Dynamical Systems 2015

Exercises — 4 — nonlinear ordinary differential equations — local results

For a vector field $v(x)$ with flow $\varphi(t,x)$, and with an equilibrium x_0, the set of points y such that $\lim_{t\to+\infty}\varphi(t,y)=x_0$ is called the *basin of attraction* of x_0.

1) Study the stability of all the equilibria and their basin of attraction for:

a) $v(x,y)=(y-\alpha x e^{-y^2},-x^3)$, with $\alpha\in\mathbf{R}$, constant.

b) $v(x,y)=((x+1)(x-1)(x-2),(y+2)(y+1)(y-1))$.

c) $v(x,y)=(3y-2\varepsilon x\cos y,-3x-\varepsilon y^3)$ with $\varepsilon\in\mathbf{R}$, constant and $x^2+y^2<1$.

2) Study the stability of the origin and its basin of attraction for

$$v(x,y,z)=(-y-x(x^2+\beta)-z^2x,-\beta y+x,-z+zx^2)$$

with $\beta\in\mathbf{R}$, constant.

3) Find the equilibria, study their stability and their basin of attraction for the equation

$$\ddot{x}+\nu(x^2-\mu)\dot{x}+x=0,\qquad \mu\in\mathbf{R}\qquad \mu\geq 0\qquad \nu\in\mathbf{R}.$$

4) Counter-example showing that in Liapunov's theorem it is essential to assume that the function is positive definite. Let $U=\mathbf{R}^2$, $v(x,y)=(x,-y)$, $f:U\longrightarrow\mathbf{R}$ given by $f(x,y)=-x^2+y^2$.

(1) Sketch the phase portrait of v and check that the origin is the only equilibrium and that it is unstable.
(2) Compute $L_vf((x,y))$ and check that $L_vf((x,y))<0$ for all $(x,y)\in\mathbf{R}^2$, $(x,y)\neq(0,0)$.
(3) Represent in the same graph the level curves of f and the phase portrait of v.

5) Study the stability of all the equilibria, find the limit cycles (if any) and sketch the phase portraits of:

a) $u(x,y)=(y-x^3,-x-y)$.

b) $v(x,y)=(2y-\dfrac{x^3}{4},\dfrac{-x}{2}-y^3)$.

c) $w(x,y)=(x^3+y^2,-2y^3-xy)$.

d) $u(x,y)=(-x^2,-y^3)$.

e) $v(x,y)=(-x^2-4xy-4y^2+2y^3,-y^3)$.

f) $v(x,y)=(-x^5-4xy,4x^2-y^3)$.

g) $w(x,y)=(-9x^3+\dfrac{1}{3}(2x+y)^2,18x^3-\dfrac{2}{3}(2x+y)^2-(2x+y)^5-3x(2x+y))$.

h) $u(x,y)=(-3x+3xy^2,y^3-3x^2y)$.

i) $v(x,y,z)=(y,-x-y^3-z^2y,-z+zy^2)$.

j) $w(x,y)=(y-x+x^3+xy^2,-x-y+x^2y+y^3)$

6) Show that $\ddot{x}=1-x\dot{x}$ does not have limit cycles.

7) Consider the equation $\dot{X} = v(X)$ in $\mathbf{R}^3$ given by:

$$(7)\begin{cases} \dot{x} &= x(1-x^2-y^2-z^2)-2xz-xz^2 \\ \dot{y} &= y(1-x^2-y^2-z^2)+2yz-yz^2 \\ \dot{z} &= z(1-x^2-y^2-z^2)-2(y^2-x^2)+z(x^2+y^2) \end{cases}$$

a) Check that the origin, $N=(0,0,1)$ and $S=(0,0,-1)$ and $(\pm\frac{\sqrt{2}}{2},\pm\frac{\sqrt{2}}{2},0)$ are equilibria of (7) and discuss their stability. Check that there are no other equilibria.

b) Show that if $X \neq (0,0,0)$ then $\omega(X)$ is contained in the sphere S^2 of center in the origin and radius 1. Check that S^2 is invariant under the flow of (7).

c) Check that the planes $x=0$ and $y=0$ are invariant under the flow of (7), i. e. that any solution of (7) with initial conditions in one of these planes remains in the plane for all $t \in \mathbf{R}$.

d) Write the differential equation of the restriction of (7) to the plane $x=0$ and sketch its phase portrait. Do the same for the plane $y=0$.

e) Show that the sector $s_+ = S^2 \cap \{x \geq 0 \text{ e } y \geq 0\}$ is invariant under the flow of (7). Check that the same is true for the other 3 sectors of S^2 where x and y have constant sign.

f) Show that if $X \in s_+$ is not an equilibrium then $\omega(X) \subset S^2 \cap (\{x=0\} \cup \{y=0\})$.

g) Sketch the phase portrait of (7).