Dynamical Systems
phase portraits of linear o.d.e.s in $\mathbf{R}^2$

Biblio: §2.3 of Arrowsmith and Place
Ordinary Differential Equations: a qualitative approach with applications
§5.3 and 5.4 of Hirsch and Smale
Differential Equations, Dynamical Systems and Linear Algebra

Isabel S. Labouriau

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Formulation

All figures correspond to solutions of $\dot{X} = AX$
for a 2×2 matrix A and $X = \begin{pmatrix} x \\ y \end{pmatrix}$.

The general solution has the form $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{At} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$
where $X(0) = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

The origin is always an equilibrium of $\dot{X} = AX$.

In each case the first frame has the phase portrait in the basis of eigenvectors, in the second frame there is a phase portrait in an arbitrary basis.

We start by the generic cases where A has two distinct eigenvalues with different real parts.

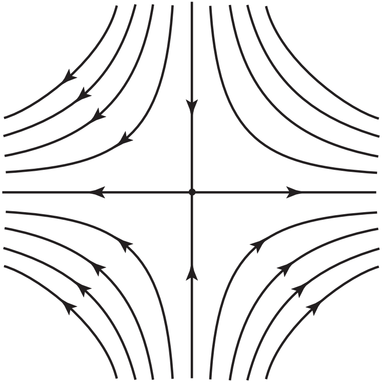
Saddle

$$A = \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix} \quad a, b > 0$$

the eigenvalues of A are:
$a > 0$ and $-b < 0$

$$e^{At} = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{-bt} \end{pmatrix}$$

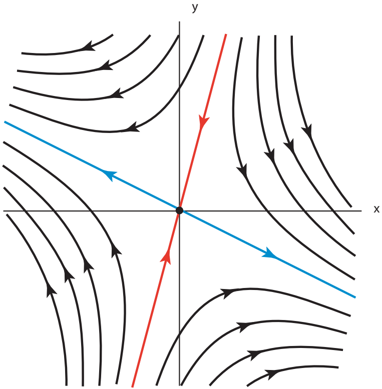
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 e^{at} \\ y_0 e^{-bt} \end{pmatrix}$$



If $x_0 \neq 0$ and $y_0 \neq 0$ then $\dfrac{y(t)}{y_0} = \left(\dfrac{x(t)}{x_0}\right)^{-b/a}$

The origin is unstable.

Saddle



A with two real eigenvalues: $a > 0$ and $-b < 0$

blue: eigenspace associated to the eigenvalue a
red: eigenspace associated to the eigenvalue $-b$

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{at} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + e^{-bt} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

The origin is unstable.

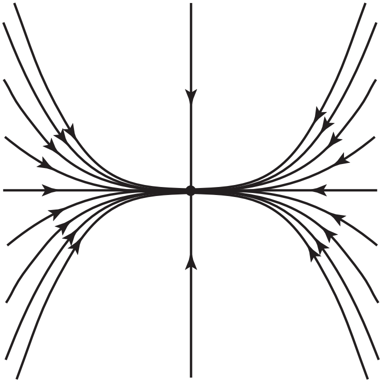
Stable node

$$A = \begin{pmatrix} -a & 0 \\ 0 & -b \end{pmatrix} \quad 0 < a < b$$

the eigenvalues of A are:
$-a < 0$ and $-b < 0$

$$e^{At} = \begin{pmatrix} e^{-at} & 0 \\ 0 & e^{-bt} \end{pmatrix}$$

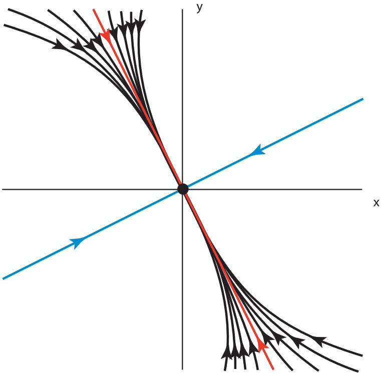
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 e^{-at} \\ y_0 e^{-bt} \end{pmatrix}$$



If $x_0 \neq 0$ and $y_0 \neq 0$ then $\dfrac{y(t)}{y_0} = \left(\dfrac{x(t)}{x_0}\right)^{b/a}$

The origin is asymptotically stable.

Stable node



A with two real eigenvalues:
$-a < 0$ and $-b < 0$
$0 < a < b$

red: eigenspace associated to the eigenvalue $-a$

blue: eigenspace associated to the eigenvalue $-b$

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-at} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + e^{-bt} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

The origin is asymptotically stable.

