Asymptotic stability of robust heteroclinic networks

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Abstract

We provide conditions guaranteeing that certain classes of robust heteroclinic networks are asymptotically stable.

We study the asymptotic stability of ac-networks — robust heteroclinic networks that exist in smooth $\mathbf{Z}_2^n$-equivariant dynamical systems defined in the positive orthant of $\mathbb{R}^n$. Generators of the group $\mathbf{Z}_2^n$ are the transformations that change the sign of one of the spatial coordinates. The ac-network is a union of hyperbolic equilibria and connecting trajectories, where all equilibria belong to the coordinate axes (not more than one equilibrium per axis) with unstable manifolds of dimension one or two. The classification of ac-networks is carried out by describing all possible types of associated graphs.

We prove sufficient conditions for asymptotic stability of ac-networks. The proof is given as a series of theorems and lemmas that are applicable to the ac-networks and to more general types of networks. Finally, we apply these results to discuss the asymptotic stability of several examples of heteroclinic networks.

1 Introduction

A large number of examples of asymptotically stable heteroclinic cycles can be found in the literature. See, for instance, Busse and Heikes [5], Jones and Proctor [11], Guckenheimer and Holmes [14], Hofbauer and So [15], Krupa and Melbourne [17] and [18], Feng [7], Postlethwaite [29] Podvigina [23] and [24], Lohse [19] or Podvigina and Chossat [27] Conditions for asymptotic stability for heteroclinic cycles in particular systems, or in certain classes in low-dimensional systems have been known for a long time, see [5, 11, 14, 15, 7, 29] and [12, 17, 18, 23, 24, 27] for the former and for the latter, respectively.

With heteroclinic networks the situation is completely different — there are just a few instances of networks whose asymptotic stability was proven. See Kirk *et al.* [16] or Afraimovich *et al.* [1]. One can point out two reasons for the rareness of such heteroclinic networks in literature. The first is the rather restrictive necessary conditions, stating that a compact asymptotically stable heteroclinic network contains the unstable manifolds of all its nodes as subsets. In Podvigina *et al.* [25] this was proven for networks where all nodes are equilibria. Applied to networks with one-dimensional connections these conditions rule out their asymptotic stability for a large variety of systems, e.g., equivariant or population dynamics. The second reason is the complexity of the problem. Given a heteroclinic cycle, the derivation of conditions for asymptotic stability involves the construction of a return map around the cycle which typically is a highly non-trivial problem. Existence of various walks along a network that can be followed by nearby trajectories makes the study of the stability of networks much more difficult than that of cycles.

A heteroclinic cycle is a union of nodes and connecting trajectories and a heteroclinic network is a union of heteroclinic cycles. Heteroclinic cycles or networks do not exist in a generic dynamical system, because small perturbations break connections between saddles. However, they may exist in systems where some constraints are imposed and be robust with respect to a constrained perturbation due to the presence of invariant subspaces. Typically, the constraints create flow-invariant subspaces where the connection is of saddle-sink type. For instance, in a Γ-equivariant system the fixed-point subspace of a subgroup of Γ is flow-invariant. In population dynamics modelled by systems in $\mathbb{R}^n_+$, the "extinction subspaces", which are the Cartesian hyperplanes, are flow-invariant. Similarly, such hyperplanes are invariant subspaces for systems on a simplex, a usual state space in evolutionary game theory. For coupled cells or oscillators, existence of flow-invariant subspaces follows from the prescribed patterns of interaction between the components.

Heteroclinic networks have very different levels of complexity. A classification of the least complex networks has been proposed by Krupa and Melbourne [17] (simple), Podvigina and Chossat [26], [27] (pseudo-simple) and Garrido-da-Silva and Castro [12] (quasi-simple).

In this paper we study heteroclinic networks emerging in a smooth $\mathbf{Z}_2^n$-equivariant dynamical system defined in the positive orthant $\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n \ : \ \min_{1\leq k\leq n} x_k \geq 0\}$. The group $\mathbf{Z}_2^n$ is the group generated by the transformations that change the sign of one of the spatial coordinates. The networks that we study, that we call *ac-networks*, are comprised of hyperbolic equilibria and connecting trajectories, where all equilibria belong to the coordinate axes (not more than one equilibrium per axis) with unstable manifold of dimension one or two.

We classify ac-networks by describing the possible structure of the associated graphs. Graphs are often employed for visualisation and/or study of heteroclinic networks. The relations between graphs and networks are discussed in a number of papers (see Ashwin and Postlethwaite [4] and Field [8]), with particular attention being given to the construction of a dynamical system that has a heteroclinic network with a given graph. Similiarly to [4], an ac-network can be realised as a network on a simplex.

We prove sufficient conditions for the asymptotic stability of heteroclinic networks. As usual, we approximate the behaviour of trajectories by local and global maps. The derived estimates for local maps near ξ_j involve the exponents ρ_j. For compact networks global maps are linear with constants that are bounded from above. We show that a network is asymptotically stable if certain products of the exponents ρ_j are larger than one. Hence, our results are applicable not only to the ac-networks, but to more general cases as well. The estimates for local maps depend on the local structure of a considered network near an equilibrium and are of a general form. We also state more cumbersome conditions for asymptotic stability that take into account possible walks that can be followed by a nearby trajectory.

Using these results we obtain conditions for asymptotic stability of two ac-networks and for two networks that do not belong to this class. We also present two examples of ac-networks whose stability was studied in [1].

The present article is organized as follows: the next section provides the necessary background for understanding our results. It is divided into three subsections, each focussing in turn on networks, graphs and stability. In Section 3, we classify ac-networks. Section 4 contains the main result providing sufficient conditions for the asymptotic stability of ac-networks. Section 5 provides some illustrative examples. The final section discusses the realisation of ac-networks and possible directions for continuation of the present study.

2 Background and definitions

2.1 Robust heteroclinic networks

Consider a smooth dynamical system in $\mathbb{R}^n$ defined by

$$\dot{\mathbf{x}} = f(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^n, \tag{1}$$

and denote by $\Phi(\tau, \mathbf{x})$ the flow generated by solutions of the system.

We say the dynamical system is Γ-equivariant, where Γ is a compact Lie group, if

$$f(\gamma.\mathbf{x}) = \gamma.f(\mathbf{x}) \quad \forall\, \mathbf{x} \in \mathbb{R}^n, \ \ \gamma \in \Gamma.$$

Recall that the *isotropy subgroup* of $\mathbf{x}$ is the subgoup of elements $\gamma \in \Gamma$ such that $\gamma\mathbf{x} = \mathbf{x}$. For a subgroup $\Sigma \subset \Gamma$ the *fixed point subspace* of Σ is

$$\mathrm{Fix}(\Sigma) = \{\ \mathbf{x} \in \mathbb{R}^n \ :\ \sigma\mathbf{x} = \mathbf{x} \ \text{ for all } \sigma \in \Sigma\ \}.$$

More detail about equivariant dynamical systems can be found in Golubitsky and Stewart [13].

For a hyperbolic equilibrium ξ of (1) its stable and unstable manifolds are

$$W^u(\xi) = \{\mathbf{x} \in \mathbb{R}^n \ :\ \alpha(\mathbf{x}) = \xi\}, \quad W^s(\xi) = \{\mathbf{x} \in \mathbb{R}^n \ :\ \omega(\mathbf{x}) = \xi\}.$$

By κ_{ij} we denote a *connecting trajectory* from ξ_i to ξ_j. Following the terminology of Ashwin *et al* [3], the full set of connecting trajectories from ξ_i to ξ_j,

$$C_{ij} = W^u(\xi_i) \cap W^s(\xi_j),$$

is called a *connection* from ξ_i to ξ_j and often also denoted by $[\xi_i \to \xi_j]$. If $i = j$ then the connection is called *homoclinic*; otherwise, it is *heteroclinic*.

A *heteroclinic cycle* is a union of a finite number of hyperbolic equilibria, $\xi_1, \ldots, \xi_m$, and heteroclinic connections $C_{j,j+1}$, $1 \le j \le m$, where $\xi_{m+1} = \xi_1$ is assumed. A *heteroclinic network* is a connected union of finitely many heteroclinic cycles. In equivariant systems equilibria and connections that belong to the same group orbit are usually identified.

A heteroclinic cycle is called *robust* if every connection is of saddle-sink type in a flow-invariant subspace, P_j. In equivariant dynamical systems we typically have $P_j := \mathrm{Fix}(\Sigma_j)$ for some subgroup $\Sigma_j \subset \Gamma$. In this case, due to Γ-invariance of fixed-point subspaces, the cycle persists with respect to Γ-equivariant perturbations of f.

The eigenvalues of the Jacobian, $\mathrm{d}f(\xi_j)$, with $\xi_j \in P_{j-1} \cap P_j$, are divided into *radial* (the associated eigenvectors belong to $L_j = P_{j-1} \cap P_j$), *contracting* (eigenvectors belonging $P_{j-1} \ominus L_j$), *expanding* (eigenvectors belonging $P_j \ominus L_j$) and *transverse* (the remaining ones), where $P \ominus L$ denotes the complementary subspace of L in P.

An equilibrium ξ_j in a heteroclinic network might belong to several heteroclinic cycles. In such a case the radial subspace is the same for all cycles, since it is the subspace that is fixed by the isotropy subgroup of ξ_j. By contracting eigenvalues we call the contracting eigenvalues of all cycles that ξ_j belongs to, the associated eigenvectors are called the contracting eigenvectors at ξ_j, the subspace spanned by these eigenvectors is called the contracting subspace at ξ_j. Similar notation is used for the expanding and transverse eigenvalues, eigenvectors or subspaces. By transverse eigenvalues we understand the eigenvalues that are not radial, contracting or expanding for any cycle through ξ_j.

A heteroclinic network has subsets that are connected unions of heteroclinic connections and equilibria and which are different from heteroclinic cycles. An invariant set of the system (1) which is called Δ-clique is defined below. The name "Δ-clique" was introduced in [3] to denote an element of a graph. The relation between these two kinds of Δ-cliques is discussed in the following subsection.

Definition 2.1. *By a Δ-clique, Δ_{ijk}, we denote the union of equilibria and connecting trajectories*

$$\Delta_{ijk} = \{\xi_i, \xi_j, \xi_k\} \cup \kappa_{ij} \cup \kappa_{jk} \cup \left(\cup_{\theta \in [0,1)} \kappa_{ik}(\theta) \right), \tag{2}$$

such that $\kappa_{ik}(\theta) = \{ \Phi(\tau, x_{ik}(\theta)) \, : \, \tau \in \mathbb{R} \}$, where $x_{ik}(\theta)$ is a continuous function of θ, $\kappa_{ik}(\theta_1) \neq \kappa_{ik}(\theta_2)$ for any $\theta_1 \neq \theta_2$ and

$$\lim_{\theta \to 1} \kappa_{ik}(\theta) = \kappa_{ij} \cup \xi_j \cup \kappa_{jk}.$$

2.2 Graphs and networks

There is a close relation between heteroclinic networks and directed graphs, which is discussed, e.g., by Ashwin and Postlethwaite [4], Field [8] or Podvigina and Lohse [28]. Given a heteroclinic network, there is an associated graph such that the vertices (or nodes) of the graph correspond to the equilibria of the network and an edge from ξ_i to ξ_j corresponds to a full set of connections C_{ij}. (This supports the frequent choice of the term *node* instead of equilibrium in the context of heteroclinic cycles and networks.) Using the terminology from graph theory[1], we call a *walk* the union of vertices $\xi_1, ..., \xi_J$ and connections $C_{j,j+1}$, $1 \leq j \leq J-1$. The respective part of the graph is also called a *walk*. A walk where all connections are distinct is called a *path*. A closed path corresponds to a heteroclinic cycle, which is a subset of the network.

