MATEMÁTICA II

COORDENADAS CILÍNDRICAS E ESFÉRICAS
RESUMO DA AULA DE 20 DE MAIO DE 2008
BIBLIOGRAFIA: S. LANG,
CALCULUS OF SEVERAL VARIABLES, CAPÍTULO XI

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Exemplo - Volume do Cilindro $C = \{(x,y,z): \quad |(x,y)| \leq \rho \quad \text{e} \quad -\rho \leq z \leq \rho\}$

$$\iiint_C 1\ dz\ dy\ dx = \int_{-\rho}^{\rho} \left(\int_{-g(x)}^{g(x)} \left(\int_{-\rho}^{\rho} 1\ dz \right)\ dy \right)\ dx$$

$g(x) = \sqrt{\rho^2 - x^2}$.

$$\int_{-g(x)}^{g(x)} \left(\int_{-\rho}^{\rho} 1\ dz \right)\ dy = \int_{-g(x)}^{g(x)} 2\rho\ dy = 4\rho g(x)$$

$$\int_{-\rho}^{\rho} 4\rho g(x)\ dx = 4\rho \int_{-\rho}^{\rho} \sqrt{\rho^2 - x^2}\ dx = 2\pi\rho^3$$

Exemplo - Volume da Esfera $S = \{(x,y,z): \quad |(x,y,z)| \leq \rho\}$

$$\iiint_S 1\ dz\ dy\ dx = \int_{-\rho}^{\rho} \left(\int_{-g(x)}^{g(x)} \left(\int_{-f(x,y)}^{f(x,y)} 1\ dz \right)\ dy \right)\ dx$$

$f(x,y) = \sqrt{\rho^2 - x^2 - y^2} \qquad g(x) = \sqrt{\rho^2 - x^2}$.

$$\int_{-f(x,y)}^{f(x,y)} 1\ dz = 2f(x,y)$$

$$\int_{-g(x)}^{g(x)} 2f(x,y)\ dy = 2\int_{-\sqrt{\rho^2-x^2}}^{\sqrt{\rho^2-x^2}} \sqrt{\rho^2 - x^2 - y^2}\ dy = \pi(\rho^2 - x^2)$$

$$\int_{-\rho}^{\rho} \pi(\rho^2 - x^2)\ dx = \pi\rho^2 x - \pi\frac{x^3}{3}\Big|_{-\rho}^{\rho} = \frac{4}{3}\pi\rho^3$$

Volume do Cone $M = \{(x,y,z): \quad |(x,y)| \leq z/2 \quad \text{e} \quad 0 \leq z \leq 2\rho\}$

$$\iiint_M 1\ dz\ dy\ dx = \int_{-\rho}^{\rho} \left(\int_{-g(x)}^{g(x)} \left(\int_{f(x,y)}^{2\rho} 1\ dz \right)\ dy \right)\ dx$$

$f(x,y) = 2\sqrt{x^2 + y^2} \qquad g(x) = \sqrt{\rho^2 - x^2}$.

$$\int_{-g(x)}^{g(x)} \left(\int_{f(x,y)}^{2\rho} 1\ dz \right)\ dy = \int_{-g(x)}^{g(x)} 2\rho - f(x,y)\ dy =$$

$$= \int_{-g(x)}^{g(x)} 2\rho - 2\sqrt{x^2+y^2}\; dy$$

2-D

Definição Dada $f(x,y)$ define-se $f^*(r,\theta) = f(r\cos\theta, r\,\mathrm{sen}\,\theta)$ ou seja $f^* = f \circ P$ com $P(r,\theta) = (r\cos\theta, r\,\mathrm{sen}\,\theta) = (x,y)$.

