Weeds in Mathematics

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This text is both a warning for prospective mathematicians who violently dislike some parts of mathematics and a praise to the connectedness of mathematics. As a warning, it tells you that, if you think some areas of mathematics are weeds in a beautiful garden, you will have to live with those weeds. Indeed, you are better off trying to make use of those weeds to produce more flowers. As a praise, it shows that, in a mathematical garden, there are indeed no weeds - everything can be put to good use, even though you may still prefer roses to wild daisies.

"Weeds are plants that are growing at the wrong place." R. Bird, *Organic Gardening*

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Algebra

Consider a group of symmetries of $\mathbf{R}^n$, a compact Lie group with an action on $\mathbf{R}^n$, with subgroups S_1, S_2 and a subgroup T

contained both in S_1 and S_2. Suppose that the spaces of points fixed by S_1 and S_2, $\mathrm{Fix}(S_1)$ and $\mathrm{Fix}(S_2)$ respectively, are one-dimensional and that $\mathrm{Fix}(T)$, which obviously contains $\mathrm{Fix}(S_1)$ and $\mathrm{Fix}(S_2)$ is of dimension two. Let the quotient groups $N(S_i)/S_i$ be isomorphic to $\mathbf{Z}_2$, where $N(S_i)$ is the normalizer of S_i. Is it then true that $N(T)/T$ contains a subgroup isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$?

Since $\mathbf{Z}_2$ acts on $\mathbf{R}$ as the flip $(x \mapsto -x)$, you would think that, since you are flipping the two axes $\mathrm{Fix}(S_1)$ and $\mathrm{Fix}(S_2)$ in $\mathrm{Fix}(T)$, then, in $\mathrm{Fix}(T)$ you will see some copy of the two flips put together. You may want to picture the coordinate axes in $\mathbf{R}^2$ and imagine that you transform each of them by multiplication by -1 (the flip). Then you expect the global transformation of a generic point (x, y) in $\mathbf{R}^2$ to be $(x, y) \mapsto (\pm x, \pm y)$, that is $\mathbf{Z}_2 \oplus \mathbf{Z}_2$.

Although this intuition is correct, if the group is abelian or $n \geq 3$, an attempt to prove it for any group takes us on the tour that follows.

Dynamical systems

That the answer to the above question is "yes" is conjectured by Melbourne, Chossat and Golubitsky in [6]. This is a paper on heteroclinic cycles in symmetric differential equations - maybe not your favourite flower but that's where this polen comes from. A heteroclinic cycle of an equation $\dot{x} = f(x)$ is a collection of constant solutions $x(t) = x_i$, $i = 1, \ldots, n$ of f together with trajectories $\phi_i(t)$ connecting them in the following way

$$x_i = \lim_{t \to -\infty} \phi_i(t) \quad x_{i+1} = \lim_{t \to +\infty} \phi_i(t).$$

By convention, $x_{n+1} = x_1$. If $n = 1$ we say the cycle is homoclinic. Although persistent heteroclinic cycles are relatively rare for problems without symmetry, they seem to appear more naturally if the equation is symmetric, that is, $f(\gamma x) = \gamma f(x)$, for all points x and γ in a given group to which we refer as the symmetries of the equation. The conjecture in Melbourne *et*

al [6] was essential in proving the existence of such a cycle.

Topology and something else

Consider the two matrices $A(p)$ and $B(q)$ with $p, q > 2$, representing p- and q-fold rotations in 3-space about perpendicular axes:

$$A(p) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\pi/p) & -\sin(2\pi/p) \\ 0 & \sin(2\pi/p) & \cos(2\pi/p) \end{pmatrix} \qquad B(q) = \begin{pmatrix} \cos(2\pi/q) & 0 & -\sin(2\pi/q) \\ 0 & 1 & 0 \\ \sin(2\pi/q) & 0 & \cos(2\pi/q) \end{pmatrix}.$$

The powers of each of these matrices form finite subgroups of the compact group $SO(3)$ of all rotations in 3-space. Is the group G of all products of powers of $A(p)$ by powers of $B(q)$ in any order a compact subgroup of $SO(3)$, for all values of p and q?