In agreement with [3], a nontransitive triangle within a graph is called Δ-*clique*: it is the union of three vertices v_i, v_j and v_k and edges $[v_i \to v_k]$, $[v_i \to v_j]$ and $[v_j \to v_k]$. In this case, in the corresponding heteroclinic network the respective equilibria, ξ_i, ξ_j and ξ_k, are connected by the trajectories κ_{ik}, κ_{ij} and κ_{ik}. If a set Δ_{ijk} (see Definition 2.1) is a subset of a network, then the respective part of the graph is a Δ-clique. Hence, we use the term Δ-clique to denote such a set. Note that a network can possibly have a subset that is represented by a Δ-clique on the graph, but it is not a Δ-clique according to Definition 2.1. However, this is not the case for the networks considered in this paper. We label equilibria and connections in a Δ-clique as follows (the notation is shown in Figure 1):

Definition 2.2. *Consider a Δ-clique Δ_{ijk}, which is an invariant set of system (1). We call the connections $[\xi_i \to \xi_j]$, $[\xi_j \to \xi_k]$ and $[\xi_i \to \xi_k]$, the* f-long, s-long *and* short *connections, respectively. We call the equilibria ξ_i, ξ_j and ξ_k, the* b-point, m-point *and* e-point, *respectively. The eigenvectors of $\mathrm{d}f(\xi_j)$ tangent to the connecting trajectories κ_{ij} and κ_{jk} are called* f-long *and* s-long vectors. *(Here there letters "f" and "s" stand for the first and second; the letters "b", "m" and "e" stand for beginning, middle and end, respectively.*

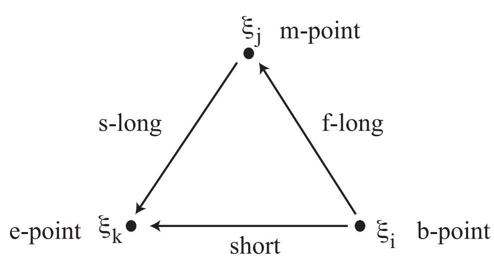


Figure 1: The Δ-clique Δ_{ijk} with its short and long connections.

2.3 Asymptotic stability

Below we recall two definitions of asymptotic stability that can be found in literature. In what follows $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$ are invariant sets of the dynamical system.

Definition 2.3. *A set X is* Lyapunov stable *if for any neighbournood U of X, there exists a neighbourhood V of X such that*

$$\Phi(\tau, \mathbf{x}) \in U \text{ for all } \mathbf{x} \in V \text{ and } \tau > 0.$$

[1]We have used Foulds [10] as a reference for graph theory. There are many other good sources.

Definition 2.4 (Meiss [20]). *A set X is* asymptotically stable *if it is Lyapunov stable and in addition the neighbourhood V can be chosen such that*

$$\lim_{\tau\to\infty} d(\Phi(\tau,\mathbf{x}),X)\to 0 \text{ for all } \mathbf{x}\in V.$$

Definition 2.5 (Oyama *et.al.* [21]). *A set X is* asymptotically stable *if it is Lyapunov stable and in addition the neighbournood V can be chosen such that*

$$\cup_{\mathbf{x}\in V}\omega(\mathbf{x})\subseteq X.$$

We refer to asymptotic stability according to Definition 2.4 as a.s. and according to Definition 2.5 as a.s.II. In this paper we adopt Definition 2.4 as the definition of asymptotic stability.

Remark 2.6. *For closed sets, such as equilibria or periodic orbits, the Definitions 2.4 and 2.5 are equivalent. However, this is not always the case. In Figure 2b we show an example of a heteroclinic network, which is not a.s.II, but is a.s. if the eigenvalues at equilibria satify the conditions stated in in Section 5 (see Example 3). Note also the following relations between these two definitions of asymptotic stability:*

$$(X \text{ is a.s.II})\Rightarrow(X \text{ is a.s.}),\qquad (X \text{ is a.s.})\Rightarrow(\overline{X} \text{ is a.s.II}),\qquad (\overline{X} \text{ is a.s.II})\Rightarrow(X \text{ is a.s.})$$

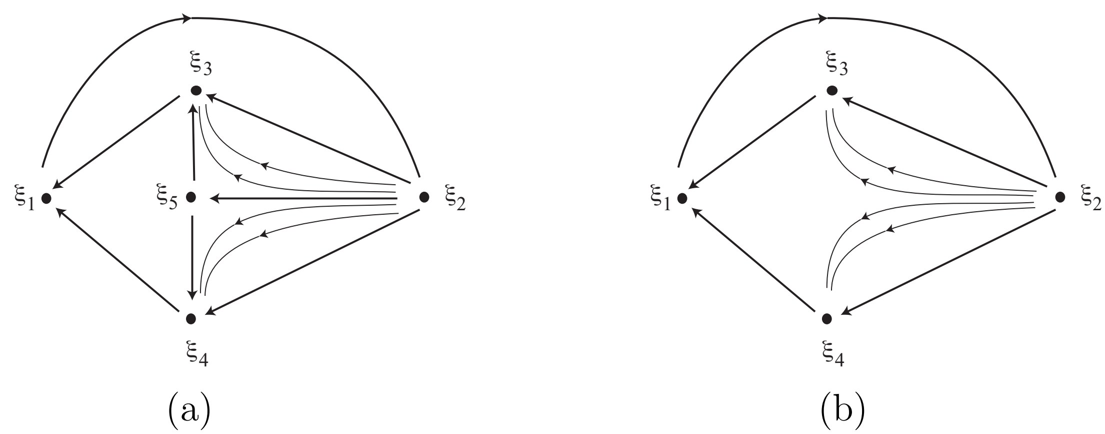


Figure 2: An a.s.II heteroclinic network (a) and its subset (b), which is a.s., but not a.s.II.

The next definition is a generalization of Definition 1.3 of Field [8], where a heteroclinic network is defined as *clean* if it is compact and equal to the union of the unstable manifolds of its nodes.

Definition 2.7. *(Adapted from [8].) A compact invariant set Y is* clean *if each of its invariant subsets X satisfies*

$$W^u(X)\subseteq Y.$$

In what follows we denote by $N_\varepsilon(Y)$ the set $N_\varepsilon(Y)=\{\mathbf{x}: d(\mathbf{x},Y)<\varepsilon\}$ for a closed set Y.

Theorem 2.8. *If a compact invariant set is asymptotically stable then it is clean.*

Proof. We show that if a compact invariant set Y is not clean then it is not Lyapunov stable.

If Y is not clean, there exists an invariant set $X \subset Y$ and $\mathbf{x} \in W^u(X)$ such that $\mathbf{x} \notin Y$. Since Y is a compact set, we know that $h = d(\mathbf{x}, Y) > 0$. Let $U = N_{h/2}(Y)$ and for any $V \supset Y$ take $\delta > 0$ such that $N_\delta(X) \subset V$. Since $\mathbf{x} \in W^u(X)$, there exists at least one $\mathbf{x}_\delta \in \Phi(\tau, \mathbf{x}) \cap N_\delta(X) \subset V$. It is clear that solutions through $\mathbf{x}$ and $\mathbf{x}_\delta$ coincide so that there exists τ_δ such that $\mathbf{x} = \Phi(\tau_\delta, \mathbf{x}_\delta) \notin U$. □

Corollary 2.9. *Suppose that the system (1) has a compact heteroclinic network, which is a union of nodes (they can be any invariant sets) and heteroclinic connections. If the network is a.s. then it is clean.*

3 Classification of ac-networks

In this section we define ac-networks and classify them by describing all possible types of associated graphs. The letters "ac" stand for *axial* and *clean*.

Definition 3.1. *An* ac-network *is a robust heteroclinic network $Y \subset \mathbb{R}^n_+$ in a $\mathbf{Z}_2^n$-equivariant system* (1)*, such that all the equilibria are hyperbolic and lie on coordinate axes, there is no more that one equilibrium per axis, the dimension of the unstable manifold of any equilibrium is either one or two, and the network is clean.*

For every equilibrium ξ_j in an ac-network, the $\mathbf{Z}_2^n$-equivariance of the vector field ensures that the eigenvalues of $\mathrm{d}f(\xi_j)$ are real and the eigenvectors are aligned with the Cartesian basis vectors. By application of the $\mathbf{Z}_2^n$ symmetries, the network may be extended from $\mathbb{R}^n_+$ to $\mathbb{R}^n$.

In the following two lemmas Y denotes an ac-network.

Lemma 3.2. *An ac-network does not have subsets that are homoclinic cycles or heteroclinic cycles with two equilibria.*

Proof. By Definition 3.1 any connection $C_{ij} \subset Y$ is robust, i.e., it belongs to a flow-invariant subspace where ξ_i is unstable and ξ_j is a sink. Since an unstable equilibrium is not a sink, an ac-network does not have homoclinic connections.

Suppose there exists a heteroclinic cycle with two equilibria ξ_i and ξ_j and connections C_{ij} and C_{ji}. Let S_{ij} be the flow-invariant subspace containing C_{ij}. In the coordinate plane $P_{ij} =< \mathbf{e}_i, \mathbf{e}_j > \subset S_{ij}$ the trajectory $\kappa_{ij} = P_{ij} \cap C_{ij}$ connects ξ_i to ξ_j. Similarly, there exists $\kappa_{ji} = P_{ij} \cap C_{ji}$ connecting ξ_j to ξ_i. Since ξ_i and ξ_j are hyperbolic equilibria, the connecting trajectories κ_{ij} and κ_{ji} cannot exist robustly and simultaneously in P_{ij}. □

Lemma 3.3. *An ac-network is a union of equilibria, one-dimensional connections and* Δ*-cliques.*

Proof. Definition 3.1 implies that all connections are 1- or 2-dimensional. Hence, due to the compactness of the network and the definition of Δ-clique it suffices to show that the closure of any 2-dimensional connection is a Δ-clique.

Let $\dim(W^u(\xi_i)) = 2$ and the expanding eigenvectors be $\mathbf{e}_j \subset L_j$ and $\mathbf{e}_k \subset L_k$. The planes $P_{ij} =< \mathbf{e}_i, \mathbf{e}_j >$ and $P_{ik} =< \mathbf{e}_i, \mathbf{e}_k >$ are flow-invariant, which implies the existence of the steady states $\xi_j = \omega(W^u(\xi_i) \cap P_{ij})$ and $\xi_k = \omega(W^u(\xi_i) \cap P_{ik})$ and the connecting trajectories $\kappa_{ij} \subset P_{ij}$ and $\kappa_{ik} \subset P_{ik}$ (see Figure 3).

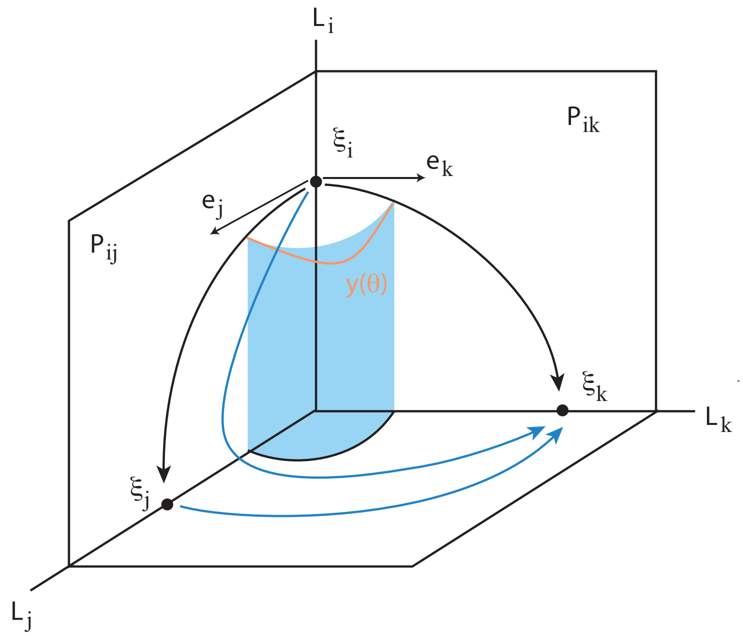


Figure 3: Conventions in the proof of Lemma 3.3 (case (i)).

