

Global Dynamics for Symmetric Planar Maps

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Abstract

We consider sufficient conditions to determine the global dynamics for equivariant maps of the plane with a unique fixed point which is also hyperbolic. When the map is equivariant under the action of a compact Lie group, it is possible to describe the local dynamics and – from this – also the global dynamics. In particular, if the group contains a reflection, there is a line invariant by the map. This allows us to use results based on the theory of free homeomorphisms to describe the global dynamical behaviour. In the absence of reflections, we use equivariant examples to show that global dynamics may not follow from local dynamics near the unique fixed point.

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1 Introduction

Dynamics of planar maps has drawn the attention of many authors. See, for instance, references such as [3], [6], [7], [9] or [15]. Problems addressed include the theory of free homeomorphism and trivial dynamics, searching for sufficient conditions for global results with topological tools or the Discrete Markus-Yamabe Problem. The latter conjectures results on global stability from local stability of the fixed point under some additional condition on the Jacobian matrix¹. Results in [3] guarantee the global stability of the unique fixed point resorting to local conditions and the existence of an invariant embedded curve joining the fixed point to infinity.

To the best of our knowledge, this problem has been addressed exclusively in a non-symmetric context. However the existence of the invariant curve in [3] made it seem natural to approach this problem in a symmetric setting, where invariant spaces for the dynamics are a key feature. In this context, we address the global dynamics of planar diffeomorphisms having a unique fixed point which is hyperbolic. We restrict our attention to the cases where the fixed point is either an attractor or a repeller. The case of a saddle point does not rely so much on symmetry. It will therefore be addressed elsewhere.

The issue of uniqueness of a fixed point has been addressed by Alarcón *et al* [4] who proved that, under mild assumptions, planar maps have the origin as a unique fixed point.

In the presence of symmetry we determine the admissible local dynamics near the fixed point. This depends only on the symmetry group. We then extend, whenever possible, the local dynamics to the whole plane using the properties determined by the symmetry. The reader may see that, in the case of $SO(2)$, $O(2)$, $\mathbb{Z}_2(\langle \kappa \rangle)$ and D_n , local dynamics determines global dynamics. However, \mathbb{Z}_n symmetry does not provide any extra information. Actually, we construct examples where more than one configuration can occur. For instance, in the case of a local attractor, the global dynamics may exhibit either several periodic points or a globally attracting set with special properties in the space of prime ends. These situations provide a comprehensive description of the possible global dynamics.

The paper is organized as follows: in the next section we transcribe preliminary results concerning dynamics of planar maps and equivariance. These

¹The Discrete Markus-Yamabe Problem also derives its fame from its relation to the Jacobian Conjecture [8].

are organized in two subsections, each of which may be skipped by the reader familiar with the subject. Section 3 is concerned with results on local dynamics that are a direct consequence of symmetry. The last section shows how the local results obtained in section 3 can be extended to global results or why they cannot be extended globally. In this latter case, we discuss examples that show the admissible dynamics.

We thus establish all possible dynamics in the presence of symmetry. Note that the aforementioned groups are the only compact subgroups of $O(2)$ acting on the plane. For the reader's convenience the results obtained are summarized in the Equivariant Table of Appendix C.

2 Preliminaries

This section contains definitions and known results about topological dynamics and equivariant theory. These are grouped in two separate subsections, which are elementary for readers in each field.

2.1 Topological Dynamics

We consider planar topological embeddings, that is, continuous and injective maps defined in \mathbb{R}^2 . The set of topological embeddings of the plane is denoted by $\text{Emb}(\mathbb{R}^2)$.

Recall that for $f \in \text{Emb}(\mathbb{R}^2)$ the equality $f(\mathbb{R}^2) = \mathbb{R}^2$ may not hold. Since every map $f \in \text{Emb}(\mathbb{R}^2)$ is open (see [17]), we will say that f is a homeomorphism if f is a topological embedding defined onto \mathbb{R}^2 . The set of homeomorphisms of the plane will be denoted by $\text{Hom}(\mathbb{R}^2)$.

When \mathcal{H} is one of these sets we denote by \mathcal{H}^+ (and \mathcal{H}^-) the subset of orientation preserving (reversing) elements of \mathcal{H} .

Given a continuous map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, we say that p is a *non-wandering point* of f if for every neighbourhood U of p there exists an integer $n > 0$ and a point $q \in U$ such that $f^n(q) \in U$. We denote the set of non-wandering points by $\Omega(f)$. We have

$$\text{Fix}(f) \subset \text{Per}(f) \subset \Omega(f),$$

where $\text{Fix}(f)$ is the set of fixed points of f , and $\text{Per}(f)$ is the set of periodic points of f .

Let $\omega(p)$ be the set of points q for which there is a sequence $n_j \rightarrow +\infty$ such that $f^{n_j}(q) \rightarrow p$. If $f \in \text{Hom}(\mathbb{R}^2)$ then $\alpha(p)$ denotes the set $\omega(p)$ under f^{-1} .

Let $f \in \text{Emb}(\mathbb{R}^2)$ and $p \in \mathbb{R}^2$. We say that $\omega(p) = \infty$ if $\|f^n(p)\| \rightarrow \infty$ as n goes to ∞ . Analogously, if $f \in \text{Hom}(\mathbb{R}^2)$, we say that $\alpha(p) = \infty$ if $\|f^{-n}(p)\| \rightarrow \infty$ as n goes to ∞ .