Before proceeding with the answer to the question, we use (rather than pluck) programming, modern technologies and the geometry of the action of the orthogonal group to obtain Figures 1 and 2. You can find the Maple commands for these in the Appendix.

Figures go near here.

Figure 1 shows the composition of two rotations $A(3)$ and $B(3)$. The result is, we claim, a map that rotates by an irrational multiple of π hence, successive images of a point never return to the original point. Figure 2 presents the images of a point on the sphere in $\mathbf{R}^3$, randomly iterated using $A(3)$ and $B(3)$.

At this point, we would like the answer to our last question to be “no”, as indeed it is. In fact, G is not a closed set.

Algebra with some Geometry

Every closed subgroup of $\mathbf{SO}(3)$ is conjugate to one of $\mathbf{SO}(3)$; $\mathbf{O}(2)$; $\mathbf{SO}(2)$; the dihedral group, $\mathbf{D_n}$ ($n \geq 2$), (symmetry group of the n-gon); the cyclic group of order n ($\mathbf{Z_n}$), the "rotational" symmetry groups of the platonic solids or the identity (See [3], p.104). Note that $\mathbf{O}(2)$ and $\mathbf{SO}(2)$ are infinite groups (of dimension 1). The remaining ones are finite.

The group G defined above is finitely generated and is not a planar group, so, if it is closed, it can only be one of the "rotational" symmetry groups of the platonic solids. Any element of these groups has, at most, order 5. If p and q are not simultaneously equal to 4, then the matrix M has order greater than 5. So, in those cases, the group G generated by $A(p)$ and $B(q)$ is not closed.

Analysis

Although the previous section answers our question, it may not be the garden path taken by the more analytically minded. This being the case, you would compute the eigenvalues of $M = A(p)B(q)$, which is surely an element of the group G. The eigenvalues are 1 and $\cos\theta \pm i \sin\theta$ where

$$2\cos\theta \quad = \quad \cos(\frac{2\pi}{p})\cos(\frac{2\pi}{q}) - \cos(\frac{2\pi}{q}) - \cos(\frac{2\pi}{p}) - 1. \tag{1}$$

If you can prove that $\cos\theta = \cos(a\pi)$ with a irrational then G is not closed and its closure is $SO(3)$, as suggested by Figure 2. Now, $\cos\theta$ is suspiciously reminiscent of the root of a Chebyshev polynomial. The curious mathematician, trying to decide whether Chebyshev polynomials are a weed or a flower, makes a digression off the main garden path.

Chebyshev polynomials

The Chebyshev polynomial of the first kind of degree n is

$$T_n(x) = cos(n\, arccos x), x \in [-1, 1].$$

Thus, $T_0(x) = 1, T_1(x) = x$ and, by the addition formula

$$cos[(n+1)\varphi] + cos[(n-1)\varphi] = 2cos\varphi cos(n\varphi)$$

it can be deduced that

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), n = 1, 2, ...$$

So, $T_n(x)$ is indeed a polynomial of degree n and it has integer coefficients.

The roots of $T_n(x)$ are

$$x_k^{(n)} = cos\left(\frac{2k-1}{2n}\pi\right), k = 1, ..., n.$$

They are all distinct and symmetrically disposed about the origin. These roots play an important role in the theory of polynomial interpolation and in gaussian quadrature. Our question would be partially answered if we could decide if $\frac{1}{2}\left(x_k^{(n)}x_l^{(m)} - x_k^{(n)} - x_l^{(m)} - 1\right)$ is the root of a Chebyshev polynomial. We look down this path but do not take it.

Number Theory

We say a bit more about the roots of a Chebyshev polynomial.