Consider the dynamics within the invariant subspace $S_{ijk} =< \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k >$. Define a curve $\mathbf{y}(\theta)$, $\theta \in [0,1]$ as the intersection of $W^u(\xi_i)$ with the surface described by $(x_j, x_k) = (\delta \sin(\pi\theta/2), \delta \cos(\pi\theta/2))$, where δ is a small number. The curve can be written as $\mathbf{y}(\theta) = (x_i(\theta), \delta \sin(\pi\theta/2), \delta(\cos \pi\theta/2))$. A trajectory through $\mathbf{y}(\theta)$ satisfies

$$\alpha(\Phi(\tau, \mathbf{y}(\theta))) = \xi_i \text{ and } \omega(\Phi(\tau, \mathbf{y}(\theta))) = \xi_j \text{ or } \xi_k.$$

Because the equilibria of an ac-network belong to coordinate axes, there can be no θ_1 such that $\omega(\Phi(\tau, \mathbf{y}(\theta_1))) = \xi' \neq \xi_j, \xi_k$. Then one of the following two alternatives holds:

(i) $\omega(\Phi(\tau, \mathbf{y}(1))) = \xi_j$ and for $0 \leq \theta < 1$ we have $\omega(\Phi(\tau, \mathbf{y}(\theta))) = \xi_k$;

(ii) $\omega(\Phi(\tau, \mathbf{y}(0))) = \xi_k$ and for $0 < \theta \leq 1$ we have $\omega(\Phi(\tau, \mathbf{y}(\theta))) = \xi_j$.

In case (i) due to the compactness of Y and the smoothness of f

$$\lim_{\theta \to 1} \Phi(\tau, \mathbf{y}(\theta)) = (\xi_j \cup \kappa_{ij} \cup \kappa_{jk}) \subset Y.$$

Therefore, the Δ-clique Δ_{ijk} is a subset of Y. In case (ii) there exists a Δ-clique Δ_{ikj} which is a subset of Y. □

Corollary 3.4. *If $C_{ij}, C_{ik} \in Y$ then either $C_{jk} \subset Y$ or $C_{kj} \subset Y$. Furthermore, if C_{ik} is not an f-long connection then $C_{jk} \subset Y$ hence $\Delta_{ijk} \subset Y$.*

We claim that an ac-network always contains at least one cycle without f-long connections. Indeed, if one starts from any node, either it has only one outgoing connection (hence not f-long), or it has one short and one f-long. Follow the short (or only) connection to the next node and repeat the reasoning there. Since the set of nodes is finite, this eventually closes into a cycle. See also Remark 3.6. The claim implies that the following theorem describes all possible ac-networks.

Theorem 3.5. *Let $Y \in \mathbb{R}^n$ be an ac-network and let $X = (\cup_{1\le j\le J}\xi_j) \cup (\cup_{1\le j\le J} C_{j,j+1})$ be a heteroclinic cycle in Y. Assume that X has no f-long connections. Then either*

(i) there is at least one connection in Y between equilibria in X which is not contained in X. Then J is odd and Y is a union of equilibria ξ_j, $1 \le j \le J$, connected by Δ-cliques.

(ii) all connections in Y between equilibria in X are contained in X. Then the equilibria $\xi \in Y$ can be grouped into disjoint sets

$$\mathcal{X}_1 = \{\xi_{1,0}, ..., \xi_{1,m_1}\}, ..., \mathcal{X}_J = \{\xi_{J,0}, ..., \xi_{J,m_J}\}$$

where $\xi_{j,0} = \xi_j \in X$. The network Y always has the connections:

$$[\xi_{j,s} \to \xi_{j+1,0}], \quad 0 \le s \le m_j, \quad [\xi_{j,s-1} \to \xi_{j,s}], \quad 1 \le s \le m_j,$$

where $j = 1, \ldots, J$. The network Y may also have one of the following connections

$$[\xi_{j,m_j} \to \xi_{j+1,1}], \qquad [\xi_{j,m_j} \to \xi_{j+2,0}], \qquad [\xi_{j,m_j} \to \xi_{j,0}].$$

It may also have the connection $[\xi_{j-1,m_{j-1}} \to \xi_{j,m_j}]$, if the connection $[\xi_{j,m_j} \to \xi_{j,0}]$ exists.

Proof. (i) Let $C_{1s} = [\xi_1 \to \xi_s] \subset Y$, $C_{1s} \not\subset X$, be the connection in the statement of the theorem. The equilibrium ξ_1 has two outgoing connections, $C_{12} \subset X$ and C_{1s}. Therefore by Corollary 3.4, since C_{12} is not f-long, there exists a further connection $C_{s2} \subset Y$ and the corresponding Δ-clique, Δ_{1s2}.

Hence, the equilibrium ξ_s has two outgoing connections, $C_{s,s+1}$ and C_{s2}. By the same arguments as above, this implies existence of the connection $C_{2,s+1}$, which in turn implies existence of the connection $C_{s+1,3}$. Proceeding like this, we prove existence of the connections $C_{3,s+2}$, $C_{4,s+3}$, and so on up to $[\xi_s \to \xi_{2s-1}]$. However, ξ_s already has two outgoing connections, $[\xi_s \to \xi_{s+1}]$ and $[\xi_s \to \xi_2]$. An equilibrium in an ac-network has one or two outgoing connections, therefore $2s-1 \pmod J = 2$, which implies that $2s-3 = J$.

Note, that we have identified other J connections $[\xi_i \to \xi_{i+(J+1)/2}]$, each of them is an f-long connection for one Δ-clique and s-long connection for another. A connection $C_{j,j+1} \subset X$ is the short connection of two Δ-cliques. Figure 10 (left panel) shows examples of this type of network.

(ii) Next we consider the case when Y does not have connections between ξ_j, $1 \le j \le J$, other than $[\xi_j \to \xi_{j+1}]$. To each equilibrium ξ_j in X we associate a class $\mathcal{X}_j$.

The grouping of the equilibria in Y into the sets $\mathcal{X}_j$ proceeds stepwise: after assigning each equilibrium ξ_j in X to a group $\mathcal{X}_j$ (step 1), subsequent steps distribute the equilibria in $Y \backslash X$ by each $\mathcal{X}_j$, or move them from one group to another, in such a way that the statement of the theorem is satisfied. In the construction we use an auxiliary network $Z \subseteq Y$, which is modified on each step by adding equilibria (in parallel with $\mathcal{X}$) and/or connections. At Step 1, $Z = X$ and, because the number of equilibria in Y is finite, this process is finite and at the last step $Z = Y$.

It follows from the assumptions of case (ii) of the theorem that there is at least one equilibrium $\xi \in Y$ but $\xi \notin X$, and a connection from some $\xi_j \in X$ to ξ so that Step 2 is always taken. There are at least two connections of the compulsory type listed in the statement of the theorem. Step 3 accounts for possible connections between the equilibria already classified in Step 2. Steps 4 and 5 are required only if there are $\xi_{\cdot,1}$ with connections to other equilibria not already in Z.

The number of connections between equilibria is restricted by the hypotheses and by Corollary 3.4.

Step 1: initiation. We initiate the sets $\mathcal{X}_j = \{\xi_{j,0}\}$, $1 \leq j \leq J$, where $[\xi_{j,0} = \xi_j]$ are the equilibria of X and define $Z = X$.

Step 2: compulsory connections for $s \leq 1$. Some of $\xi_{j,0} = \xi_j$ have outgoing connections other than $[\xi_j \to \xi_{j+1}]$, i.e., connections $C_{jk} \subset Y$ where $k > J$. For each j there is at most one such connection due to the definition of ac-networks. Existence of a connection C_{jk} implies that there are no connections C_{ik} from other $\xi_i \in X$, $i \neq j$, to ξ_k.[2] So, if a connection $[\xi_j \to \xi_k]$ exists, we assign ξ_k to $\mathcal{X}_j$: $\xi_{j,1} = \xi_k$. We include in Z all such equilibria, the connections $[\xi_j \to \xi_k]$ and $[\xi_k \to \xi_{j+1}]$. The latter exists due to Corollary 3.4 since the connection $[\xi_j \to \xi_{j+1}]$ is not an f-long connection of a Δ-clique. Note, that at the end of this step all outgoing connections from $\xi_{j,0}$ belong to Z.

Step 3: optional connections for $m_j = 1$. We add to Z connections $[\xi_{j,1} \to \xi_i]$, such that $\xi_i \in Z$ and the connection was not included in Z at the previous step. These are of two types: either from $\xi_{j,1}$ back to an equilibrium in X or from $\xi_{j,1}$ to an equilibrium $\xi_{s,1}$.

By Lemma 3.3, if a connection from $\xi_{j,1}$ back to an equilibrium in X exists, it must be $[\xi_{j,1} \to \xi_{j+2,0}]$. In this case $m_j = 1$, since $\xi_{j,1}$ already has two outgoing connections, $[\xi_{j,1} \to \xi_{j+1,0}]$ (identified in Step 2) and $[\xi_{j,1} \to \xi_{j+2,0}]$ as in Figure 4 (a).

Lemma 3.3 gives two possibilities for $[\xi_{j,1} \to \xi_{s,1}]$: they are $[\xi_{j,1} \to \xi_{j+1,1}]$ or $[\xi_{j,1} \to \xi_{j-1,1}]$. (There are no other possibilities due to Corollary 3.4 and absence in Y of additional connections between $\xi_j \in X$.) In the former case (Figure 4 (b)) we have $m_j = 1$. In the latter case (Figure 4 (c)) we move $\xi_{j-1,1}$ from $\mathcal{X}_{j-1}$ to $\mathcal{X}_j$ where we set $\xi_{j,2} = \xi_{j-1,1}$ (Figure 4

[2]Since $C_{j,j+1}$ is not an f-long connection for any Δ-clique, by Corollary 3.4, existence of the connections C_{jk} and $C_{j,j+1}$ implies the existence of the connection $C_{k,j+1}$. Similarly, existence of the connections C_{ik} and $C_{i,i+1}$ implies the existence of $C_{k,i+1}$. Hence, the corollary implies existence of a connection between C_{j+1} and C_{i+1}, which contradicts the statement of the theorem unless $i = j + 1$ or $j = i + 1$. But in the case, e.g. $i = j + 1$, there exist both connections $C_{j+1,k}$ and $C_{k,j+1}$, which is not possible in an ac-network.

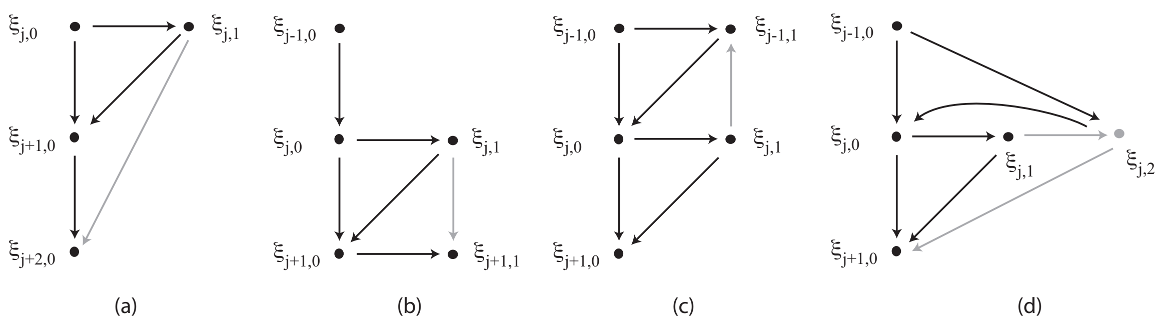


Figure 4: Possible cases for optional connections in Step 3 of the construction of the network Z, new optional connections and consequent compulsory connections shown in grey. The structure in (c) is moved to that in (d) generating another compulsory connection. In all these examples the set $\mathcal{X}_j$ is complete at this step.

(d)). Then there are two outgoing connections from $\xi_{j,1}$ and by Corollary 3.4 there is a connection between $\xi_{j,2}$ and $\xi_{j+1,0}$. The connection $[\xi_{j+1,0} \to \xi_{j,2}]$ is not compatible with the existing $[\xi_{j-1,0} \to \xi_{j,2}]$, as shown in Step 2. Hence, the second connection is $[\xi_{j,2} \to \xi_{j+1,0}]$, that we add to Z, and $m_j = 2$, $m_{j-1} = 0$.

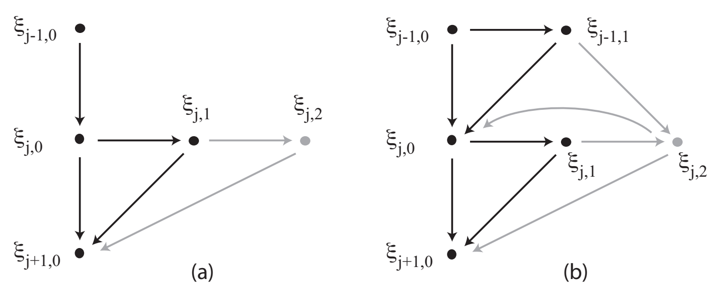


Figure 5: Compulsory (a) and optional (b) connections in Step 4 of the construction of the network Z, new connections and consequent compulsory connections shown in grey. In case (b) the set $\mathcal{X}_j$ is complete at this step.