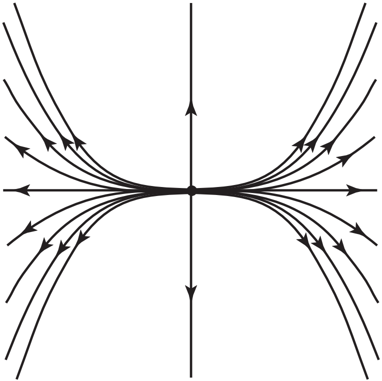
Unstable node

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad 0 < a < b$$

the eigenvalues of A are:
$a > 0$ and $b > 0$

$$e^{At} = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{pmatrix}$$

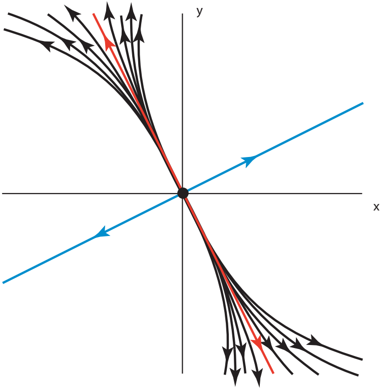
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 e^{at} \\ y_0 e^{bt} \end{pmatrix}$$



If $x_0 \neq 0$ and $y_0 \neq 0$ then $\dfrac{y(t)}{y_0} = \left(\dfrac{x(t)}{x_0}\right)^{b/a}$

The origin is unstable.

Unstable node



A with two real eigenvalues:
$a > 0$ and $b > 0$
$0 < a < b$

red: eigenspace associated to the eigenvalue a

blue: eigenspace associated to the eigenvalue b

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{at} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + e^{bt} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

The origin is unstable.

Centre

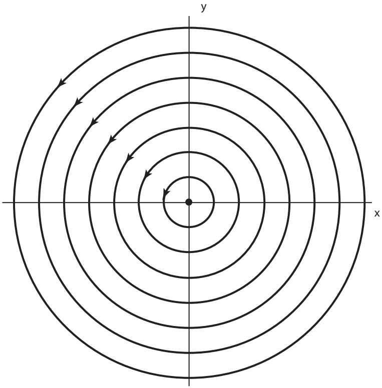
$$A = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \quad a > 0$$

the eigenvalues of A are: $\pm ai$

$$e^{At} = \begin{pmatrix} \cos at & -\sin at \\ \sin at & \cos at \end{pmatrix}$$

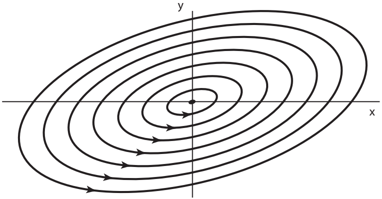
$$X(t) = \begin{pmatrix} x_0 \cos at - y_0 \sin at \\ \\ x_0 \sin at + y_0 \cos at \end{pmatrix}$$

$$(x(t))^2 + (y(t))^2 = (x_0)^2 + (y_0)^2$$



The origin is stable but not asymptotically stable.

Centre



A with two imaginary eigenvalues:
$\pm b$i with $b \neq 0$

$$X(t) = \begin{pmatrix} x(t) \\ \\ y(t) \end{pmatrix} = \cos bt \begin{pmatrix} x_1 \\ \\ y_1 \end{pmatrix} + \sin bt \begin{pmatrix} x_2 \\ \\ y_2 \end{pmatrix}$$

The origin is stable but not asymptotically stable.

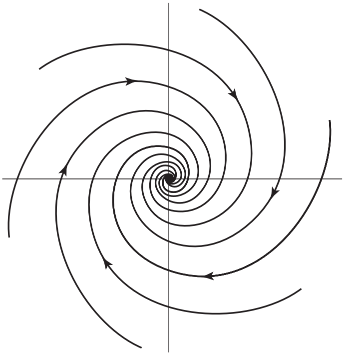
Stable focus

$$A = \begin{pmatrix} -a & -b \\ b & -a \end{pmatrix} \quad a, b > 0$$

the eigenvalues of A are:

$-a \pm b\mathrm{i}$

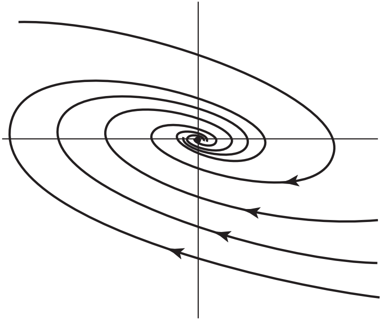
$$\mathrm{e}^{At} = \mathrm{e}^{-at} \begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix}$$



$$X(t) = \mathrm{e}^{-at} \begin{pmatrix} x_0 \cos bt - y_0 \sin bt \\ \\ x_0 \sin bt + y_0 \cos bt \end{pmatrix}$$

The origin is asymptotically stable.

