Dynamical Systems 2015

Exercises — 6 — difference equations

1) Study of the dynamics of linear difference equations in $\mathbf{R}^2$. For the matrices:

$$A_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -2 & 0 \\ 1 & 3 \end{pmatrix} \qquad A_3 = \begin{pmatrix} -1/2 & 1 \\ 0 & 4 \end{pmatrix} \qquad A_4 = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

a) Plot the different types of trajectories of $f(x) = A_j x$ for each of the matrices A_j above.

b) Identify the invariant subspaces E^s and E^u for $f(x) = A_j x$ for each of the matrices A_j above.

c) Show that the trajectory under $f(x, y) = A_1(x, y)^T$ of every point (x, y) with $x \neq 0$ accumulates on the xx axis as it approaches the origin.

d) For $X_0 = (x, y)^T \neq (0, 0)^T$ let $X_n = \dfrac{1}{|A_2 X_{n-1}|} A_2 X_{n-1}$. Show that if $y \neq 0$ then $\lim_{n\to\infty} X_n = \pm(0, 1)^T$.

2) Study of the symbolic dynamics of the map $f : [0, 1] \longrightarrow \mathbf{R}$ given by:

$$f(x) = \begin{cases} 3x & \text{if } 0 \leq x < 2/3 \\ 3(x - 2/3) & \text{if } 2/3 \leq x \leq 1 \end{cases}$$

We are concerned with the dynamics in the set $\Omega = \{x \in [0, 1] : \ f^n(x) \in [0, 1]\ \forall n \in \mathbf{N}\}$.

a) Plot the graph of f, find all its fixed points and discuss their stability.

b) Show that $\Omega \subset [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$. Extend your proof to get

$$\Omega \subset \bigcup_{j=0}^{(3^n-1)/2} \left[\frac{2j}{3^n}, \frac{2j+1}{3^n}\right] \qquad \forall n \in \mathbf{N}$$

c) Show that if the symbol 1 never occurs in the base 3 expression of $x_0 \in [0, 1]$ then $x_0 \in \Omega$.

d) Show that the base 3 expression of $x_0 \in \Omega$ defines a symbolic dynamics for f.

e) Show that if the symbol 1 ever occurs in the base 3 expression of $x_0 \in \Omega$ then all the symbols after it are equal, and they are either all 0 or all 2.

f) Show that Ω is not countable. *Hint: find a bijection from a subset of Ω into the numbers $[0, 1[$ written in base 2.*

g) Find all the points $x \in \Omega$ such that $f^2(x) = x$.

h) Find the approximate value with 3 correct decimal places of a point $x_0 \in \Omega$ such that $f^5(x_0) = x_0$ and $f^4(x_0) \neq x_0$.

i) Find the base 3 expression of a point $x_0 \in \Omega$ such that $\forall n > 0, n \in \mathbf{N}\ f^n(x_0) \neq x_0$.