Step 4: compulsory connections for $s = 2$. Let $\xi_{j,1}$ have an outgoing connection $[\xi_{j,1} \to \xi_i]$ where $\xi_i \notin Z$ and assume there are no other connections $[\xi_{k,1} \to \xi_i]$. Then we add ξ_i to $\mathcal{X}_j$: $\xi_{j,2} = \xi_i$. As well, we add to Z the equilibrium $\xi_{j,2} = \xi_i$ and the connections $[\xi_{j,1} \to \xi_i]$ and $[\xi_i \to \xi_{j+1}]$ (Figure 5 (a)). If some $\xi_i \in Y$ has several such incoming connections from the equilibria $\xi_{j,1}$, then there are two such connections and they are $[\xi_{j-1,1} \to \xi_i]$ and $[\xi_{j,1} \to \xi_i]$. (This can be shown by arguments similar to the ones applied at Step 2.) Then we add ξ_i to $\mathcal{X}_j$: $\xi_{j,2} = \xi_i$ (Figure 5 (b)). We add to Z the equilibrium ξ_i and the connections $[\xi_{j-1,1} \to \xi_i]$, $[\xi_i \to \xi_j]$, $[\xi_{j,1} \to \xi_i]$ and $[\xi_i \to \xi_{j+1}]$. Note that at the end of this step all outgoing connections from $\xi_{j,1}$ that are in Y belong to Z.

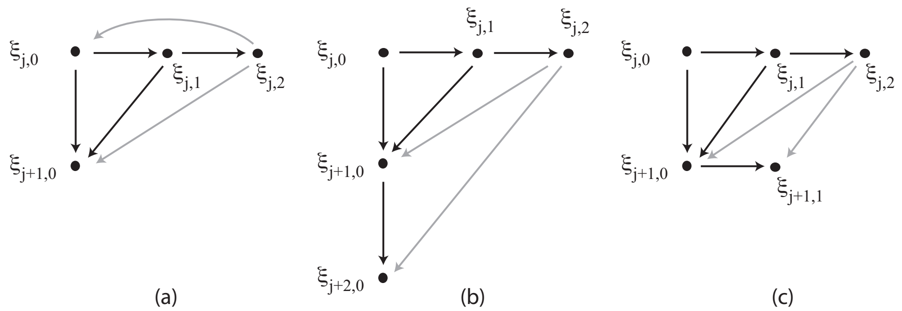


Figure 6: Some possible cases for optional connections in Step 5 of the construction of the network Z, new connections and consequent compulsory connections shown in grey. In all these examples the set $\mathcal{X}_j$ is complete at this step.

Step 5: optional connections for $m_j = 2$. We add to Z outgoing connections from $\xi_{j,2}$ to other equilibria that are already in Z, namely $\xi_{s,0}$, $\xi_{s,1}$ and $\xi_{s,2}$. Lemma 3.3 implies that the only (mutually exclusive) possibilities are: $[\xi_{j,2} \to \xi_{j,0}]$, $[\xi_{j,2} \to \xi_{j+2,0}]$, $[\xi_{j,2} \to \xi_{j+1,1}]$, $[\xi_{j,2} \to \xi_{j-1,1}]$ or $[\xi_{j,2} \to \xi_{j-1,2}]$ (Figure 6 (a)–(c) and Figure 7 (a),(c)). If a connection $[\xi_{j,2} \to \xi_{j-1,1}]$ exists (Figure 7 (a)), then we move $\xi_{j-1,1}$ from $\mathcal{X}_{j-1}$ to $\mathcal{X}_j$ where we set $\xi_{j,3} = \xi_{j-1,1}$ (Figure 7 (b)). If a connection $[\xi_{j,2} \to \xi_{j-1,2}]$ exists (Figure 7 (c)), then we move $\xi_{j-1,2}$ from $\mathcal{X}_{j-1}$ to $\mathcal{X}_j$ where we set $\xi_{j,3} = \xi_{j-1,2}$ (Figure 7 (d)). If one of the first three connections exists then $m_j = 2$, otherwise $m_j = 3$.

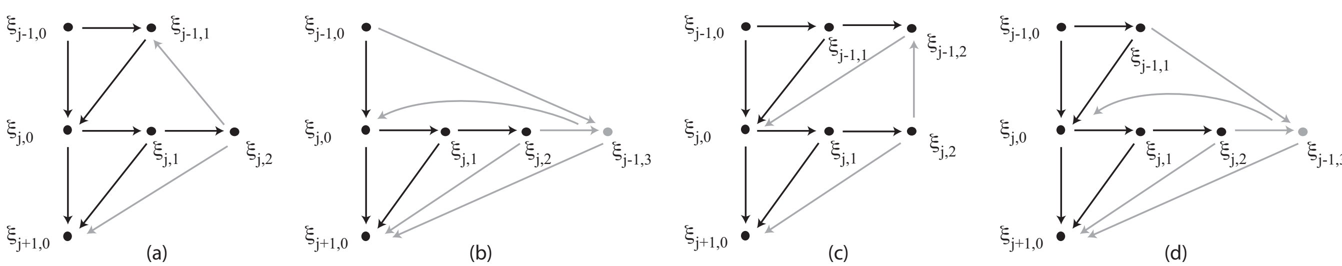


Figure 7: Some possible cases for optional connections in Step 5 of the construction of the network Z, where some ξ_i is moved from $\mathcal{X}_{j-1}$ to $\mathcal{X}_j$. New connections and consequent compulsory connections shown in grey.The structure in (a) is moved to that in (b) and contains a compulsory connection. Similarly, (c) is moved to (d) In all these examples the set $\mathcal{X}_j$ is complete at this step.

Steps 4 and 5 are repeated with larger s until all equilibria of Y become equilibria of Z. Since Y has a finite number of equilibria the process will be terminated after a finite number of steps.

Therefore, we have obtained the network Z that coincides with Y. By construction, the network Z has the structure stated in the theorem. □

The decomposition of the sets of nodes in Theorem 3.5 implies that an ac-network Y contains at most two heteroclinic cycles without f-long connections.

Remark 3.6. *The structure of the graphs associated with ac-networks implies that for a given ac-network Y, the cycle X without f-long connections of Theorem 3.5 is unique, except for the case when Y is comprised of equilibria $\xi_{j,0}$ and $\xi_{j,1}$ and the connections*

$$[\xi_{j,0} \to \xi_{j+1,0}], \quad [\xi_{j,0} \to \xi_{j,1}], \quad [\xi_{j,1} \to \xi_{j+1,0}], \quad [\xi_{j,1} \to \xi_{j+1,1}],$$

where $1 \le j \le J$. For such Y the cycle $[\xi_{1,1} \to \xi_{2,1} \to ... \to \xi_{J,1} \to \xi_{1,1}]$ also has no f-long connections. See Example 5, right panel.

The sets $\mathcal{X}_j$ in Theorem 3.5 depend on the cycle X, so when the network contains two cycles without f-long connections this yields two different groupings.

4 Stability of ac-networks

In this section we derive sufficient conditions for asymptotic stability of an ac-network Y, comprised of equilibria ξ_j, $1 \le j \le J$, and heteroclinic connections C_{ij}. According to Lemma 3.3, the connections are either one or two-dimensional and any two-dimensional connection is accompanied by two one-dimensional connections that belong to a Δ-clique.

For $\varepsilon > 0$, we say that a trajectory $\Phi(\tau, x)$ passes *ε-near* ξ_j if $\Phi(\tau_0, x) \in N_\varepsilon(\xi_j)$ for some $\tau_0 > 0$. We say that a trajectory $\Phi(\tau, x)$ *ε-follows* a connection C_{ij} if for some $\tau_2 > \tau_1 > 0$ we have $\Phi(\tau_1, x) \in N_\varepsilon(\xi_i)$, $\Phi(\tau_2, x) \in N_\varepsilon(\xi_j)$, $\Phi(\tau, x) \in N_\varepsilon(\kappa_{ij})$, $\tau \in (\tau_1, \tau_2)$ and, moreover, $\Phi(\tau, x) \notin N_\varepsilon(\xi_k)$ for any $\tau \in (\tau_1, \tau_2)$ $k \ne i, j$. When ε is clear from the context, we just say the trajectory passes near ξ_j, or follows κ_{ij}.

To study stability, the behaviour of a trajectory near a heteroclinic cycle or network is approximated by local and global maps. Namely, near an equilibrium ξ_j the local map $\mathbf{x}_j^{out} = \phi_j \mathbf{x}_j^{in}$ relates the point $\mathbf{x}_j^{out}$ where a trajectory $\Phi(\tau, x)$ exits the box $N_{\tilde\delta}(\xi_j)$ with the point $\mathbf{x}_j^{in}$ where it enters. Outside the boxes, a connecting diffeomorphism ψ_{ij} relates the point $\mathbf{x}_j^{in}$ where at $t = t_j^{in}$ a trajectory enters $N_{\tilde\delta}(\xi_j)$ with the point $\mathbf{x}_i^{out}$ where at $t = t_i^{out}$ it exits $N_{\tilde\delta}(\xi_i)$, assuming that for $t \in [t_i^{out}, t_j^{in}]$ the trajectory does not pass near any other equilibria. We consider trajectories in the neighbourhoods $N_\delta(Y)$, for arbitrarily small δ and use the following strategy: we fix $\tilde\delta > 0$ assuming $\tilde\delta$ to be small, but $\delta \ll \tilde\delta$.

In what follows we prove results applicable not only to ac-networks, but also in a more general setting, e.g., for compact networks, if estimates for local maps are known. To obtain estimates for ac-networks, instead of the distance $d(\cdot,\cdot)$ based on the maximum norm, it is convenient to use the complementary distance, $\mathfrak{d}(\cdot,\cdot)$, that we define below.

Definition 4.1. *Consider a robust heteroclinic network $Y \subset \mathbb{R}^n$. For any connection $C_{ij} \subset Y$ there exists a flow-invariant subspace $S_{ij} \supset C_{ij}$. The* complementary distance *between a point $x \in \mathbb{R}^n$ and Y is*

$$\mathfrak{d}(x, Y) = \min d(x, S_{ij}),$$

where the minimum is taken over all connections C_{ij} in Y.

Note that $d(x, S_{ij}) = \Pi_{ij}^\perp(x)$, where $\Pi_{ij}^\perp(x)$ denotes the orthogonal projection into $\mathbb{R}^n \ominus S_{ij}$, the orthogonal complement to S_{ij} in $\mathbb{R}^n$. This is why we call it *complementary distance.*

4.1 Estimates for local maps

Choose $\tilde{\delta}$ sufficiently small so that near ξ_j the behaviour of trajectories can be approximated by

$$\dot{x}_k = \lambda_{jk} x_k, \tag{3}$$

where λ_{jk} are eigenvalues of $\mathrm{d}f(\xi_j)$ and x_k are coordinates in the local basis comprised of eigenvectors of $\mathrm{d}f(\xi_j)$. For the derivation of this approximation and its validity see, e.g. [2, 30].