Stable focus



A with two complex eigenvalues: $-a \pm bi$ with $a > 0$ and $b \neq 0$ with negative real part.

$$X(t) = \begin{pmatrix} x(t) \\ \\ y(t) \end{pmatrix} = e^{-at} \cos bt \begin{pmatrix} x_1 \\ \\ y_1 \end{pmatrix} + e^{-at} \sin bt \begin{pmatrix} x_2 \\ \\ y_2 \end{pmatrix}$$

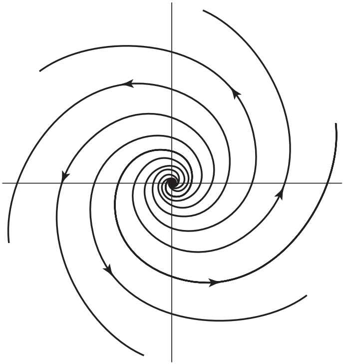
The origin is asymptotically stable.

Unstable focus

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad a, b > 0$$

the eigenvalues of A are:
$a \pm b\mathrm{i}$

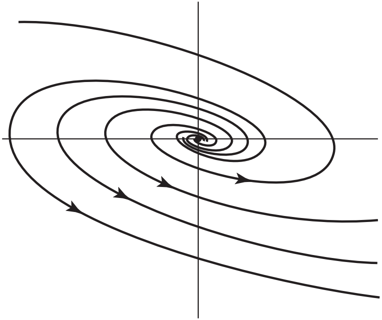
$$e^{At} = e^{at} \begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix}$$



$$X(t) = e^{at} \begin{pmatrix} x_0 \cos bt - y_0 \sin bt \\ \\ x_0 \sin bt + y_0 \cos bt \end{pmatrix}$$

The origin is unstable.

Unstable focus



A with two complex eigenvalues:
$a \pm bi$ with $a > 0$ and $b \neq 0$
with positive real part.

$$X(t) = \begin{pmatrix} x(t) \\ \\ y(t) \end{pmatrix} = e^{at} \cos bt \begin{pmatrix} x_1 \\ \\ y_1 \end{pmatrix} + e^{at} \sin bt \begin{pmatrix} x_2 \\ \\ y_2 \end{pmatrix}$$

The origin is unstable.

Generic cases:

- the eigenvalues of A have nonzero real parts (A is hyperbolic);
- A has two distinct eigenvalues.

A small change in the entries of A does not change much the phase portrait.

Special cases:

One special situation (already seen):
the eigenvalues of A have zero real parts (A is not hyperbolic).

Other special situations occur when:

- A has two real identical eigenvalues different from zero;
- zero is an eigenvalue of A.

In all these situations, a small change in the entries of A completely changes the phase portrait.

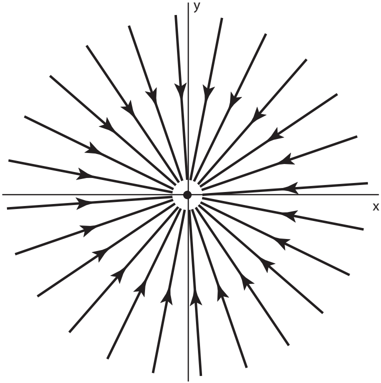
Sink

$$A = \begin{pmatrix} -b & 0 \\ 0 & -b \end{pmatrix} \quad b > 0$$

A has two identical real eigenvalues equal to $-b$

$$e^{At} = \begin{pmatrix} e^{-bt} & 0 \\ 0 & e^{-bt} \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 e^{-bt} \\ y_0 e^{-bt} \end{pmatrix}$$



Any vector $\begin{pmatrix} x \\ y \end{pmatrix}$
with $x, y \in \mathbf{R}$ $x \neq 0$ or $y \neq 0$ is an eigenvector of A.

The origin is asymptotically stable.

