - 16)(x^2 - 9)$ on the plane $(x, \dot{x})$.

7) Obtain a first integral for the vector field $v_1(x, y) = (y, x^2 + 1)$ and sketch its phase portrait.

8) Let $v_1(x, y) = (x, -y)$ and $v_2(x, y) = (x(1 - y), -y(1 - y))$.

a) Sketch the phase portrait of v_1.

b) Find a function f that is a first integral for both v_1 and v_2.

c) Sketch the phase portrait of v_2.

9) Sketch the phase portrait of $\ddot{x} = -x + \alpha x^2$ for $\alpha > 0$.

10) Study of the equation for the nonlinear pendulum with friction $\ddot{\theta} = -mg\sin\theta$ where $\theta(t)$ is the angle that the pendulum forms with the vertical at the instant t, m is the pendulum mass and g is the gravity acceleration.

a) Find the equilibria and discuss their stability.

b) Check that solutions with initial condition $(x, \dot{x}) = (x_0, 0)$, with $0 < x_0 < \pi$ are periodic.

c) Let $T(x_0)$ be the minimum period of the solution through $(x_0, 0)$. Obtain estimates for $\lim_{x_0 \to \pi} T(x_0)$ and $\lim_{x_0 \to 0} T(x_0)$.

11) Obtain a first integral for $v_2(x,y) = (g(y), h(x))$.

12) The animals of a certain population x (for instance, plaice — the *prey*) are eaten by the animals of the population y (for instance, sharks — the *predators*) and the two populations evolve according to the equation:

$$(12)\begin{cases} \dot{x} = \alpha x - \gamma xy \\ \dot{y} = -\beta y + \delta xy \end{cases}$$

where $\alpha, \beta, \gamma, \delta$ are positive constants. For simplicity, assume α and β are integers.

a) Find the equilibria of (12).

b) Show that the first quadrant $\{(x,y) : x \geq 0 y \geq 0\}$ is invariant for the flow of (12) (hint: show that the axes are invariant).

c) Compute $L_v f(x,y)$ for the vector field $v(x,y)$ associated to (12) and for $f(x,y) = x^\alpha y^\beta \mathrm{e}^{-(\delta x + \gamma y)}$.

d) Find the critical points of $f(x,y)$ in the first quadrant and obtain the local description of the graph of f near these points (i.e.: decide if they are local maxima, local minima, saddle points or other types of graphs).

e) Sketch the phase portrait of (12) on the first quadrant.

Note: the equation (12) is a classical in mathematical biology, proposed independently by Vito Volterra and Alfred Lotka, and is known as the *Volterra-Lotka equation*.