For a heteroclinic cycle X comprised of one-dimensional connections the expression for a local map was derived in [17, 18]; we give here a brief description. Denote by $(u, v, w, \mathbf{z})$ the local coordinates in the basis comprised of radial, contracting, expanding and transverse eigenvectors, respectively. The incoming trajectory κ_{ij} crosses the boundary of $N_{\tilde{\delta}}(\xi_j)$ at a point $(u, v, w, \mathbf{z}) = (u_0, v_0, 0, 0)$, the outgoing trajectory κ_{jk} crosses the boundary at $(u, v, w, \mathbf{z}) = (u_1, 0, w_1, 0)$. A trajectory close to the connection κ_{ij} enters and exits $N_{\tilde{\delta}}(\xi_j)$ at

$$\mathbf{x}^{in} = (u_0 + u^{in}, v_0 + v^{in}, w^{in}, \mathbf{z}^{in}) \text{ and } \mathbf{x}^{out} = (u_1 + u^{out}, v^{out}, w_1 + w^{out}, \mathbf{z}^{out}), \tag{4}$$

respectively. Since $u^{in}, v^{in}, u^{out}, w^{out} \ll u_0, v_0, u_1, w_1$, in the expressions for $\mathbf{x}^{in}$ and $\mathbf{x}^{out}$ they are ignored. The radial direction is ignored because it is irrelevant in the study of stability. Therefore, the local map $(v^{out}, \mathbf{z}^{out}) = \phi_j(w^{in}, \mathbf{z}^{in})$ can be approximated as

$$v^{out} = v_0 w_1^{c/e} (w^{in})^{-c/e} \text{ and } \mathbf{z}^{out} = \mathbf{z}^{in} w_1^{\mathbf{t}/e} (w^{in})^{-\mathbf{t}/e}, \tag{5}$$

where $c, e, \mathbf{t}$ are the contracting, expanding and transverse eigenvalues, respectively.

Since the radial variable in $\mathbf{x}^{out}$ depends on w^{in} as $u^{out} = u_0 w_1^{c/e} (w^{in})^{-r/e}$, we use the robust distances as the measure of closeness of $\Phi(\tau, x)$ to a cycle, or to a network. Let $\mathfrak{d}^{in} = \mathfrak{d}(\mathbf{x}^{in}, X) = |(w^{in}, \mathbf{z}^{in})|$ and $\mathfrak{d}^{out} = \mathfrak{d}(\mathbf{x}^{out}, X) = |(v^{out}, \mathbf{z}^{out})|$ be the robust distances between the entry and exit points of $\Phi(\tau, x)$ and X. From (5) we obtain the estimate

$$\mathfrak{d}^{out} < A_j (\mathfrak{d}^{in})^{\rho_j}, \text{ where } \rho_j = \min(-c/e, 1 + \min_s(-t_s/e)) \tag{6}$$

and $A_j > 0$ is a constant independent of w^{in} and $\mathbf{z}^{in}$.

Remark 4.2. *Let $\xi_j \in Y$, where Y is an ac-network, be an equilibrium with one-dimensional incoming and outgoing connections. Then the approximations (5) and the estimate (6) for the local map near ξ_j hold true. Note that for an ac-network all transverse eigenvalues are negative.*

From now on we deal with the case when there are 2-dimensional connections.

We divide the estimates into lemmas taking into account whether the equilibrium is, or is not, an m-point for a Δ-clique and on whether it has one or two expanding eigenvectors. The combinations of these two concerns cover all possibilities for a node in an ac-network.

Let ξ_j be an equilibrium in an ac-network Y. The notions of radial, contracting, expanding and transverse eigenvalues of the Jacobian matrix at an equilibrium are given in Subsection 2.1. Denote by c_s, $1 \le s \le S$, the contracting eigenvalues of $\mathrm{d}f(\xi_j)$; by

e_p, $1 \le p \le P$, the expanding eigenvalues; finally, by t_q, $1 \le q \le Q$, the transverse eigenvalues. Note that P is either 1 or 2. We write

$$c = \max_{1\le s\le S} c_s \quad \text{and} \quad e = \max_{1\le p\le P} e_p.$$

We use $\mathbf{e}_s^{co}$ to denote the contracting eigenvectors ($1 \le s \le S$) and $\mathbf{e}_p^{ex}$ ($p = 1$ or 2) the expanding eigenvectors. We use the superscripts $.^{in}$ and $.^{out}$ to indicate the quantities at the moments when a trajectory $\Phi(\tau, x)$ enters and exits $N_{\tilde{\delta}}(\xi_j)$, respectively. As above, the corresponding time moments are τ^{in} and τ^{out} and we define $\mathfrak{d}^{in} = \mathfrak{d}(\mathbf{x}^{in}, Y)$ and $\mathfrak{d}^{out} = \mathfrak{d}(\mathbf{x}^{out}, Y)$.

Lemma 4.3. *Suppose that an equilibrium $\xi_j \in Y$ is not an m-point for any of the Δ-cliques of Y. Let ξ_j have S contracting eigenvectors and P expanding eigenvectors. Then*

$$\mathfrak{d}^{out} < A_j(\mathfrak{d}^{in})^{\rho_j}, \text{ where } \rho_j = \min(-c/e, 1 + \min_q(-t_q/e)),\ A_j \in \mathbb{R}. \tag{7}$$

Proof. Since ξ_j is not an m-point, the expanding eigenvectors $\mathbf{e}_p^{ex}$ belong to the orthogonal complement to any of P_d^{in}, where P_d^{in} are the invariant subspaces that the incoming connections belong to. Therefore, $\max_{1\le p\le P} |w_p^{in}| < \mathfrak{d}^{in}$. If $P = 2$ then the outgoing connections (short and f-long connections of a Δ-clique) intersect with $N_{\tilde{\delta}}(\xi_j)$ along the faces $w_1 = \tilde{\delta}$ and $w_2 = \tilde{\delta}$. For the trajectory that exits through the face $w_1 = \tilde{\delta}$ the time of flight $\tau^* = \tau^{out} - \tau^{in}$ satisfies

$$\mathrm{e}^{\tau^*} = \tilde{\delta}^{1/e_1}|w_1^{in}|^{-1/e_1}.$$

For the trajectory that exits through the face $w_2 = \tilde{\delta}$ it satisfies

$$\mathrm{e}^{\tau^*} = \tilde{\delta}^{1/e_2}|w_2^{in}|^{-1/e_2}.$$

Therefore, we have

$$\mathrm{e}^{\tau^*} > C_1(\mathfrak{d}^{in})^{-1/e},$$

where C_1 is a constant. If $P = 1$ then the above estimate evidently holds true.

There exists a constant $C_2 > 0$ such that $v_{s,0} < C_2$ for all incoming connections. Therefore, for the contracting coordinates

$$v_s^{out} = v_{s,0}\mathrm{e}^{c\tau^*} < C_2(C_1)^c(\mathfrak{d}^{in})^{-c/e}. \tag{8}$$

For the transverse ones we have

$$z_q^{out} = z_{q,0}\mathrm{e}^{t_q\tau^*} < (C_1)^{t_q}(\mathfrak{d}^{in})^{1-t_q/e}. \tag{9}$$

The statement of the lemma follows from (8), (9) and the definition of the robust distance. □

Lemma 4.4. *Suppose that an equilibrium $\xi_j \in Y$ is m-point for a Δ-clique of Y, ξ_j has one contracting eigenvector and one expanding, which are the f-long and s-long vectors of the Δ-clique. Then the trajectories passing through $N_{\tilde{\delta}}(\xi_j)$ satisfy*

$$\mathfrak{d}^{out} \le \mathfrak{d}^{in}. \tag{10}$$

Proof. The expanding and contracting eigenvectors of ξ_j belong to the invariant subspace P_j that contains the Δ-clique. Therefore, $P_j^{\perp}$ is spanned by the transverse eigenvectors of $\mathrm{d}f(\xi_j)$. Since the transverse coordinates satisfy $z_q^{out} = z_{q,0}\mathrm{e}^{t_q\tau^*}$, where again $\tau^* = \tau^{out} - \tau^{in}$, and all the transverse eigenvalues t_q are negative the statement of the lemma holds true. □

Lemma 4.5. *Suppose that an equilibrium $\xi_j \in Y$ is m-point for several (one or more) Δ-cliques of Y and it has one expanding eigenvector. For $1 \leq s \leq S_0$ let $\mathbf{e}_s^{co}$ be the f-long vector for a Δ-clique, while the remaining contracting eigenvectors are not f-long. Then*

$$\mathfrak{d}^{out} < A_j(\mathfrak{d}^{in})^{\rho_j}, \ \textit{where } \rho_j = \min(-c/e, 1) \textit{ and } A_j \in \mathbb{R}. \tag{11}$$

Proof. First we consider trajectories that enter the δ-neighbourhood of a Δ-clique. Since all eigenvalues of $\mathrm{d}f(\xi_j)$, except for one expanding, are negative, then a trajectory $\Phi(\tau, x)$ that at $\tau = \tau^{in}$ belongs to the δ-neighbourhood of a Δ-clique remains in this neighbourhood for $\tau \in [\tau^{in}, \tau^{out}]$ and satisfies

$$\mathfrak{d}^{out} < \mathfrak{d}^{in}. \tag{12}$$

If a trajectory does not belong to the δ-neighbourhood of any of Δ-cliques then it satisfies $|w^{in}| < \mathfrak{d}^{in}$. Hence, the time of flight $\tau^* = \tau^{out} - \tau^{in}$ satisfies

$$\mathrm{e}^{\tau^*} > C_1(\mathfrak{d}^{in})^{-1/e}.$$

(This can be shown by the same arguments as the ones used in the proof of Lemma 4.3.) Let $C_2 > 0$ be a constant such that $v_{s,0} < C_2$ for all incoming connections. Then

$$v_s^{out} = v_{s,0}\mathrm{e}^{c\tau^*} < C_2(C_1)^c(\mathfrak{d}^{in})^{-c/e}, \tag{13}$$

where $c = \max_{S_0 < s \leq S} c_s$. For the transverse coordinates we have

$$z_q^{out} = z_{q,0}\mathrm{e}^{t_q\tau^*} < (C_1)^{t_q}(\mathfrak{d}^{in})^{1-t_q/e}. \tag{14}$$

The statement of the lemma follows from (12)-(13) and the definition of the robust distance. □

Lemma 4.6. *Suppose that $\lambda_1 < 0$ and $\lambda_2, q, \varepsilon > 0$. Then*

$$\min(q^{\lambda_1}, \varepsilon q^{\lambda_2}) \leq \varepsilon^{\lambda_1/(\lambda_1-\lambda_2)}.$$

Proof.

If $q^{\lambda_1} \leq \varepsilon q^{\lambda_2}$ then $q^{\lambda_1-\lambda_2} \leq \varepsilon$, which implies that $q^{\lambda_1} \leq \varepsilon^{\lambda_1/(\lambda_1-\lambda_2)}$.
If $\varepsilon q^{\lambda_2} \leq q^{\lambda_1}$ then $q^{\lambda_2-\lambda_1} \leq \varepsilon^{-1}$, which implies that $\varepsilon q^{\lambda_2} \leq \varepsilon^{\lambda_1/(\lambda_1-\lambda_2)}$.

□

Lemma 4.7. *Suppose that an equilibrium $\xi_j \in Y$ has one contracting eigenvector, $\mathbf{e}^{co}$, two expanding eigenvectors, $\mathbf{e}_1^{ex}$ and $\mathbf{e}_2^{ex}$, ξ_j is an m-point for just one Δ-clique, $\mathbf{e}^{co}$ is the f-long vector of the Δ-clique and $\mathbf{e}_1^{ex}$ is the s-long vector of the Δ-clique. Then*

$$\mathfrak{d}^{out} < A_j(\mathfrak{d}^{in})^{\rho_j}, \ \textit{where } \rho_j = \frac{c}{c - e_2}, \ A_j \in \mathbb{R}. \tag{15}$$

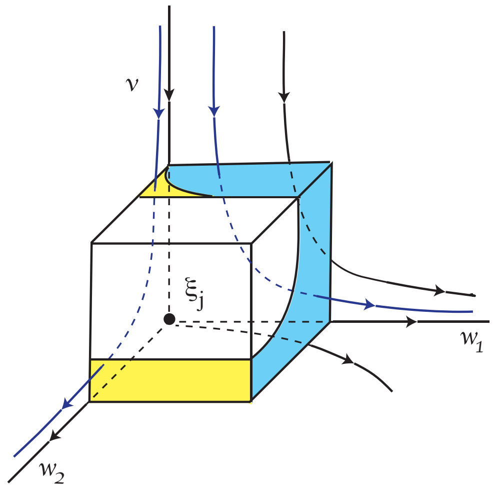


Figure 8: The neighbourhood of ξ_j considered in Lemma 4.7. Trajectories in Y are black, while the ones near Y are dark blue. Blue color indicates the entry and exit points of the trajectories that exit through the face $w_1 = \tilde{\delta}$ while the yellow indicates the ones that exit through the face $w_2 = \tilde{\delta}$.