All numbers of the form $x_k^{(n)}$ are *radical algebraic*. They are *algebraic*, of course, because they are the roots of $T_n(x)$, which has rational coefficients. The word *radical* means that they can be expressed from a finite extension of the rationals by radicals. They are the real parts of powers of the $4n^{\underline{\text{th}}}$ roots of unity $\zeta_{4n} = e^{2\pi i/4n}$. For a fixed n the $x_k^{(n)}$ belong to the subfield of the complex numbers which is generated, over $\mathbf{Q}$, by ζ_{4n}. This field, $\mathbf{Q}(\zeta_{4n})$, is called a *cyclotomic* field.

The radical ζ_{4n} is a root of the polynomial $x^{4n} - 1$ but there are polynomials over $\mathbf{Q}$, of lower degree, having ζ_{4n} as a root.

Any algebraic number α satisfies a unique polynomial having leading coefficient 1, of least degree, called the *minimal polynomial* of α. The degree of this polynomial is called the *degree* of α.

For ζ_{4n} the minimal polynomial is the polynomial whose zeros are the primitive $4n^{\underline{\text{th}}}$ roots of unity. It can be written in the form

$$F_{4n}(x) = \prod_{\substack{k=0 \\ (k,4n)=1}}^{4n-1} \left(x - e^{2\pi ik/4n}\right),$$

where $(k, 4n)$ means the greater common divisor of k and $4n$.

The degree of F_{4n} is the exact number of primitive $4n^{\underline{\text{th}}}$ roots of unity. This is given by the number of positive integers $j < 4n$ such that $(j, 4n) = 1$, called the Euler function $\phi(4n)$.

The polynomial $F_{4n}(x)$ is obtained from $x^{4n} - 1$ by dividing it by all $F_d(x)$ where d is a divisor of $4n$ other than $4n$. It turns out that $F_{4n}(x)$ is irreducible over the field of rational numbers, so, the degree of ζ_{4n} is $\phi(4n)$.

Next we obtain the degree of $\cos(2\pi k/2n)$. Let $n > 1$ and define $m = \phi(2n)/2$. It can be proved that $x^{-m}F_{2n}(x)$ is a monic polynomial in $x + x^{-1}$, say $\psi_{2n}(x + x^{-1})$, with rational coefficients. Also $\psi_{2n}(x)$ is irreducible of degree m. If $n > 1$ and $(k, 2n) = 1$, the number $2\cos(2\pi k/2n) = e^{2\pi ik/2n} + e^{-2\pi ik/2n}$ is a root of $\psi_{2n}(x)$. So, the algebraic number $2\cos(2\pi k/2n)$ has degree $\phi(2n)/2$. See [7], p. 37 for the details. These notions will be used as fertilizer for the next section.

Chebyshev polynomials with number theory

After proving that roots of Chebyshev polynomials are radical algebraic, our question becomes: can we tell when a radical algebraic number in $[-1, 1]$ is the cosine of a rational multiple of π? Can we at least tell it for the radical algebraic number $\frac{1}{2}(xy - x - y - 1)$, if x and y are such cosines?

The question arises naturally from the well-known fact that the set of zeros of Chebyshev polynomials is dense on $[-1, 1]$ (See [1]). We start by answering the question for rational numbers, the algebraic numbers of degree 1. The number $\cos(2\pi k/2n)$ as above is rational if and only if $\phi(2n) = 2$, and the Euler function takes the value 2 precisely for $n = 2, 3$. Thus the only rational values of $\cos(2\pi k/2n)$ are $0, \pm 1/2, \pm 1$.

With this we construct an example where θ in (1) is an irrational multiple of π: if $\cos\theta \in \mathbf{Q}$ it suffices to show that $\cos\theta$ does not take the above values of $0, \pm 1/2, \pm 1$. Indeed, for $p = q = 6$, $\cos(2\pi/p) = \cos(2\pi/q) = 1/2$ and $\cos\theta = -7/8$, which shows that θ is an irrational multiple of π and G is not closed. The answer in this case is not as complete as the one obtained using Geometry but it is nevertheless an answer.

As for the question at the beginning of this section, westill have no general answer.

Final remarks

You may of course read this text back-to-front if Number Theory is your main interest. In this case, starting with the question about radical algebraic numbers, the text provides a geometric answer in the first sections.