We say that $0 \in \text{Fix}(f)$ is a *local attractor* if its basin of attraction $\mathcal{U} = \{p \in \mathbb{R}^2 : \omega(p) = \{0\}\}$ contains an open neighbourhood of 0 in \mathbb{R}^2 and that 0 is a *global attractor* if $\mathcal{U} = \mathbb{R}^2$. The origin is a *stable fixed point* if for every neighborhood U of 0 there exists another neighborhood V of 0 such that $f(V) \subset V$ and $f(V) \subset U$. Therefore, the origin is an *asymptotically local (global) attractor* or a *(globally) asymptotically stable fixed point* if it is a stable local (global) attractor. See [5] for examples.

We say that $0 \in \text{Fix}(f)$ is a *local repeller* if there exists a neighbourhood V of 0 such that $\omega(p) \notin V$ for all $0 \neq p \in \mathbb{R}^2$ and a *global repeller* if this holds for $V = \mathbb{R}^2$. We say that the origin is an *asymptotically global repeller* if it is a global repeller and, moreover, if for any neighbourhood U of 0 there exists another neighbourhood V of 0, such that, $V \subset f(V)$ and $V \subset f(U)$.

When the origin is a fixed point of a C^1 -map of the plane we say the origin is a *local saddle* if the two eigenvalues of Df_0 , α, β , are both real and verify $0 < |\alpha| < 1 < |\beta|$.

We also need the following theorem of Murthy [16], to be applied to parts of the domain of our maps with no fixed points:

Theorem 2.1 (Murthy [16]). *Let $f \in \text{Emb}^+(\mathbb{R}^2)$. If $\text{Fix}(f) = \emptyset$, then $\Omega(f) = \emptyset$.*

We say that $f \in \text{Emb}(\mathbb{R}^2)$ has *trivial dynamics* if $\omega(p) \subset \text{Fix}(f)$, for all $p \in \mathbb{R}^2$. Moreover, we say that a planar homeomorphism has trivial dynamics if both $\omega(p), \alpha(p) \subset \text{Fix}(f)$, for all $p \in \mathbb{R}^2$.

Let $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a continuous map. Let $\gamma : [0, \infty) \rightarrow \mathbb{R}^2$ be a topological embedding of $[0, \infty)$. As usual, we identify γ with $\gamma([0, \infty))$. We will say that γ is an *f -invariant ray* if $\gamma(0) = (0, 0)$, $f(\gamma) \subset \gamma$, and $\lim_{t \rightarrow \infty} |\gamma(t)| = \infty$, where $|\cdot|$ denotes the usual Euclidean norm.

Proposition 2.2 (Alarcón et al. [3]). *Let $f \in \text{Emb}^+(\mathbb{R}^2)$ be such that $\text{Fix}(f) = \{0\}$. If there exists an f -invariant ray γ , then f has trivial dynamics.*

Corollary 2.3. *Let $f \in \text{Hom}^+(\mathbb{R}^2)$ be such that $\text{Fix}(f) = \{0\}$. If there exists an f -invariant ray γ , then for each $p \in \mathbb{R}^2$, as n goes to $\pm\infty$, either $f^n(p)$ goes to 0 or $\|f^n(p)\| \rightarrow \infty$.*

2.2 Equivariant Planar Maps

Let Γ be a compact Lie group acting on \mathbb{R}^2 , that is, a group which has the structure of a compact C^∞ -differentiable manifold such that the map $\Gamma \times \Gamma \rightarrow \Gamma$, $(x, y) \mapsto xy^{-1}$ is of class C^∞ . The following definitions and results are taken from Golubitsky *et al.* [10], especially Chapter XII, to which we refer the reader interested in further detail.

We think of a group mostly through its action or representation on \mathbb{R}^2 . A *linear action* of Γ on \mathbb{R}^2 is a continuous mapping

$$\begin{aligned} \Gamma \times \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ (\gamma, p) &\mapsto \gamma p \end{aligned}$$

such that, for each $\gamma \in \Gamma$ the mapping ρ_γ that takes p to γp is linear and, given $\gamma_1, \gamma_2 \in \Gamma$, we have $\gamma_1(\gamma_2 p) = (\gamma_1 \gamma_2)p$. Furthermore the identity in Γ fixes every point. The mapping $\gamma \mapsto \rho_\gamma$ is called the *representation* of Γ and describes how each element of Γ transforms the plane.

We consider only standard group actions and representations. A representation of a group Γ on a vector space V is *absolutely irreducible* if the only linear mappings on V that commute with Γ are scalar multiples of the identity. Since every compact Lie group in $GL(2)$ can be identified with a subgroup of the orthogonal group $O(2)$, we need only be concerned with the groups we list below.