Proof. Trajectories that enter $N_{\tilde{\delta}}(\xi_j)$ belong to the δ-neighbourhood of the Δ-clique, hence initially $w_2^{in} < \mathfrak{d}^{in}$. Therefore, the trajectories that exit through the face $w_2 = \tilde{\delta}$ satisfy

$$v^{out} < C(\mathfrak{d}^{in})^{-c/e_2} \text{ and } z_q^{out} < C(\mathfrak{d}^{in})^{1-t_q/e_2}. \tag{16}$$

(This follows from arguments similar to the ones employed in the proofs of Lemmas 4.3 and 4.5.)

For trajectories that exit through the face $w_1 = \tilde{\delta}$ the time of flight satisfies

$$w_1^{in}\mathrm{e}^{\tau^*} = \tilde{\delta}.$$

At the exit point

$$v^{out} = v_0\mathrm{e}^{c\tau^*} \text{ and } w_2^{out} = w_2^{in}\mathrm{e}^{e_2\tau^*}.$$

Therefore (see Lemma 4.6)

$$\min(v^{out}, w_2^{out}) < C(\mathfrak{d}^{in})^{c/(c-e_2)}. \tag{17}$$

The statement of the lemma follows from (16), (17) and the fact that $-c/e_2 > c/(c-e_2)$ and that

$$z_q^{out} = z_q^{in}\mathrm{e}^{t_q\tau^*} \text{ where } t_q < 0.$$

□

Lemma 4.8. *Suppose that an equilibrium $\xi_j \in Y$ has several contracting eigenvectors and two expanding eigenvectors. Let the eigenvectors $\mathbf{e}_1^{co}, ..., \mathbf{e}_q^{co}$ be f-long vectors of Δ-cliques for which $\mathbf{e}_1^{ex}$ is the s-long vector. Assume also that the eigenvectors $\mathbf{e}_h^{co}, ..., \mathbf{e}_d^{co}$, where $h \le q+1$, are f-long vectors of Δ-cliques where $\mathbf{e}_2^{ex}$ is the s-long vector. Then*

$$\begin{aligned}&\mathfrak{d}^{out} < A_j(\mathfrak{d}^{in})^{\rho_j},\ \textit{where } \rho_j = \min(a_1, a_2, a_3),\ a_1 = -\max\nolimits_{q+1\le s\le S} c_s/\max\nolimits_{p=1,2} e_p\\ &a_2 = e_1/(e_1-\tilde{c}_1),\ a_3 = e_2/(e_2-\tilde{c}_2),\ \tilde{c}_1 = \max\nolimits_{1\le s\le q} c_s,\ \tilde{c}_2 = \max\nolimits_{h\le s\le d} c_s,\ A_j \in \mathbb{R}.\end{aligned} \tag{18}$$

Proof. Proceeding as in the proof of Lemma 4.7, we first obtain estimates of $\mathfrak{d}^{out}$ for trajectories that when they enter $N_{\tilde{\delta}}(\xi_j)$ are not in the δ-neighbourhoods of any Δ-cliques, then we consider trajectories in each of the Δ-cliques individually. □

4.2 Sufficient conditions for asymptotic stability

In this section we prove a theorem providing sufficient conditions for asymptotic stability of certain heteroclinic networks. The networks that are considered in theorems[3] and lemmas of this section are robust heteroclinic networks in (1), comprised of a finite number of hyperbolic equlibria ξ_j and heteroclinic connections C_{ij} that belong to flow-invariant subspaces S_{ij}. In each lemma and theorem we explicitly state the other assumptions that are made. The ac-networks satisfy these assumptions, therefore the lemmas and theorems are applicable to them.

In this subsection we consider a set $N_{\tilde{\delta}}(\xi_j)$ bounded by $|a_m| = \tilde{\delta}$, where the a_m coordinate is taken in a basis comprised of eigenvectors of $df(\xi_j)$. Thus, $N_{\tilde{\delta}}(\xi_j)$ is not a box, as it is for ac-networks, but a bounded region. The times τ^{in} and τ^{out} are taken with reference to $N_{\tilde{\delta}}(\xi_j)$.

Lemma 4.9. *Given a clean robust heteroclinic network Y, there exist $s > 0$ and $\varepsilon' > 0$ such that for any $0 < \varepsilon < \varepsilon'$*

$$d(\Phi(\tau^{in}, x), Y) < \varepsilon^s \text{ implies that } d(\Phi(\tau, x), Y) < \varepsilon \text{ for all } \tau \in [\tau^{in}, \tau^{out}].$$

Proof. Since near ξ_j a flow of (1) can be approximated by a linearised system, in $N_{\tilde{\delta}}(\xi_j)$ the difference $\Delta\mathbf{x} = \mathbf{x}(t, \mathbf{x}_0) - \mathbf{y}(t, \mathbf{y}_0)$ between two trajectories is

$$\Delta\mathbf{x} = (\mathbf{x}_0 - \mathbf{y}_0)\mathrm{e}^{\Lambda\tau}, \tag{19}$$

where $\Lambda = (\lambda_1, ..., \lambda_n)$ are the eigenvalues of $\mathrm{d}f(\xi_j)$. For simplicity we assume that all eigenvalues are real. In the case of complex eigenvalues the proof is similar and in the expressions for s_j the eigenvalues should be replaced by their real parts.

Let $\mathbf{x}_0$ be the point where $\Phi(\tau, x)$ enters $N_{\tilde{\delta}}$ and $\mathbf{y}_0$ be the nearest to $\mathbf{x}_0$ point in Y. Split contracting eigenvectors of $\mathrm{d}f(\xi_j)$ into two groups: belonging to the subspace $S_{ij} \supset C_{ij} \ni \mathbf{y}_0$, and those that do not belong. The respective coordinates are denoted by $\mathbf{v}$ and $\mathbf{q}$ and the associated eigenvalues by $\mathbf{c}_v$ and $\mathbf{c}_q$. The expanding eigenvectors are splitted similarly, into belonging to S_{ij} and do not. The respective coordinates are denoted by $\mathbf{h}$ and $\mathbf{w}$, associated eigenvalues by $\mathbf{e}_h$ and $\mathbf{e}_w$. Due to (19)

$$d(\Phi(\tau, \mathbf{x}_0), \Phi(\tau, \mathbf{y}_0)) = \max(|\Delta\mathbf{u}\mathrm{e}^{\mathbf{r}\tau}|, |\Delta\mathbf{v}\mathrm{e}^{\mathbf{c}_v\tau}|, |\Delta\mathbf{h}\mathrm{e}^{\mathbf{e}_h\tau}|, |\mathbf{q}_0\mathrm{e}^{\mathbf{c}_q\tau}|, |\mathbf{w}_0\mathrm{e}^{\mathbf{e}_w\tau}|, |\mathbf{z}_0\mathrm{e}^{\mathbf{t}\tau}|),$$

where we have denoted

$$\mathbf{x}_0 = (\mathbf{u}_0, \mathbf{v}_0, \mathbf{h}_0, \mathbf{q}_0, \mathbf{w}_0, \mathbf{z}_0),$$
$$\mathbf{y}_0 = (\mathbf{u}_0 - \Delta\mathbf{u}, \mathbf{v}_0 - \Delta\mathbf{v}, \mathbf{h}_0 - \Delta\mathbf{h}, 0, 0, 0).$$

[3] We state two theorems providing sufficient conditions for asymptotic stability. Since the proofs are similar, we prove only one of them.

Similarly we write

$$d(\Phi(\tau,\mathbf{x}_0),W^u(\xi_j))=\max(|\mathbf{u}_0\mathrm{e}^{\mathbf{r}\tau}|,|\mathbf{v}_0\mathrm{e}^{\mathbf{c}_v\tau}|,|\mathbf{q}_0\mathrm{e}^{\mathbf{c}_q\tau}|,|\mathbf{z}_0\mathrm{e}^{\mathbf{t}\tau}|).$$

Since all transverse eigenvalues are negative, using Lemma (4.6) we obtain that if

$$d(\mathbf{x}_0,\mathbf{y}_0)=\min(|\Delta\mathbf{u}|,|\Delta\mathbf{v}|,|\Delta\mathbf{h}|,|\mathbf{q}_0|,|\mathbf{h}_0|,|\mathbf{z}_0|)<\varepsilon^s$$

then for all $\tau\in[\tau^{in},\tau^{out}]$

$$d(\Phi(\tau,\mathbf{x}_0),Y)<\min\Big(d(\Phi(\tau,\mathbf{x}_0),\Phi(\tau,\mathbf{y}_0)),d(\Phi(\tau,\mathbf{x}_0),W^u(\xi_j)\Big)<\varepsilon^{sd_j},$$

where $d_j=c'/(c'-e)$, $c'=\max(r,c)$, $r=\max_{1\le k\le K}r_k$ and r_k are the radial eigenvalues of ξ_j. (Recall that we do not assume that Y is an ac-network, hence its equilibria can have several radial eigenvalues). Taking s to be the maximux of $1/d_j$ we prove the lemma. □

Lemma 4.10. *Let Y be the network from Lemma 4.9. Then there exist $s>0$ and $\varepsilon'>0$ such that for any $0<\varepsilon<\varepsilon'$*

$$d(\Phi(\tau,x),Y)<\varepsilon^s \textit{ for some } \tau\in[\tau^{in},\tau^{out}] \textit{ implies that } d(\Phi(\tau^{out},x),Y)<\varepsilon.$$

The proof is similar to the proof of Lemma 4.9 and we do not present it.

Lemma 4.11. *Given a compact heteroclinic network Y, for any heteroclinic connection C_{ij} there exists $B_j>0$ such that*

$$d(\Phi(\tau^{out},x),Y)<\varepsilon \textit{ implies that } d(\Phi(\tau,x),Y)<B_j\varepsilon \textit{ for all } \tau\in[\tau^{out},\tau^{in}],$$

where τ^{out} is the time moment when the trajectory exits the neighbourhood $N_{\tilde\delta}(\xi_i)$, τ^{in} is the time moment when it enters $N_{\tilde\delta}(\xi_j)$, $\Phi_t(x)\in N_{\tilde\delta}(Y)$ for any $\tau\in[\tau^{out},\tau^{in}]$ and $\Phi(\tau,x)\notin N_{\tilde\delta}(\xi_k)$ for any $\tau\in(\tau^{out},\tau^{in})$.

Proof. Due to the smoothness of f (see (1)) for small $|x-y|$ the difference $\Phi(\tau,x)-\Phi(\tau,y)$ satisfies

$$|\Phi(\tau,x)-\Phi(\tau,y)|<B(\tau,x)|x-y|, \tag{20}$$

where the constant B depends on the time τ and the initial condition x. Taking $\tau_i^{out}=0$ and $B(x)=\max_{0<\tau<\tau^{in}}B(\tau,x)$ we obtain a finite bound $B(x)$ for an individual trajectory.

Consider the intersection $\mathcal{W}=W^u(\xi_j)\cap\partial N_{\tilde\delta}(\xi_j)$. Due to the compactness of Y, $\mathcal{W}$ is a compact set. Denoting $B_j=\max_{x\in\mathcal{W}}B(x)$ we prove the lemma. (The maximum exists and is finite because $\mathcal{W}$ is compact.) □

Lemma 4.12. *Let Y be the network from Lemma 4.11. For any heteroclinic connection C_{ij} there exists $B_{ij}>0$ such that*

$$d(\Phi(\tau,x),Y)<\varepsilon \textit{ for some } \tau\in[\tau^{out},\tau^{in}] \textit{ implies that } d(\Phi(\tau^{in},x),Y)<B_{ij}\varepsilon.$$

The proof is similar to the proof of Lemma 4.11 and we do not present it.

