Dynamical Systems 2015

Exercises — 3 — Lie derivatives, changes of coordinates, conservative systems

1) Show that $f : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ given by $f(x,y) = (x+y, x-y)$ is a diffeomorphism and compute $f_*v(x,y)$ for the vector field in the plane given by $v(x,y) = (x-y^2, x^2+y)$.

Solution:

f is a linear map represented by the matrix $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Since $\det M \neq 0$ then f has a linear inverse f^{-1}. Both f and f^{-1} are differentiable, hence f is a diffeomorphism.

By definition, $f_*v(X) = Df\left(f^{-1}(X)\right) v\left(f^{-1}(X)\right)$.

We compute $f^{-1}(X) = \left(\frac{1}{2}(x+y), \frac{1}{2}(x-y)\right)$ for $X = (x,y)$.

Since f is linear, $Df(Z) = f$ at every point Z. Hence:

$$f_*v(X) = Df\left(f^{-1}(X)\right) v\left(f^{-1}(X)\right) = f\left(v\left(f^{-1}(X)\right) = v\left(f^{-1}(X)\right)\right)$$

i.e.

$$f_*v(X) = M\left(\frac{1}{2}(x+y) - \left(\frac{1}{2}(x-y)\right)^2, \left(\frac{1}{2}(x+y)\right)^2 + \frac{1}{2}(x-y)\right)\right)$$

and

$$f_*v(X) = M\left(\frac{1}{2}(x+y+xy) - \frac{1}{4}(x^2+y^2), \frac{1}{2}(x-y+xy) + \frac{1}{4}(x^2+y^2)\right)$$

hence

$$f_*v(X) = \left(x+xy, y - \frac{1}{2}(x^2+y^2)\right)$$

It is also possible to compute $f_*v(X)$ directly using the change of coordinates.

From $(\dot{x}, \dot{y}) = v(x,y)$ we have

$$\dot{x} = x - y^2 \qquad \dot{y} = x^2 + y$$

and if

$$(x_1, y_1) = f(x,y) = (x+y, x-y) \quad \Rightarrow \quad (x,y) = \left(\frac{1}{2}(x_1+y_1), \frac{1}{2}(x_1-y_1)\right)$$

then

$$\dot{x}_1 = \dot{x} + \dot{y} = (x-y^2) + (x^2+y) = x_1 + \frac{1}{4}(x_1+y_1)^2 - \frac{1}{4}(x_1-y_1)^2 = x_1 + x_1y_1$$

and

$$\dot{y}_1 = \dot{x} - \dot{y} = (x-y^2) - (x^2+y) = y_1 - \frac{1}{4}(x_1+y_1)^2 - \frac{1}{4}(x_1-y_1)^2 = x_1 - \frac{1}{2}(x_1^2+y_1^2)$$