Compact subgroups of $O(2)$

- $O(2)$, acting on $\mathbb{R}^2 \simeq \mathbb{C}$ as the group generated by θ and κ given by

$$\theta.z = e^{i\theta}z, \quad \theta \in S^1 \quad \text{and} \quad \kappa.z = \bar{z}.$$

- $SO(2)$, acting on $\mathbb{R}^2 \simeq \mathbb{C}$ as the group generated by θ given by

$$\theta.z = e^{i\theta}z, \quad \theta \in S^1.$$

- D_n , $n \geq 2$, acting on $\mathbb{R}^2 \simeq \mathbb{C}$ as the finite group generated by ζ and κ given by

$$\zeta.z = e^{\frac{2\pi i}{n}} z \quad \text{and} \quad \kappa.z = \bar{z}.$$

- \mathbb{Z}_n , $n \geq 2$, acting on $\mathbb{R}^2 \simeq \mathbb{C}$ as the finite group generated by ζ given by

$$\zeta.z = e^{\frac{2\pi i}{n}} z.$$

- $\mathbb{Z}_2(\langle \kappa \rangle)$, acting on \mathbb{R}^2 as

$$\kappa.(x, y) = (x, -y).$$

Given a map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, we say that $\gamma \in \Gamma$ is a *symmetry* of f if $f(\gamma x) = \gamma f(x)$. We define the *symmetry group* of f as the biggest closed subset of $GL(2)$ containing all the symmetries of f . It will be denoted by Γ_f .

We say that $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is Γ -*equivariant* or that f *commutes* with Γ if

$$f(\gamma x) = \gamma f(x) \quad \text{for all} \quad \gamma \in \Gamma.$$

It follows that every map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is equivariant under the action of its symmetry group, that is, f is Γ_f -equivariant.

Let Σ be a subgroup of Γ . The *fixed-point subspace* of Σ is

$$\text{Fix}(\Sigma) = \{p \in \mathbb{R}^2 : \sigma p = p \text{ for all } \sigma \in \Sigma\}.$$

If Σ is generated by a single element $\sigma \in \Gamma$, we write $\text{Fix}\langle \sigma \rangle$.

We note that, for each subgroup Σ of Γ , $\text{Fix}(\Sigma)$ is invariant by the dynamics of a Γ -equivariant map ([10], XIII, Lemma 2.1).

When f is Γ -equivariant, we can use the symmetry to generalize information obtained for a particular point. This is achieved through the *group orbit* Γx of a point x , which is defined to be

$$\Gamma x = \{\gamma x : \gamma \in \Gamma\}.$$

Lemma 2.4. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be Γ -equivariant and let p be a fixed point of f . Then all points in the group orbit of p are fixed points of f .*

Proof. If $f(p) = p$ it follows that $f(\gamma p) = \gamma f(p) = \gamma p$, showing that γp is a fixed point of f for all $\gamma \in \Gamma$. \square

Since most of our results depend on the existence of a unique fixed point for f , it is worthwhile noting that the group actions we are concerned with are such that $\text{Fix}(\Gamma) = \{0\}$. Therefore, if f is Γ -equivariant then $f(0) = 0$.

3 Symmetric Local Dynamics

In this section, we describe the consequences for the local dynamics arising from the fact that a map is equivariant under the action of a compact Lie group Γ . These are patent in the structure of the Jacobian matrix at the origin. The relation between the group action and the Jacobian matrix of an equivariant map f is obtained through the following

Lemma 3.1. *Let $f : V \rightarrow V$ be a Γ -equivariant map differentiable at the origin. Then $Df(0)$, the Jacobian of f at the origin, commutes with Γ .*

Proof. Since f is Γ -equivariant we have $f(\gamma.v) = \gamma f(v)$ for all $\gamma \in \Gamma$ and $v \in V$. Differentiating both sides of the equality with respect to v , we obtain $Df(\gamma.v)\gamma = \gamma Df(v)$ and, evaluating at the origin gives $Df(0)\gamma = \gamma Df(0)$. \square

If the representation is absolutely irreducible, we know that $Df(0)$ is a multiple of the identity and thus it has one real eigenvalue of geometric multiplicity two. Therefore, the origin is locally either an attractor or a repeller. We have the following

Lemma 3.2. *The standard representation on \mathbb{R}^2 is absolutely irreducible for $O(2)$ and D_n with $n \geq 3$ and for no other subgroup of $O(2)$.*

Proof. The proof follows by direct computation.

- $O(2)$: the generators of this group are θ and κ and it suffices to find the linear matrices that commute with both. A real matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

commutes with κ if and only if $b = c = 0$. In order for such a matrix to commute with any rotation it must be

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

which holds when $a = d$ or $\sin \theta = 0$ for all $\theta \in S^1$. Therefore, the action of $O(2)$ is absolutely irreducible.

- $SO(2)$: the elements of $SO(2)$ are rotation matrices which commute with any other rotation matrix, also non-diagonal ones.
- $D_n, n \geq 3$: see the proof for $O(2)$. In the last step, we must have $a = d$ or $\sin 2\pi i/n = 0$ which is never satisfied for $n \geq 3$. Hence, the action is absolutely irreducible.
- $\mathbb{Z}_n, n \geq 3$: as for $SO(2)$, any rotation matrix commutes with the rotation of $2\pi/n$, including non-diagonal ones.
- $\mathbb{Z}_2(\langle \kappa \rangle)$: see the proof for $\kappa \in O(2)$ to conclude that linear commuting matrices are diagonal but not necessarily linear multiples of the identity.
- \mathbb{Z}_2 : all linear maps commute with $-Id$.
- $D_2 = \mathbb{Z}_2 \oplus \mathbb{Z}_2(\langle \kappa \rangle)$: as above, \mathbb{Z}_2 introduces no restrictions and for commuting with κ it suffices that the map is diagonal.

□

The following result is then a straightforward consequence of the previous proof.