Lemma 4.13. *Let Y be the network from Lemma 4.9 and $\xi_j \in Y$ be such that $\bar{W}^u(\xi_j) \cap W^u(\xi_i) = \xi_j$ or $\varnothing$ for any $i \neq j$. Then there exist $s > 0$ and $\varepsilon' > 0$ such that for any $0 < \varepsilon < \varepsilon'$*

$$\mathfrak{d}(\Phi(\tau^{in}, \mathbf{x}), Y) < \varepsilon^s \textit{ implies that } |\mathbf{u}^{out}| < \varepsilon.$$

Proof. The conditions of the lemma ensure the absence of $\mathbf{h}$ coordinated introduced in the proof of Lemma 4.9. Since $|\mathbf{w}| < \varepsilon^s$, for a trajectory exiting through the face $w_p = \tilde{\delta}$ we have

$$|\mathbf{u}^{out}| < |\mathbf{u}^{in}| \mathrm{e}^{-r\tau^*} < \tilde{\delta}^{1+r/c_p} \varepsilon^{-rs/c}.$$

Therefore, for $s = -c/2r$ the statement of the lemma holds true. □

Remark 4.14. *In an ac-network any equilibrium which is not an m-point for a Δ-cliques satisfies the conditions of Lemma 4.13.*

Theorem 4.15. *Let Y be a clean heteroclinic network such that for any equilibrium the nearby trajectories satisfy*

$$d(\Phi(\tau^{out}, x), Y) < A_j (d(\Phi(\tau^{in}, x), Y))^{\rho_j} \textit{ for some } A_j > 0.$$

Suppose also that for any heteroclinic cycle $X \subset Y$ we have

$$\rho(X) > 1 \qquad \textit{where} \qquad \rho(X) = \prod_{1 \le j \le m} \rho_j.$$

Then Y is asymptotically stable.

Proof. According to the definition of asymptotic stability, to prove that Y is a.s., for any given $\delta > 0$ we should find $\varepsilon > 0$ such that any trajectory $\Phi(\tau, x)$, with $d(x, Y) < \varepsilon$, satisfies

(i) $d(\Phi(\tau, x), Y) < \delta$ for any $\tau > 0$;

(ii) $\lim_{\tau \to \infty} d(\Phi(\tau, x), Y) = 0$.

Since Y is clean, by Lemmas 4.9-lem52n it is sufficient to check (i) only at time instances when $\Phi(\tau, x)$ enters the boxes $N_{\tilde{\delta}}(\xi_j)$. For a trajectory which is staying in a δ-neighbourhood of Y for all $\tau > 0$ there are two possibilities: it can either be attracted to an equilibrium $\xi_j \in Y$, or it can make infinite number of crossings of the boxes $N_{\tilde{\delta}}(\xi_j)$. Therefore, the condition (ii) also may be checked only at the time instances when the trajectory enters the boxes. So, we consider only times when $\Phi(\tau, x)$ enters a box and approximate a trajectory $\Phi(\tau, x)$ by a sequence of maps $g_{ij} = \psi_{ij}\phi_i(x)$, where $x \in \partial N_{\tilde{\delta}}(\xi_i)$. For any connection C_{ij} Lemma 4.11 ensures the existence of $B_{ij} > 0$ such that

$$d(\Phi(\tau^{in}, x), Y) < B_{ij} d(\Phi(\tau^{out}, x), Y).$$

Since for any cycle $X = [\xi_1 \to ... \to \xi_m \to \xi_1]$ we have $\rho(X) > 1$, there exists $q > 0$ such that

$$\tilde{\rho}(X) = \prod_{1 \le j \le m} \tilde{\rho}_j > 1, \text{ where } \tilde{\rho}_j = \rho_j - q.$$

Given a path $\mathcal{P} = [\xi_1 \to ... \to \xi_m]$ define

$$\rho(\mathcal{P}) = \prod_{1 \le j \le m} \rho_j \quad \text{and} \quad \tilde{\rho}(\mathcal{P}) = \prod_{1 \le j \le m} \tilde{\rho}_j$$

Denote $A = \max_j A_j$ and $B = \max_{ij} B_{ij}$, where the maxima are taken over all equilibria and all connections in Y. We choose ε according to the following rules:

I) $0 < \varepsilon < \delta$;

II) $\varepsilon^{(\tilde{\rho}(X)-1)\tilde{\rho}(\mathcal{P})} < 1/2$ for any heteroclinic cycle $X \subset Y$ and any path $\mathcal{P} \subset Y$;

III) $\varepsilon^{\tilde{\rho}(\mathcal{P})} < \tilde{\varepsilon}$ for any path $\mathcal{P} \subset Y$, where $0 < \tilde{\varepsilon} < \delta$ satisfies $\tilde{\varepsilon}^q AB < 1$.

Consider a trajectory $\Phi(\tau, x)$ at $\tau \in [0, T]$ where $d(\Phi(0, x), Y) < \varepsilon$ and ε satisfies I-III. Suppose that $x \in \partial N_{\tilde{\delta}}(\xi_{i_1})$ and the trajectory passes near equilibria with the indices $[i_1, ..., i_S]$. Denote $d_s = d(g_{i_{s-1}} ... g_{i_1}(x), Y)$ and $d_1 = d(x, Y)$. The proof employs the estimate $d_{s+1} < AB(d_s)^{\rho_{i_s}} < (d_s)^{\tilde{\rho}_{i_s}}$, which follows from III under the condition that $d_s < \tilde{\varepsilon}$.

For d_2 we have

$$d_2 < AB(d_1)^{\rho_{i_1}} < (d_1)^{\tilde{\rho}_{i_1}} < \tilde{\varepsilon} < \delta.$$

Similarly

$$d_{s+1} < AB(...AB(AB(d_1)^{\rho_{i_1}})^{\rho_{i_2}} ...)^{\rho_{i_s}} < (d_1)^{\tilde{\rho}_{i_s} \cdots \tilde{\rho}_{i_1}} < \tilde{\varepsilon} < \delta,$$

as long as the indices $i_1, ..., i_{s-1}$ are distinct.

If we have $i_s = i_d$ for some $1 \le d \le s-2$, then the sequence of equilibria $[\xi_{i_{d+1}}, ..., \xi_{i_s}]$ belongs to a heteroclinic cycle $X \subset Y$. Hence

$$d_{s+2} = d(g_{i_{s+1}} ... g_{i_{d+1}}(g_{i_d} ... g_{i_1}(x)), Y) < (d_1)^{\tilde{\rho}(X)\tilde{\rho}_{i_{s+1}}\tilde{\rho}_{i_d} \cdots \tilde{\rho}_{i_1}} < (d_1)^{\tilde{\rho}_{i_{s+1}}\tilde{\rho}_{i_d} \cdots \tilde{\rho}_{i_1}} < \tilde{\varepsilon} < \delta. \tag{21}$$

We proceed as for the path for i_{s+3}, i_{s+4}, etc., provided that all indices in the sequence $i_1, ..., i_d, i_{s+1}, ..., i_{s+k}$ are distinct. If i_{s+k} equals to some $i_p \in \{i_1, ..., i_d, i_{s+1}, ..., i_{s+k-2}\}$, we similarly remove the subsequence $\{i_{p+1}, ..., i_{s+k}\}$ and obtain that

$$d_{s+k+2} < (d_1)^{\tilde{\rho}_{i_1} \cdots \tilde{\rho}_{i_p} \tilde{\rho}_{s+k+1}} < \tilde{\varepsilon} < \delta.$$

Applying this procedure to the sequence of equilibria that is followed by the trajectory $\Phi(\tau, x)$, we obtain that $d(\Phi(\tau, x), Y) < \delta$ for any crossing of $N_{\tilde{\delta}}(\xi_j)$ at positive τ.

To prove that a trajectory is attracted by Y, we recall that any trajectory that is not attracted by $\xi_j \in Y$ passes near an infinite number of equilibria. Thus the sequence of equilibria visited by the trajectory must have repetitions. The first repetition follows a cycle $X \subset Y$. Due to our choice of ε (condition II) the estimate (21) may be refined to yield

$$d_{s+2} < (d_1)^{\tilde{\rho}(X)\tilde{\rho}_{i_{s+1}}\tilde{\rho}_{i_d} \cdots \tilde{\rho}_{i_1}} < (d_1)^{(\tilde{\rho}(X)-1)\tilde{\rho}_{i_{s+1}}\tilde{\rho}_{i_d} \cdots \tilde{\rho}_{i_1}} (d_1)^{\tilde{\rho}_{i_{s+1}}\tilde{\rho}_{i_d} \cdots \tilde{\rho}_{i_1}} < \frac{1}{2}(d_1)^{\tilde{\rho}_{i_{s+1}}\tilde{\rho}_{i_d} \cdots \tilde{\rho}_{i_1}}.$$

After removing of k such subsequences we have

$$d_S < \frac{1}{2^k}(d_0)^{\tilde{\rho}_{i'} \cdots \tilde{\rho}_{i''}},$$

where due to (III) $(d_1)^{\tilde\rho_{i'}\cdots\tilde\rho_{i''}} < \tilde\varepsilon$. As $S \to \infty$ the number of removed cycles also tends to infinity we have $\lim_{S\to\infty} d_S = 0$. □

Corollary 4.16. *Let Y be an ac-network such that any heteroclinic cycle $X \subset Y$ satisfies $\rho(X) > 1$, where the exponents ρ_j are the ones derived in Lemmas 4.3-4.8. Then Y is asymptotically stable.*

Recalling that the quantities ρ_j are defined in Lemmas 4.3, 4.5 or 4.8, the proof follows from the property that $\rho_j > 1$ only if ξ_j is not an m-point and Remark 4.14.

The conditions for asymptotic stability proven in Theorem 4.15 are not very accurate, in particular, in the employed estimates the exponents ρ_j for local maps do not take into account which faces a trajectory enters and exits $N_{\tilde\delta}(\xi_j)$ through. As can be seen from the proofs of Lemmas 4.3, 4.5 and 4.8, different values of ρ_j can be obtained for different prescribed entry and exit faces. Based on this, we propose more accurate and more cumbersome, compared to Theorem 4.15, sufficient conditions for asymptotic stability of a heteroclinic network. We start by saying that a walk $\mathcal{P} = [\xi_1 \to ... \to \xi_m]$ is *semi-linear* if it does not have repeating triplets $(\xi_{s-1} = \xi_{d-1}, \xi_s = \xi_d, \xi_{s+1} = \xi_{d+1})$.

Decompose $\partial N_{\tilde\delta}(\xi_j) = \cup_{1\le s\le n}\mathcal{F}_s$, where $\mathcal{F}_s$ is(are) the face(s) $|y_s| = \tilde\delta$ and $\mathbf{y}$ are the local coordinates near ξ_j. Since we consider networks in $\mathbb{R}^n_+$, there is just one face for each coordinate, except the radial one. For a walk $\mathcal{P} = [\xi_1 \to ... \to \xi_m]$ and an equilibrium $\xi_j \in \mathcal{P}$ denote by $\mathcal{F}_j^{in}$ the union of faces that intersect with the connection $\xi_{j-1} \to \xi_j$ and by $\mathcal{F}_j^{out}$ the faces that intersect with the connection $\xi_j \to \xi_{j+1}$. For ac-networks the sets $\mathcal{F}_j^{in}$ and $\mathcal{F}_j^{out}$ are unions of one or two faces, depending on whether the respective connections are one- or two-dimensional. Let $\Phi(\tau, x)$ enters $N_{\tilde\delta}(\xi_j)$ though $\mathcal{F}_j^{in}$ and exits through $\mathcal{F}_j^{out}$. Then, by the same arguments that used to prove Lemmas 4.3-4.8, we have the estimate

$$\mathfrak{d}(\Phi(\tau^{out}, x), Y) < A_j(\mathfrak{d}(\Phi(\tau^{in}, x), Y))^{\rho^*(\mathcal{P},\xi_j)},$$

with the exponent $\rho^*(\mathcal{P}, \xi_j) \ge \rho_j$. The expressions for $\rho^*(\mathcal{P}, \xi_j)$ are similar to the ones for ρ_j derived in the lemmas and depend on dimensions of incoming and outgoing connections and whether the eigenvectors are f-long/s-long vectors of some Δ-cliques. (To obtain $\rho^*(\mathcal{P}, \xi_j)$ one just ignores the parts of the lemmas that are not relevant to $\mathcal{F}_j^{in}$ or $\mathcal{F}_j^{out}$.)