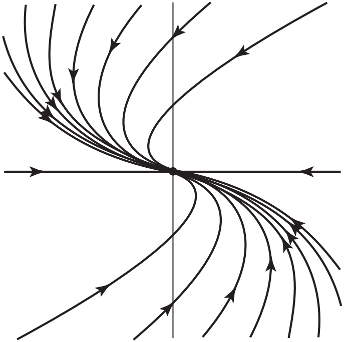
Degenerate stable node

$$A = \begin{pmatrix} -b & 1 \\ 0 & -b \end{pmatrix} \quad b > 0$$

A has two identical real eigenvalues equal to $-b$

$$e^{At} = e^{-bt} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} (x_0 + ty_0)e^{-bt} \\ y_0 e^{-bt} \end{pmatrix}$$

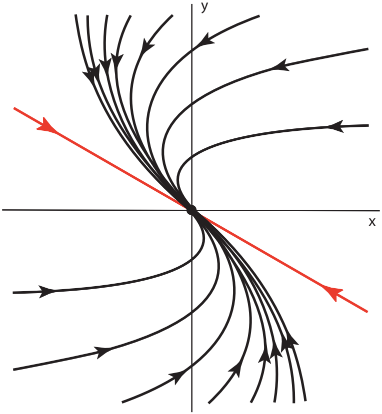


The eigenvectors of A are of the form $\begin{pmatrix} x \\ 0 \end{pmatrix}$ with $x \in \mathbf{R}$ $x \neq 0$

and no others.

The origin is asymptotically stable.

Degenerate stable node



A has two identical real eigenvalues equal to $-b$ with $b > 0$.

red:
eigenspace associated to the eigenvalue $-b$

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-bt} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + t e^{-bt} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

The origin is asymptotically stable.

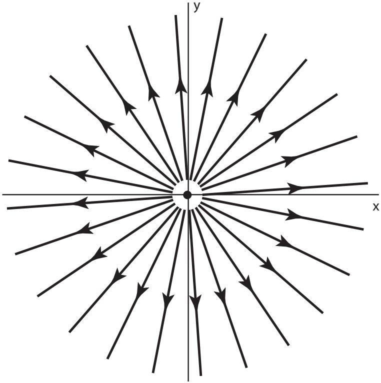
Source

$$A = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \quad b > 0$$

A has two identical real eigenvalues equal to b

$$e^{At} = \begin{pmatrix} e^{bt} & 0 \\ 0 & e^{bt} \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 e^{bt} \\ y_0 e^{bt} \end{pmatrix}$$



Any vector $\begin{pmatrix} x \\ y \end{pmatrix}$ with $x, y \in \mathbf{R}$ $x \neq 0$ or $y \neq 0$ is an eigenvector of A.

The origin is unstable.

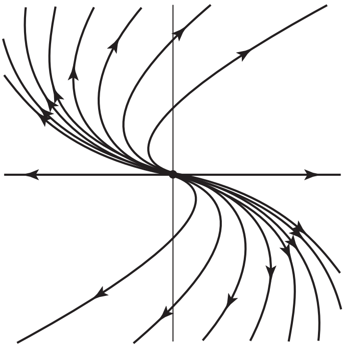
Degenerate unstable node

$$A = \begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix} \quad b > 0$$

A has two identical real eigenvalues equal to b

$$e^{At} = e^{bt} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} (x_0 + ty_0)e^{bt} \\ y_0 e^{bt} \end{pmatrix}$$

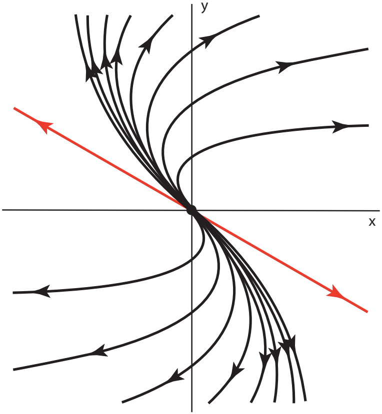


The eigenvectors of A are of the form $\begin{pmatrix} x \\ 0 \end{pmatrix}$ with $x \in \mathbf{R}$ $x \neq 0$

and no others.

The origin is unstable.