Lemma 3.3. *The linear maps that commute with the standard representations of the subgroups of $O(2)$ are rotations and homotheties (and their compositions) for $SO(2)$ and $\mathbb{Z}_n, n \geq 3$, linear multiples of the identity for $O(2)$ and $D_n, n \geq 3$, any linear map for \mathbb{Z}_2 and maps represented by diagonal matrices for the remaining groups.*

Proof. The only linear maps that were not already explicitly calculated in the previous proof are those that commute with rotations. We have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

if and only if either $\sin \theta = 0$ for all $\theta \in S^1$ or $a = d$ and $b = -c$. Hence, the only maps commuting with either $SO(2)$ or $\mathbb{Z}_n, n \geq 3$, are rotations and homotheties and their compositions. □

With the results obtained so far, we are able to describe the Jacobian matrix at the origin for maps equivariant under each of the groups above.

Symmetry group	$Df(0)$	hyperbolic local dynamics
$O(2)$	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \alpha \in \mathbb{R}$	attractor / repellor
$SO(2)$	$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \alpha, \beta \in \mathbb{R}$	attractor / repellor
$D_n, n \geq 3$	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \alpha \in \mathbb{R}$	attractor / repellor
$\mathbb{Z}_n, n \geq 3$	$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \alpha, \beta \in \mathbb{R}$	attractor / repellor
\mathbb{Z}_2	any matrix	saddle / attractor / repellor
$\mathbb{Z}_2(\langle \kappa \rangle)$	$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \alpha, \beta \in \mathbb{R}$	saddle / attractor / repellor
$D_2 = \mathbb{Z}_2 \oplus \mathbb{Z}_2(\langle \kappa \rangle)$	$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \alpha, \beta \in \mathbb{R}$	saddle / attractor / repellor

Table 1: Compact subgroups of $O(2)$ and the admissible forms of the Jacobian at the origin of maps equivariant under the standard action of each group. If in addition the Jacobian at the origin is hyperbolic, then this determines the local stability.

Proposition 3.4. *Let f be a planar map differentiable at the origin. The admissible forms for the Jacobian matrix of f at the origin are those given in Table 1 depending on the symmetry group of f .*

The relation between the stability of the origin and the admissible forms of the Jacobian at that point is well known, but it is not usually clear that it holds for continuous maps that are not necessarily C^1 . The precise hypotheses are stated in the following result, and its proof is given in Appendix A.

Proposition 3.5. *Let $U \subset \mathbb{R}^n$ be an open set containing the origin and let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous map, differentiable at the origin and such that $f(0) = 0$. If all the eigenvalues of $Df(0)$ have norm strictly smaller than one then the origin is locally asymptotically stable. If all the eigenvalues of $Df(0)$ have norm strictly greater than one then the origin is a local repeller.*

4 Symmetric Global Dynamics

In this section we study the global dynamics of a symmetric discrete dynamical system with a unique and hyperbolic fixed point. We establish conditions for the hyperbolic local dynamics in Table 1 to become global dynamics. As can be seen from Table 1, for most symmetry groups, the local dynamics is restricted to either an attractor or a repeller. Saddle points only occur for very small symmetry groups and therefore the study of these points does not depend so much on symmetry issues and requires additional tools. As pointed out before, this will be the object of a separate article.

We first address the dynamics under $SO(2)$ symmetry. This group is special since we can use polar coordinates to reduce this problem to that of dynamics on the line. We then study the dynamics when the groups involved possess an element acting as a reflection (flip). In this case, we make use of the fact that there exists an invariant ray for the dynamics (either of f itself or of f^2) from which results follow. Finally, we show examples of different dynamics for \mathbb{Z}_n -equivariant maps with $n \geq 2$.

4.1 Symmetry Group $SO(2)$

For this group, the dynamics on the plane may be studied via dynamics on the line.

Lemma 4.1. *Let $g : [0, 1) \rightarrow [0, 1)$ be a continuous and injective map such that $\text{Fix}(g) = \{0\}$. The following holds:*

- a) *If 0 is a local attractor for g , then 0 is a global attractor for g .*
- b) *If 0 is a local repeller for g , then 0 is a global repeller for g .*

Proof. Assume g is a local attractor. Since g is a continuous map, g is increasing at 0. Because g is injective, g is increasing in $[0, 1)$. Moreover, $\text{Fix}(g) = \{0\}$ so the graph of g does not cross the diagonal of the first quadrant and one of the following holds:

- i) $g(x) > x$, for all $x \in (0, 1)$;
- ii) $g(x) < x$, for all $x \in (0, 1)$.

Only ii) can happen when 0 is a local attractor. Then $g(x) < x$, for all $x \in [0, 1)$ and 0 is a global attractor for g .

The proof of b) follows in a similar fashion. □

Proposition 4.2. *Let $f \in \text{Emb}(\mathbb{R}^2)$ be an embedding with symmetry group $SO(2)$ such that $\text{Fix}(f) = \{0\}$. The following holds:*

- a) *If 0 is a local attractor, then 0 is a global attractor.*
- b) *If 0 is a local repeller, then 0 is a global repeller.*

Proof. Write f in polar coordinates as $f(\rho, \theta) = (R(\rho, \theta), T(\rho, \theta))$. Since f is $SO(2)$ -equivariant, for all $\alpha \in \mathbb{R}$, $f(\rho, \theta + \alpha) = (R(\rho, \theta), T(\rho, \theta) + \alpha)$. Then

$$f(\rho, \theta - \theta) = (R(\rho, 0), T(\rho, 0)) = (R(\rho, \theta), T(\rho, \theta) - \theta).$$

So $R(\rho, \theta)$ only depends on ρ and $R \in \text{Emb}(\mathbb{R}^+)$.