Teorema 1. *Seja $S \subset \mathbf{R}^2$ o conjunto*

$$S = \{(x,y) : x = r\cos\theta \quad y = r\,\mathrm{sen}\,\theta \quad a \le \theta \le b \quad c \le r \le d\}$$

e seja S^* o retângulo

$$S^* = \{(r,\theta) : a \le \theta \le b \quad c \le r \le d\}.$$

Se $f : S \longrightarrow \mathbf{R}$ for uma função seccionalmente contínua então

$$\iint_S f(x,y)dydx = \iint_{S^*} f^*(r,\theta)\; r\; drd\theta$$

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Volume do Cone $M = \{(x,y,z) : \quad |(x,y)| \le z/2 \quad \text{e} \quad 0 \le z \le 2\rho\}$
$M^* = \{(r,\theta,z) \quad 0 \le r \le z/2 \quad \text{e} \quad 0 \le z \le 2\rho \quad \text{e} \quad 0 \le \theta \le 2\pi\}$

$$\iiint_M 1\; dz\; dy\; dx = \iiint_{M^*} r\; dz\; dr\; d\theta = \int_0^{2\pi}\int_0^{2\rho}\int_0^{z/2} r\; dr\; dz\; d\theta$$

$$\int_0^{z/2} r\; dr = \frac{z^2}{8}$$

$$\int_0^{2\rho}\int_0^{z/2} r\; dr\; dz = \frac{\rho^3}{3}$$

$$\iiint_{M^*} r\; dz\; dr\; d\theta = \frac{2}{3}\pi\rho^3$$

Teorema 2 (Arquimedes ≈ -250)**.** *Para uma esfera e um cone perfeitamente inscritos em um cilindro, os volumes do cone, da esfera e do cilindro estão na relação 1:2:3.*

Coordenadas cilíndricas (r,θ,z) $x = r\cos\theta \qquad y = r\,\mathrm{sen}\,\theta \qquad z = z$

$$G(r,\theta,z) = (r\cos\theta, r\,\mathrm{sen}\,\theta, z)$$

transforma um paralelogramo $\mathcal{P}$ em uma casca cilíndrica $\mathcal{C} = G(\mathcal{P})$. $\mathcal{P} = \{(r,\theta,z) : \; r_1 \le r \le r_2 \quad \theta_1 \le \theta \le \theta_2 \quad z_1 \le z \le z_2\}$

$$\text{Volume de } \mathcal{P} = (r_2 - r_1)(\theta_2 - \theta_1)(z_2 - z_1)$$

$$\text{Volume de } \mathcal{C} = \frac{(r_2^2 - r_1^2)}{2}(\theta_2 - \theta_1)(z_2 - z_1)$$

$$\text{Volume de } \mathcal{C} = \frac{(r_2 + r_1)}{2}(r_2 - r_1)(\theta_2 - \theta_1)(z_2 - z_1)$$

Coordenadas cilíndricas (r, θ, z) $x = r\cos\theta \qquad y = r\,\mathrm{sen}\,\theta \qquad z = z$

$$G(r, \theta, z) = (r\cos\theta, r\,\mathrm{sen}\,\theta, z)$$

Dada $f(x, y, z)$ define-se $f^* = f \circ G$ isto é $f^*(r, \theta, z) = f(r\cos\theta, r\,\mathrm{sen}\,\theta, z)$.

Teorema 3. *Dado* $A \subset \mathbf{R}^3$ *seja* A^* *o conjunto*

$$A^* = \{(r, \theta, z) : G(r, \theta, z) \in A\}.$$

Se $f : A \longrightarrow \mathbf{R}$ *for uma função seccionalmente contínua então*

$$\iiint_A f(x, y, z)\ dz\ dy\ dx = \iiint_{A^*} f^*(r, \theta, z)\ r\ dz\ dr\ d\theta$$

Teorema 4. *Se* $A \subset \mathbf{R}^3$ *for dado por*

$$A = \left\{ \begin{array}{l} (x, y, z) : x = r\cos\theta,\ y = r\,\mathrm{sen}\,\theta, \quad a \le \theta \le b \\ 0 \le g_1(\theta) \le r \le g_2(\theta),\ h_1(r, \theta) \le z \le h_2(r, \theta) \end{array} \right\}$$

com $b \le a + 2\pi$ *e se* $f : A \longrightarrow \mathbf{R}$ *for uma função seccionalmente contínua, então*

$$\iiint_A f = \int_a^b \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r,\theta)}^{h_2(r,\theta)} f^*(r, \theta, z) r\ dz\ dr\ d\theta$$

porque

$$A^* = \left\{ \begin{array}{l} (r, \theta, z) : \ a \le \theta \le b, \quad 0 \le g_1(\theta) \le r \le g_2(\theta), \\ h_1(r, \theta) \le z \le h_2(r, \theta) \end{array} \right\}$$