It may be worthwhile pointing out that this text arose from our research on the existence of heteroclinic cycles in dynamical systems and represents (the sensible) part of our thoughts while trying to answer the question in the first section.

Reading for the curious

There are many excellent books for learning the basics of the material we used. The list we give here is rather a statement about the contents of the library accessible to us at present than one about the references we do not include.

The basics of Algebra can be learnt from the book by Green [4]. If you want a bit more than the basics you may want to look for Lang [5]. You can learn about dynamical systems from Jacob Palis and Wellington de Melo in [8] and how symmetry comes into the subject by using [3]. If you are patient you can wait for the forthcoming book by Marty Golubitsky and Ian Stewart [2]. For Chebyshev polynomials try Rivlin [9] while you can get through number theory with Niven [7].

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Appendix — Maple programs for figures

These are the Maple commands used to produce Figure 1:

```
> with(linalg):
>
>A3:=matrix([[1,0,0],[0,cos(2*Pi/3),-sin(2*Pi/3)],[0,sin(2*Pi/3),cos(2*Pi/3)]]):
>>
>B3:=matrix([[cos(2*Pi/3),0,-sin(2*Pi/3)],[0,1,0],[sin(2*Pi/3),0,cos(2*Pi/3)]]):
>> II:=matrix([[1,0,0],[0,1,0],[0,0,1]]):
> A3B3:=evalm(A3&*B3):
> B3A3:=evalm(B3&*A3):
> s[0] := vector( [sqrt(3)/2,0,1/2] ):
> t[0] := vector( [0,sqrt(3)/2,1/2] ):
> poin:=[]:
> newpoin:=[]:
> s[1]:=multiply(II,s[0]):
> t[1]:=multiply(II,t[0]):
> for i from 1 by 1 to 100 do
> s[i+1]:=multiply(A3B3,s[i]);poin:=[op(poin),s[i]]; od:
> for k from 1 by 1 to 100 do
```

```
> t[k+1]:=multiply(B3A3,t[k]);newpoin:=[op(newpoin),t[k]]; od:
> with(plots):
> aa:=pointplot3d(poin,orientation=[-170,80],color=black,symbol=POINT):
> bb:=pointplot3d(newpoin,orientation=[-170,80],color=black,symbol=POINT):
> cc:=plots[spacecurve]({[cos(s),sin(s),0],[cos(s),0,sin(s)]},
 s=0..2*Pi,color=black):
> display(aa,bb,cc);
```

The Maple commands for Figure 2 are:

```
> with(linalg):
>
>A3:=matrix([[1,0,0],[0,cos(2*Pi/3),-sin(2*Pi/3)],[0,sin(2*Pi/3),cos(2*Pi/3)]]):
>>
>B3:=matrix([[cos(2*Pi/3),0,-sin(2*Pi/3)],[0,1,0],[sin(2*Pi/3),0,cos(2*Pi/3)]]):
>> II:=matrix([[1,0,0],[0,1,0],[0,0,1]]):
> aa := vector( [0,1,0] ):
> v:=multiply(II,aa):
> poin:=[]:
> for i from 1 by 1 to 5000 do  d := rand(0..1):
if (d()=1) then v:=multiply(A3,v); else v:=multiply(B3,v); fi;
 poin:=[op(poin),[evalf(v[1]),evalf(v[2]),evalf(v[3])]]; od:
> with(plots):
> fig2:=pointplot3d(poin,orientation=[109,177],color=black,symbol=POINT):
> display(fig2);
```

Figure 1: The composition of two rotations of $2\pi/3$ about perpendicular axes is not a finite rotation. Solid lines are the intersections of the planes of rotation of the matrices $A(3)$ and $B(3)$ with a sphere. The dots are 100 successive images by $A(3)B(3)$ and by $B(3)A(3)$ of two points on the sphere.

Figure 2: Successive images obtained by random rotations of $2\pi/3$ about two perpendicular axes. Each point is obtained from the previous one by either $A(3)$ or $B(3)$ chosen ramdomly.