Moreover, $\text{Fix}(f) = \{0\}$ implies $\text{Fix}(R) = \{0\}$ and the result follows by Lemma 4.1. □

4.2 Symmetry Groups with a flip

For the groups addressed in this subsection, we use the existence of an invariant ray to obtain information concerning the dynamics. Results in this section concern the groups $O(2)$, D_n ($n \geq 2$) and $\mathbb{Z}_2(\langle \kappa \rangle)$.

Lemma 4.3. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a map with symmetry group Γ . If $\kappa \in \Gamma$, then $\text{Fix}\langle\kappa\rangle$ is an f -invariant line. Moreover, $\text{Fix}\langle\kappa\rangle$ contains an f^2 -invariant ray.*

Proof. By Lemma XIII, 2.1 and Theorem XIII, 2.3 in [10], $\text{Fix}\langle\kappa\rangle$ is a vector subspace of dimension one such that $f(\text{Fix}\langle\kappa\rangle) \subseteq \text{Fix}\langle\kappa\rangle$. Let γ denote one of the two half-lines in $\text{Fix}\langle\kappa\rangle$, then γ is an f^2 -invariant ray. \square

Although the next result is only required to hold for planar maps, we present it for \mathbb{R}^n , as the proof is the same.

Let $p \in \mathbb{R}^n$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous map. We denote by $\omega_2(p)$ the ω -limit of p with respect to f^2 , given by

$$\omega_2(p) = \{q \in \mathbb{R}^n : \lim f^{2n_k}(p) = q, \text{ for some sequence } n_k \rightarrow \infty\}.$$

Lemma 4.4. *Let $f \in \text{Emb}(\mathbb{R}^n)$ be such that $f(0) = 0$.*

For $p \in \mathbb{R}^n$,

- a) *if $\omega_2(p) = \{0\}$, then $\omega(p) = \{0\}$;*
- b) *if $\omega_2(p) = \infty$, then $\omega(p) = \infty$.*

Proof. Let $p \in \mathbb{R}^n$.

a) Suppose $\omega_2(p) = \{0\}$ and suppose also that $\omega(p) \neq \{0\}$. Then there exists an $r \neq 0$ such that $r \in \omega(p)$. In that case $r = \lim f^{n_k}(p)$, so there exists a $k_0 \in \mathbb{N}$ such that $\forall k > k_0$, n_k is odd because $\omega_2(p) = \{0\}$. Then, $f(r) = \lim f^{n_k+1}(p)$ with $n_k + 1$ even. So $f(r) \in \omega_2(p)$ hence $f(r) = 0$ with $r \neq 0$, which is impossible because f is an injective map such that $f(0) = 0$.

b) Suppose now $\omega_2(p) = \infty$ and also that $\omega(p) \neq \infty$. Then there exists an $r \in \mathbb{R}^n$ such that $r \in \omega(p)$. In that case $r = \lim f^{n_k}(p)$, so there exists a $k_0 \in \mathbb{N}$ such that $\forall k > k_0$, n_k is odd because $\omega_2(p) = \infty$. Then, $f(r) = \lim f^{n_k+1}(p)$ with $n_k + 1$ even. So $f(r) \in \omega_2(p)$ which is impossible, since $\omega_2(p) = \infty$. \square

Proposition 4.5. *Let $f \in \text{Emb}(\mathbb{R}^2)$ have symmetry group Γ with $\kappa \in \Gamma$, such that $\text{Fix}(f) = \{0\}$. Suppose one of the followings holds:*

- a) *$f \in \text{Emb}^+(\mathbb{R}^2)$ and f does not interchange connected components of $\mathbb{R}^2 \setminus \text{Fix}\langle\kappa\rangle$.*

b) $\text{Fix}(f^2) = \{0\}$.

Then for each $p \in \mathbb{R}^2$ either $\omega(p) = \{0\}$ or $\omega(p) = \infty$.

Proof. Suppose a) holds. By Lemma 4.3, $\mathbb{R}^2 \setminus \text{Fix}\langle\kappa\rangle$ is the disjoint union of two open subsets $U_1, U_2 \subset \mathbb{R}^2$ homeomorphic to \mathbb{R}^2 . Moreover, $f|_{U_i} : U_i \rightarrow U_i$ for $i = 1, 2$ is an orientation preserving embedding without fixed points. Then by Theorem 2.1, $\Omega(f|_{U_i}) = \emptyset$ for $i = 1, 2$ and it then follows that $\Omega(f) \subseteq \text{Fix}\langle\kappa\rangle$.

The subspace $\text{Fix}\langle\kappa\rangle \setminus \{0\}$ is the disjoint union of two subsets homeomorphic to $(0, 1)$. Even if f interchanges these components, f^2 does not. Then applying Lemma 4.1 to the restriction of f^2 to each component, it follows that for $p \in \text{Fix}\langle\kappa\rangle$ either $\omega_2(p) = 0$ or $\omega_2(p) = \infty$. By Lemma 4.4, it follows that for $p \in \text{Fix}\langle\kappa\rangle$ either $\omega(p) = 0$ or $\omega(p) = \infty$.

