Dynamical Systems 2015

Exercises — 5 — nonlinear ordinary differential equations

1) Are the dynamical systems $(\dot{x}, \dot{y}) = (x\mathrm{e}^x, y\mathrm{e}^x)$ and $(\dot{x}, \dot{y}) = (x, y)$ topologically equivalent? If so, exhibit the homeomorphism that maps the trajectories of one into the trajectories of the other.

2) Show that if $v(x, y)$ is a C^∞ vector field in an open subset U of the plane, such that at all points of U the trace of $Dv(x, y)$ is positive, then $(\dot{x}, \dot{y}) = v(x, y)$ has no nonconstant periodic solutions in U. *Hint: Suppose there is a periodic solution $\gamma(t)$, apply Green's theorem to $\int_\gamma (v_2, -v_1)$ where $v(x, y) = (v_1(x, y), v_2(x, y))$.* This result is called Bendixson's criterium.

3) Explain why an equilibrium surrounded by a continuum of periodic orbits cannot be asymptotically stable, and if this takes place in the plane, then the equilibrium must be Liapunov stable.

4) Sketch phase portraits in the plane consistent with the following information:

a) Two equilibria, a saddle and a stable node.

b) Three equilibria, one saddle and two stable nodes.

c) An unstable limit cycle and three equilibria, one saddle and two stable nodes.

d) A stable focus and two limit cycles, one stable and one unstable.

5) For each of the following equations show that the indicated region R is positively invariant (meaning that a solution that starts inside the region stays in the region for all positive time):

a) $\dot{x} = 2xy$, $\dot{y} = y^2$, $R = \{(x, y) : y \geq 0\}$.

b) $\dot{x} = -3x + y$, $\dot{y} = (\beta - 3)y$, β constant, $R = \{(x, y) : y = \beta x\}$.

c) $\dot{x} = -x + y + x(x^2 + y^2)$, $\dot{y} = -x - y + y(x^2 + y^2)$, $R = \{(x, y) : x^2 + y^2 < 1\}$.

d) $\dot{x} = x(y^2 - x)$, $\dot{y} = y(y^2 - x)$, $R = \{(x, y) : x > y^2\}$.

6) Show that the equation $\dot{x} = -y + x(1 - x^2 - y^2)$, $\dot{y} = x + y(1 - x^2 - y^2)$ in polar coordinates (r, θ) takes the form $\dot{r} = r(1 - r^2)$, $\dot{\theta} = 1$. Find the general form of the solution of the equation in polar coordinates and plot the graphs of $r(t)$ for:
a) $0 < r(0) < 1$ b) $r(0) = 1$ c) $r(0) > 1$.
Use this information to obtain the phase portrait of the original system.

7) Show that there is an $r > 0$ such that the region $R = \{(x, y) : x^2 + y^2 < r^2\}$ is positively invariant for the equation $\dot{x} = -\alpha y + x(1 - x^2 - y^2)$, $\dot{y} = \alpha x + y(1 - x^2 - y^2) - \beta$ where $\alpha \neq 0$ and β are constants. Show that the equation has a limit cycle when $\beta = 0$.