Exemplo Volume de um cone cuja geratriz tem comprimento ρ e faz um ângulo φ com o eixo do cone.
Raio da base: $a = \rho\cos\varphi$ altura: $b = \rho\,\mathrm{sen}\,\varphi$ Cone:
$M = \{(x, y, z) : \quad |(x, y)| \le z\,\mathrm{tg}\,\varphi \quad \text{e} \quad 0 \le z \le \rho\cos\varphi\}$
$M^* = \{(r, \theta, z) \quad 0 \le r \le z\,\mathrm{tg}\,\varphi \text{ e } 0 \le z \le \rho\cos\varphi \text{ e } 0 \le \theta \le 2\pi\}$

$$\iiint_M 1\, dz\, dy\, dx = \iiint_{M^*} r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^{\rho\cos\varphi} \int_0^{z\,\mathrm{tg}\,\varphi} r\, dr\, dz\, d\theta$$

$$\int_0^{z\,\mathrm{tg}\,\varphi} r\ dr = \frac{z^2}{2}\,\mathrm{tg}^2\,\varphi \qquad \int_0^{\rho\cos\varphi} \int_0^{z\,\mathrm{tg}\,\varphi} r\ dr\ dz = \frac{\rho^3}{6}\,\mathrm{sen}^2\,\varphi\cos\varphi$$

$$\iiint_{M^*} r\ dz\ dr\ d\theta = \frac{\pi}{3}\rho^3\,\mathrm{sen}^2\,\varphi\cos\varphi = \frac{\pi}{3}ab^2$$

Exercício Volume de uma calota da esfera de raio ρ cortada por um plano a uma distância $b = \rho \operatorname{sen} \varphi$ do centro.
Calota esférica:
$S = \{(x, y, z) : \quad |(x, y, z)| \leq \rho \quad \text{e} \quad b \leq z \leq \rho\}$
$S^* = \{(r, \theta, z) \quad 0 \leq r \leq \sqrt{\rho^2 - z^2} \quad \text{e} \quad b \leq z \leq \rho \quad \text{e} \quad 0 \leq \theta \leq 2\pi\}$

$$\iiint_S 1\ dz\ dy\ dx = \int_{-\rho}^{\rho} \left(\int_b^{\rho} \left(\int_0^{f(z)} r\ dr \right)\ dz \right)\ d\theta$$

$f(z) = \sqrt{\rho^2 - z^2}$.

$$\int_0^{f(z)} r\ dr = \frac{\rho^2 - z^2}{2}$$

porque $\rho^2 - z^2 > 0$ para $b \leq z \leq \rho$.

$$\iiint_S 1\ dz\ dy\ dx = \pi \left(\frac{2}{3}\rho^3 - \rho^3 \cos \varphi + \frac{1}{3}\rho^3 \cos^3 \varphi \right)$$

Coordenadas esféricas (ρ, θ, φ)

$$x = \rho \operatorname{sen} \varphi \cos \theta \quad y = \rho \operatorname{sen} \varphi \operatorname{sen} \theta \quad z = \rho \cos \varphi$$

$G : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$

$$G(\rho, \theta, \varphi) = (\rho \operatorname{sen} \varphi \cos \theta, \rho \operatorname{sen} \varphi \operatorname{sen} \theta, \rho \cos \varphi) = (x, y, z)$$

Dada $f(x, y, z)$ define-se $f^* = f \circ G$ isto é $f^*(r, \theta, z) = f(\rho \operatorname{sen} \varphi \cos \theta, \rho \operatorname{sen} \varphi \operatorname{sen} \theta, \rho \cos \varphi)$.

Teorema 5. *Dado* $A \subset \mathbf{R}^3$ *seja* A^* *o conjunto*

$$A^* = \{(\rho, \theta, \varphi) : G(\rho, \theta, \varphi) \in A\}.$$

Se $f : A \longrightarrow \mathbf{R}$ *for uma função seccionalmente contínua então*

$$\iiint_A f(x, y, z)\ dz\ dy\ dx = \iiint_{A^*} f^*(\rho, \theta, \varphi)\ \rho^2 \operatorname{sen} \varphi\, d\rho\, d\theta\, d\varphi$$