Let $p \in \mathbb{R}^2 \setminus \text{Fix}\langle\kappa\rangle$. Since $\Omega(f) \subseteq \text{Fix}\langle\kappa\rangle$, we have that $\omega(p) \subseteq \text{Fix}\langle\kappa\rangle$. We show next that $\omega(p) \neq \text{Fix}\langle\kappa\rangle$.

Suppose there is an $r \in \omega(p) \cap (\text{Fix}\langle\kappa\rangle \setminus \{0\})$ and an open neighbourhood K of r such that $0 \notin K$ and $\text{Fix}\langle\kappa\rangle \cap K$ is an embedded segment and $K \setminus \text{Fix}\langle\kappa\rangle$ is the union of two disjoint disks W_1 and W_2 homeomorphic to \mathbb{R}^2 . Suppose without loss of generality that $p \in U_1$, then the positive orbit of p accumulates at r and this positive orbit meets W_1 infinitely many times. Since $r \in \Omega(f) \setminus \{0, \infty\}$ is not a fixed point, taking K (and hence W_1) sufficiently small, there exists an open disk $V \subset W_1$ and a positive integer n , with $n \geq 2$, such that for some $s \in V$, we have that $f^n(s) \in V$, while $V \cap f^\ell(V) = \emptyset$, for $\ell = 1, 2, \dots, n-1$. Then, by Theorem 3.3 in [16], f has a fixed point in V which contradicts the uniqueness of the fixed point. So the orbit of p does not accumulate at $\text{Fix}\langle\kappa\rangle$ and hence either $\omega(p) = 0$ or $\omega(p) = \infty$.

Suppose b) holds. By Lemma 4.3 there exists an f^2 -invariant ray $\gamma \subset \text{Fix}\langle\kappa\rangle$. Moreover, $f^2 \in \text{Emb}^+(\mathbb{R}^2)$ and $\text{Fix}(f^2) = \{0\}$, so by Proposition 2.2 we have that for each $p \in \mathbb{R}^2$ either $\omega_2(p) = \{0\}$ or $\omega_2(p) = \infty$ and therefore, by Lemma 4.4, either $\omega(p) = \{0\}$ or $\omega(p) = \infty$. \square

The next example shows that assumption b) in Proposition 4.5 is necessary in the case where f interchanges connected components of $\mathbb{R}^2 \setminus \text{Fix}\langle\kappa\rangle$.

Example 4.6. Consider the map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$f(x, y) = \left(-ax^3 + (a-1)x, -\frac{y}{2} \right) \quad 0 < a < 1.$$

It is easily checked that f has symmetry group D_2 and verifies (see Figure 1):

1. $f \in Emb^+(\mathbb{R}^2)$ is an orientation-preserving diffeomorphism.
2. $\text{Spec}(f) \cap [0, \infty) = \emptyset$.
3. 0 is a local hyperbolic attractor.
4. $\text{Fix}(f^2) \neq \{0\}$.

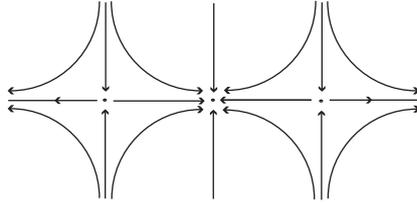


Figure 1: A local attractor which is not a global attractor due to the existence of periodic orbits.

The main results, Theorems 4.7 and 4.9, are obtained as a direct consequence of Proposition 4.5 under additional assumptions.

We say that a map f is *dissipative* if there is an open set V such that $\mathbb{R}^2 \setminus V$ is compact and $\omega(p) \notin V$ for all $p \in \mathbb{R}^2$.

Theorem 4.7. *Let $f \in Emb(\mathbb{R}^2)$ be dissipative with symmetry group Γ with $\kappa \in \Gamma$ such that $\text{Fix}(f) = \{0\}$. Suppose in addition that one of the following holds:*

- a) $f \in Emb^+(\mathbb{R}^2)$ and f does not interchange connected components of $\mathbb{R}^2 \setminus \text{Fix}(\kappa)$.
- b) There exist no 2-periodic orbits.

Then 0 is a global attractor.

Proof. Follows from Proposition 4.5 since being dissipative excludes $\omega(p) = \infty$. □

Corollary 4.8. *Suppose the assumptions of Theorem 4.7 are verified and f is differentiable at 0. If every eigenvalue of $Df(0)$ has norm strictly less than one, then 0 is a global asymptotic attractor.*

Proof. Follows by Theorem 4.7 and Proposition 3.5. \square

Theorem 4.9. *Let $f \in \text{Emb}(\mathbb{R}^2)$ be a map with symmetry group Γ such that $\text{Fix}(f) = \{0\}$. Suppose in addition that one of the following holds:*

- a) $f \in \text{Emb}^+(\mathbb{R}^2)$ and f does not interchange connected components of $\mathbb{R}^2 \setminus \text{Fix}\langle \kappa \rangle$.
- b) *There exist no 2-periodic orbits.*

Then, if 0 is a local repeller, then 0 is a global repeller.

Proof. Follows from Proposition 4.5, since a local repeller excludes $\omega(p) = \{0\}$. \square

Corollary 4.10. *Suppose the assumptions of Theorem 4.9 are verified and f is differentiable at 0. If every eigenvalue of $Df(0)$ has norm strictly greater than one, then 0 is a global asymptotic repeller.*

Proof. Follows from Theorem 4.9 and Proposition 3.5. \square

4.3 Symmetry Group \mathbb{Z}_n , $n \geq 2$

In this case, we can have a local attractor or repeller if $n \geq 3$ and a local attractor, repeller or saddle in the case of \mathbb{Z}_2 . For a local attractor, examples of dissipative \mathbb{Z}_n -equivariant diffeomorphism with a periodic orbit of period n are given in [2] for all $n \geq 2$.