Degenerate unstable node



A has two identical real eigenvalues equal to $b > 0$.

red:
eigenspace associated to the igenvalue b

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{bt} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + te^{bt} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

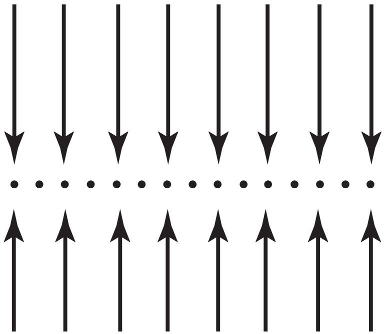
The origin is unstable.

One zero eigenvalue

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -b \end{pmatrix} \quad b > 0$$

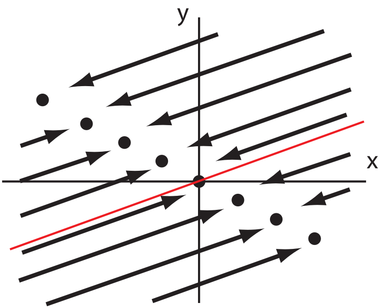
the eigenvalues of A are:
0 and $-b < 0$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 e^{-bt} \end{pmatrix}$$



The points $\begin{pmatrix} x \\ 0 \end{pmatrix}$ with $x \in \mathbf{R}$ are stable equilibria of $\dot{X} = AX$ that are not asymptoticallty stable.

One zero eigenvalue



the eigenvalues of A are:
0 and $-b < 0$

red:
eigenspace associated
to the eigenvalue $-b$

equilibria lie in the
eigenspace associated
to the eigenvalue 0

$$X(t) = \begin{pmatrix} x(t) \\ \\ y(t) \end{pmatrix} = e^{-bt} \begin{pmatrix} x_1 \\ \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ \\ y_2 \end{pmatrix}$$

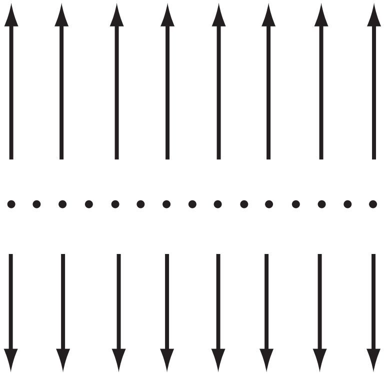
All equilibria of $\dot{X} = AX$ are stable but not asymptotically stable.

One zero eigenvalue

$$A = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \quad b > 0$$

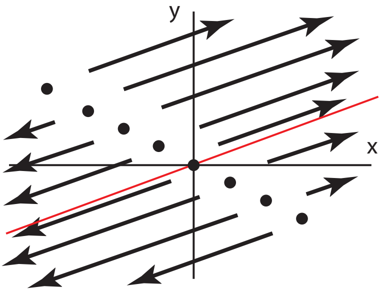
the eigenvalues of A are:
0 and $b > 0$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 e^{bt} \end{pmatrix}$$



The points $\begin{pmatrix} x \\ 0 \end{pmatrix}$ with $x \in \mathbf{R}$ are unstable equilibria of $\dot{X} = AX$.

One zero eigenvalue



the eigenvalues of A are:
0 and $b > 0$

red:
eigenspace associated
to the eigenvalue b

equilibria lie in the eigenspace associated
to the eigenvalue 0

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{bt} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

Equilibria of $\dot{X} = AX$ are unstable.

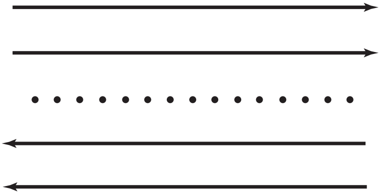
Very degenerate case

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

A has a double zero eigenvalue

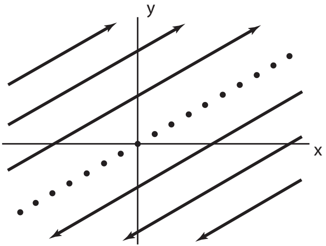
$$e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 + t y_0 \\ y_0 \end{pmatrix}$$



The points $\begin{pmatrix} x \\ 0 \end{pmatrix}$ with $x \in \mathbf{R}$ are unstable equilibria of $\dot{X} = AX$.

Very degenerate case



A has a double zero eigenvalue

equilibria lie in the eigenspace associated to the eigenvalue 0

$$X(t) = \begin{pmatrix} x(t) \\ \\ y(t) \end{pmatrix} = \begin{pmatrix} x_1 \\ \\ y_1 \end{pmatrix} + t \begin{pmatrix} x_2 \\ \\ y_2 \end{pmatrix}$$

Equilibria of $\dot{X} = AX$ are unstable.