Alarcón [1] proves the existence of a Denjoy map of the circle with symmetry group \mathbb{Z}_n . This is used to construct orientation preserving homeomorphisms of the plane with symmetry group \mathbb{Z}_n having the origin, the unique fixed point, as an asymptotic local attractor. Moreover, for these homeomorphisms there exists a compact and connected subset Δ of \mathbb{R}^2 verifying:

- (i) $0 \in \Delta$;
- (ii) Δ has null Lebesgue measure;

(ii) for all $p \in \mathbb{R}^2$, $\omega(p) \subset \Delta$, thus Δ is a global attractor.

For the \mathbb{Z}_2 -equivariant local saddle, the symmetry properties do not provide any extra information about the global dynamics, since \mathbb{Z}_2 does not contain any reflection. We therefore need additional hypotheses to guarantee the existence of a global saddle. This case will be treated in another article.

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A Proof of Proposition 3.5

In order to prove Proposition 3.5 we show that the hypotheses guarantee that f is either a uniform contraction or expansion in a neighbourhood of the origin. For this we need the following result, similar to a Lemma in Chapter 9 §1 of Hirsch and Smale [14].

Lemma A.1. *Let $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be linear and let $\alpha, \beta \in \mathbb{R}$ satisfy*

$$0 < \alpha < |\lambda| < \beta$$

for all eigenvalues λ of A . Then there exists a basis for \mathbb{R}^n such that, in the corresponding inner product and norm,

$$\alpha \|x\| \leq \|Ax\| \leq \beta \|x\| \quad \forall x \in \mathbb{R}^n. \quad (1)$$

and moreover,

$$|\langle Ax, y \rangle| \leq \beta \|x\| \|y\| \quad \forall x, y \in \mathbb{R}^n. \quad (2)$$

Proof. We will prove that

$$\alpha^2 \|x\|^2 \leq \|Ax\|^2 \leq \beta^2 \|x\|^2 \quad \forall x \in \mathbb{R}^n$$

by transfinite induction on the Jordan canonical form of A . This is done in three steps:

Step 1: The statement is true if A has a single Jordan block, with a single real eigenvalue λ . To prove this we choose the basis where A is represented by

$$A \sim J = \begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 & 0 \\ \varepsilon & \lambda & 0 & \cdots & 0 & 0 \\ 0 & \varepsilon & \lambda & \cdots & 0 & 0 \\ \vdots & & & & \vdots & \\ 0 & 0 & 0 & \cdots & \varepsilon & \lambda \end{pmatrix}$$

with $\varepsilon > 0$ to be chosen later. Writing the coordinates on this basis as $x = (x_1, \dots, x_n)$, we get $Ax = \lambda x + \varepsilon(0, x_1, \dots, x_{n-1})$ and thus

$$\|Ax\|^2 = \langle Ax, Ax \rangle = \lambda^2 \|x\|^2 + 2\lambda\varepsilon(x_1x_2 + \cdots + x_{n-1}x_n) + \varepsilon^2(x_1^2 + x_2^2 + \cdots + x_{n-1}^2).$$

For $x \neq 0$ let

$$f(x) = \frac{x_1x_2 + \cdots + x_{n-1}x_n}{\|x\|^2} \quad g(x) = \frac{x_1^2 + x_2^2 + \cdots + x_{n-1}^2}{\|x\|^2}$$

so

$$\frac{\|Ax\|^2}{\|x\|^2} = \lambda^2 + 2\lambda\varepsilon f(x) + \varepsilon^2 g(x).$$

For $x \in S^{n-1}$, the unit sphere in \mathbb{R}^n , we have $0 \leq g(x) \leq 1$ and by compactness, there are real constants m and M such that $m \leq f(x) \leq M$. These estimates hold for all $x \neq 0 \in \mathbb{R}^n$ since for any $a \neq 0 \in \mathbb{R}$ and any $x \neq 0 \in \mathbb{R}^n$ we have $f(ax) = f(x)$, $g(ax) = g(x)$. Thus for any $x \neq 0$, and for ε sufficiently small we have

$$\alpha^2 \leq \lambda^2 + 2\lambda\varepsilon m \leq \frac{\|Ax\|^2}{\|x\|^2} \leq \lambda^2 + 2\lambda\varepsilon M + \varepsilon^2 \leq \beta^2$$

completing the proof of Step 1.

Step 2: The statement is true if A has two Jordan blocks, corresponding to a pair of complex eigenvalues $a \pm ib$. The proof is similar to Step 1, after noticing that for

$$A \sim J = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

we have for $x = (x_1, x_2)$

$$\|Ax\|^2 = \langle Ax, Ax \rangle = (ax_1 - bx_2)^2 + (bx_1 + ax_2)^2 = (a^2 + b^2) \|x\|^2 .$$

Step 3: If the statement is true for two matrices B_1 and B_2 of dimensions $m_1 \times m_1$ and $m_2 \times m_2$ respectively, then it is true for the block-diagonal matrix

$$A = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} .$$

For this, write $x = (X, Y)$ with coordinates $X \in \mathbb{R}^{m_1}$, $Y \in \mathbb{R}^{m_2}$ where the result holds for the B_j , and get $A(X, Y) = (B_1X, B_2Y)$ so

$$\|A(X, Y)\|^2 = \|(B_1X, B_2Y)\|^2 = \|B_1X\|^2 + \|B_2Y\|^2$$

and the result follows if $\alpha < |\lambda_j| < \beta$ for all the eigenvalues λ_j of B_j . This completes the induction and proves assertion (1).

Finally, (2) follows from (1) and Schwarz inequality:

$$|\langle Ax, y \rangle| \leq \|Ax\| \|y\| \leq \beta \|x\| \|y\| .$$

□

Proof of Proposition 3.5. Write $f(x) = Df(0).x + r(x)$. If f is linear then $r(x) \equiv 0$. Otherwise, using the norm of the previous lemma, since f is differentiable at the origin, we know that

$$\lim_{x \rightarrow 0} \frac{\|r(x)\|}{\|x\|} = 0,$$

that is,

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : \quad \|x\| < \delta \Rightarrow \|r(x)\| < \varepsilon \|x\| .$$

Assume the eigenvalues of $Df(0) = A$ have absolute value smaller than $\beta < 1$. Then, we have, if $\|x\| < \delta$:

$$\begin{aligned} \|f(x)\|^2 &= \langle f(x), f(x) \rangle = \langle Ax + r(x), Ax + r(x) \rangle \\ &= \langle Ax, Ax \rangle + 2\langle Ax, r(x) \rangle + \langle r(x), r(x) \rangle \\ &\leq \beta^2 \|x\|^2 + 2\beta \|x\| \|r(x)\| + \|r(x)\|^2 \\ &\leq \beta^2 \|x\|^2 + 2\varepsilon\beta \|x\|^2 + \varepsilon^2 \|x\|^2 \\ &\leq (\beta^2 + 2\varepsilon\beta + \varepsilon^2) \|x\|^2 \end{aligned}$$

showing that f is a uniform contraction in a ball of radius δ around the origin provided that $\beta^2 + 2\beta\varepsilon + \varepsilon^2 < 1$. Since β is fixed and smaller than one, such an ε always exists.

Assume now that

$$1 < \alpha < |\lambda| < \beta,$$

for all eigenvalues λ of $Df(0) = A$. Then

$$\begin{aligned} \|f(x)\|^2 &= \langle f(x), f(x) \rangle = \langle Ax + r(x), Ax + r(x) \rangle \\ &= \langle Ax, Ax \rangle + 2\langle Ax, r(x) \rangle + \langle r(x), r(x) \rangle \\ &\geq \alpha^2 \|x\|^2 + 2\langle Ax, r(x) \rangle \end{aligned}$$

since $\|r(x)\|^2 > 0$. It then follows that

$$\begin{aligned} \|f(x)\|^2 &\geq \alpha^2 \|x\|^2 + 2\langle Ax, r(x) \rangle \\ &\geq \alpha^2 \|x\|^2 - 2|\langle Ax, r(x) \rangle| \\ &\geq (\alpha^2 - 2\beta\varepsilon) \|x\|^2 , \end{aligned}$$

showing that f is a uniform expansion in a ball of radius δ around the origin provided that

$$\varepsilon < \frac{\alpha^2 - 1}{2\beta}.$$

Since $\alpha^2 - 1 > 0$ such an ε always exists. □

B The equivariant table

This appendix contains a table summarizing the bulk of results in the paper. The first two columns concern the group of symmetries and its action. The third column provides information about the existence of an invariant ray. The fourth and fifth column concern the dynamics by providing the form of the jacobian matrix at the origin and a list of possible local dynamics. Finally, the last column lists hypotheses required to go from local to global dynamics.

EQUIVARIANT TABLE					
Symmetry Group	contains κ ?	# Fix dim 1	$Df(0)$	Hyperbolic Local Stability	Hypothesis for Hyperbolic Global Stability.
$O(2)$	yes	1	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ $\alpha \in \mathbb{R}$	attractor repellor	Emb(\mathbb{R}^2) differentiable and dissipative. Emb(\mathbb{R}^2) differentiable.
$SO(2)$	no	0	$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$ $\alpha, \beta \in \mathbb{R}$	attractor repellor	Emb(\mathbb{R}^2) differentiable. Emb(\mathbb{R}^2) differentiable.
$D_n, n \geq 3,$	yes	1	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ $\alpha \in \mathbb{R}$	attractor repellor	Emb(\mathbb{R}^2) differentiable and dissipative. Emb(\mathbb{R}^2) differentiable.
D_2	yes	2	$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ $\alpha, \beta \in \mathbb{R}$	attractor repellor saddle	Emb(\mathbb{R}^2) differentiable and dissipative. Emb(\mathbb{R}^2) differentiable.
$\mathbb{Z}_2((\kappa))$	yes	1	$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ $\alpha, \beta \in \mathbb{R}$	attractor repellor saddle	Emb(\mathbb{R}^2) differentiable and dissipative. Emb(\mathbb{R}^2) differentiable.
\mathbb{Z}_2	no	0	any matrix	attractor / repellor saddle	Other configurations.
$\mathbb{Z}_n, n \geq 3$	no	0	$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$ $\alpha, \beta \in \mathbb{R}$	attractor / repellor	Other configurations.