Theorem 4.17. *Let Y be a clean heteroclinic network such that for any closed semi-linear walk $\mathcal{P} \subset Y$, $\mathcal{P} = [\xi_1 \to ... \to \xi_m \to \xi_1]$, the local maps*

$$\phi(\mathcal{P}, j) : F_{\tilde\delta}(\xi_j, \kappa_{j-1,j}) \to F_{\tilde\delta}(\xi_j, \kappa_{j,j+1}),$$

where $F_{\tilde\delta}(\xi_j, \kappa_{j-1,j})$ and $F_{\tilde\delta}(\xi_j, \kappa_{j,j+1})$ are the faces of $\partial N_{\tilde\delta}(\xi_j)$ that intersect with respective connections, satisfy

$$d(\Phi(\tau^{out}, x), Y) < A_j(d(\Phi(\tau^{in}, x), Y))^{\rho^*(\mathcal{P},j)}, \ A_j \in \mathbb{R}.$$

Moreover, for any such walk

$$\rho^*(\mathcal{P}) > 1 \qquad \text{where} \qquad \rho^*(\mathcal{P}) = \prod_{1\le j\le m} \rho^*(\mathcal{P}, j).$$

Then Y is asymptotically stable.

The proof is similar to the proof of Theorem 4.15 and therefore is omitted.

Remark 4.18. *Any network discussed in part (i) of Theorem 3.5 has a subcycle X, such that any equilibrium $\xi_j \in X$ is an m-point for a Δ-clique, where the f-long and s-long vectors are the contracting and expanding eigenvectors of ξ_j, respectively. This is also the case for the networks discussed in Remark 3.6. Instances of such networks are given in Example 5 in the next section. Hence, by Lemma 4.4, $\rho^*(X, \xi_j) = 1$, which implies that $\rho^*(X) = 1$ and neither Theorem 4.15 nor Theorem 4.17 can be used to derive conditions for asymptotic stability of these networks. Such networks were obtained in [1] as examples of heteroclinic networks in the Lotka-Volterra system under the condition that all equilibria have two-dimensional unstable manifolds. As it was shown ibid a sufficient condition of asymptotic stability of these networks is that for each equilibrium $\xi_j \in Y$*

$$|c_j/e_j| > 1, \text{ where } c_j = \min_s c_{js} \text{ and } e_j = \min_q e_{jq},$$

c_{js} are the contracting eigenvalues of $\mathrm{d}f(\xi_j)$ and e_{jq} are the expanding ones.

For networks which were derived in part (ii) of Theorem 3.5 and are not the ones of Remark 3.6, there exists at least one ξ_j such that the connection $[\xi_{j,1} \to \xi_{j+1,1}]$ does not exist. Therefore, any semi-linear walk involves at least one of the connections $[\xi_{j,s} \to \xi_{j+1,0}]$ or $[\xi_{j,s} \to \xi_{j+2,0}]$, which is not an f-long connection for any Δ-clique. Depending on the sequence of equilibria followed, either for $\xi_{j+1,0}$ or for $\xi_{j+2,0}$, the exponent $\rho^*(\mathcal{P}, \cdot)$ is calculated by Lemma 4.3. Therefore, by prescribing the respective expanding eigenvalues to be sufficiently small, we can obtain $\rho^*(\mathcal{P}) > 1$ for any semi-linear walk in any such network. By any network we understang any given grouping and set of connections allowed by part (ii) of Theorem 3.5, except for the ones of Remark 3.6.

5 Examples

In this section we present five examples addressing the stability of six clean heteroclinic networks, four of which are ac-networks. For the networks in Examples 1, 3 and 4 sufficient conditions for asymptotic stability are derived from Theorem 4.15. The theorem, however, cannot be employed to derive such conditions for the network in Example 2, because the network contains a cycle X, where all equilibria $\xi_j \in X$ are m-points of some Δ-cliques. By Lemmas 4.5 and 4.8 for m-points $\rho_j \leq 1$, which implies that $\rho(X) \leq 1$. For this network the sufficient conditions are derived from Theorem 4.17. The last two networks are the ones discussed in the first part of Remark 4.18.

Example 1. Consider a heteroclinic network in $\mathbb{R}^4$ comprised of equilibria $\xi_j \in L_j$, where L_j are the coordinate axes, one-dimensional connections $[\xi_3 \to \xi_4]$ and $[\xi_4 \to \xi_1]$ and a Δ-clique Δ_{123} (see Figure 9a). By Lemmas 4.3 and 4.4 the values of ρ_j for the equilibria are

$$\rho_1 = \frac{-\lambda_{14}}{\max(\lambda_{12}, \lambda_{13})}, \quad \rho_2 = 1, \quad \rho_3 = \frac{-\max(\lambda_{31}, \lambda_{32})}{\lambda_{34}}, \quad \rho_4 = \min\left(\frac{-\lambda_{43}}{\lambda_{41}}, 1 - \frac{\lambda_{42}}{\lambda_{41}}\right),$$

where λ_{ij} is the eigenvalue of $\mathrm{d}f(\xi_i)$ associated with the eigenvector $\mathbf{e}_j$. The network is a union of two cycles, $[\xi_1 \to \xi_2 \to \xi_3 \to \xi_4 \to \xi_1]$ and $[\xi_1 \to \xi_3 \to \xi_4 \to \xi_1]$, hence by Theorem 4.15 the sufficient condition for asymptotic stability of the network is

$$\frac{\lambda_{14} \max(\lambda_{31}, \lambda_{32})}{\lambda_{34} \min(\lambda_{12}, \lambda_{13})} \max\left(\frac{-\lambda_{43}}{\lambda_{41}}, 1 - \frac{\lambda_{42}}{\lambda_{41}}\right) > 1.$$

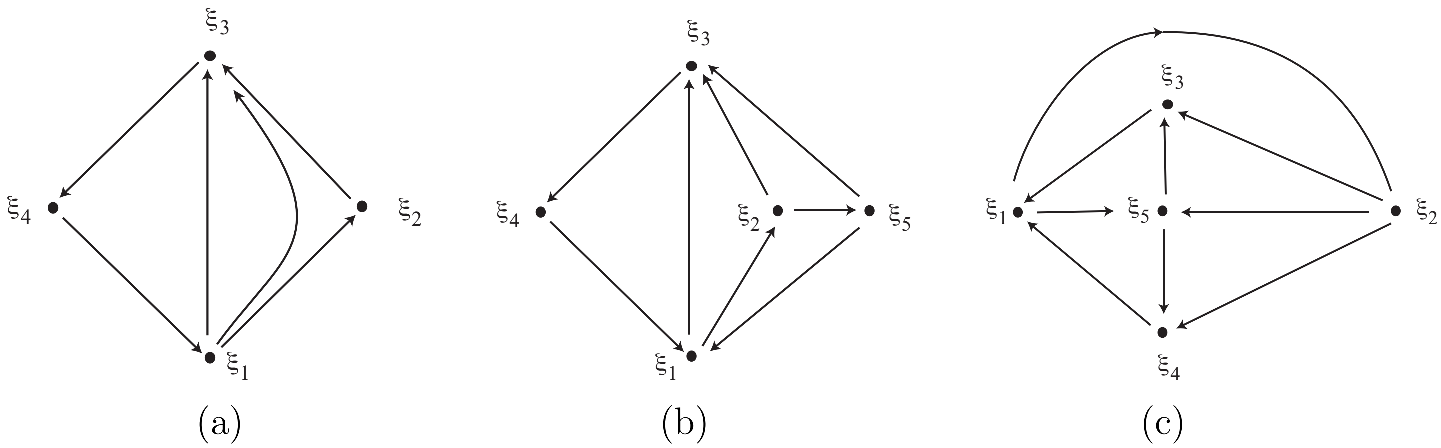


Figure 9: The networks of Examples 1, 2 and 4.

Example 2. Consider the network $Y \subset \mathbb{R}^5$ shown in Figure 9b. Its graph has as a subset the graph of the network considered in Example 1. It also has one more equilirium $\xi_5 \in L_5$ and three more connections: $[\xi_2 \to \xi_5]$, $[\xi_5 \to \xi_1]$ and $[\xi_5 \to \xi_3]$. Thus, the network has three Δ-cliques Δ_{123}, Δ_{253}, and Δ_{513}. For the cycle $X = [\xi_1 \to \xi_2 \to \xi_5 \to \xi_1]$ the equilibria are m-points of some Δ-cliques, therefore $\rho(X) \le 1$ by Lemmas 4.5 and 4.8. Hence, we cannot use Theorem 4.15 to find conditions for asymptotic stability of Y. The values of $\rho^*(\mathcal{P})$ calculated for all semi-linear paths $\mathcal{P} \subset Y$ are presented in the table below:

$\mathcal{P}$	$\rho^*(\mathcal{P})$
$(\xi_1 \to \xi_2 \to \xi_5 \to \xi_1)$	$q_{125}q_{251}q_{512}$
$(\xi_4 \to \xi_1 \to \xi_3 \to \xi_4)$	$q_{413}q_{134}q_{341}$
$(\xi_4 \to \xi_1 \to \xi_2 \to \xi_3 \to \xi_4)$	$q_{412}q_{123}q_{234}q_{341}$
$(\xi_4 \to \xi_1 \to \xi_2 \to \xi_5 \to \xi_3 \to \xi_4)$	$q_{412}q_{125}q_{253}q_{534}q_{341}$
$(\xi_4 \to \xi_1 \to \xi_2 \to \xi_5 \to \xi_1 \to \xi_3 \to \xi_4)$	$q_{412}q_{125}q_{251}q_{513}q_{134}q_{341}$
$(\xi_4 \to \xi_1 \to \xi_2 \to \xi_5 \to \xi_1 \to \xi_2 \to \xi_3 \to \xi_4)$	$q_{412}q_{125}q_{251}q_{512}q_{123}q_{234}q_{341}$
$(\xi_4 \to \xi_1 \to \xi_2 \to \xi_5 \to \xi_1 \to \xi_2 \to \xi_5 \to \xi_3 \to \xi_4)$	$q_{412}q_{125}q_{251}q_{512}q_{125}q_{253}q_{534}q_{341}$
$(\xi_4 \to \xi_1 \to \xi_2 \to \xi_5 \to \xi_1 \to \xi_2 \to \xi_5 \to \xi_1 \to \xi_3 \to \xi_4)$	$q_{412}q_{125}q_{251}q_{512}q_{125}q_{251}q_{513}q_{134}q_{341}$

where q_{ijk} denotes $\rho^*(\mathcal{P}, \xi_j)$, i.e., the exponent of the local map near ξ_j with the contracting eigenvector $\mathbf{v}_i$ and the expanding $\mathbf{v}_k$. They were calculated from Lemmas 4.3-4.4:

$$q_{412} = \min\left(\frac{-\lambda_{14}}{\lambda_{12}}, 1 - \frac{\lambda_{15}}{\lambda_{12}}\right), \quad q_{413} = \min\left(\frac{-\lambda_{14}}{\lambda_{13}}, 1 - \frac{\lambda_{15}}{\lambda_{13}}\right), \quad q_{512} = \min\left(\frac{-\lambda_{15}}{\lambda_{12}}, 1 - \frac{\lambda_{14}}{\lambda_{12}}\right),$$
$$q_{513} = \frac{\lambda_{15}}{\lambda_{15} - \lambda_{13}}, \quad q_{123} = \frac{\lambda_{21}}{\lambda_{21} - \lambda_{23}}, \quad q_{125} = \min\left(\frac{-\lambda_{21}}{\lambda_{25}}, 1 - \frac{\lambda_{24}}{\lambda_{25}}\right),$$
$$q_{134} = \min\left(\frac{-\lambda_{31}}{\lambda_{34}}, 1 - \frac{\max(\lambda_{32}, \lambda_{35})}{\lambda_{34}}\right), \quad q_{234} = \min\left(\frac{-\lambda_{32}}{\lambda_{34}}, 1 - \frac{\max(\lambda_{31}, \lambda_{35})}{\lambda_{34}}\right)$$
$$q_{534} = \min\left(\frac{-\lambda_{35}}{\lambda_{34}}, 1 - \frac{\max(\lambda_{31}, \lambda_{32})}{\lambda_{34}}\right), \quad q_{341} = \min\left(\frac{-\lambda_{43}}{\lambda_{41}}, 1 - \frac{\max(\lambda_{42}, \lambda_{45})}{\lambda_{41}}\right),$$
$$q_{251} = \min\left(\frac{-\lambda_{52}}{\lambda_{51}}, 1 - \frac{\lambda_{54}}{\lambda_{51}}\right), \quad q_{253} = \frac{\lambda_{52}}{\lambda_{52} - \lambda_{51}}.$$

By Theorem 4.17 a sufficient condition for the stability of the network is that $\rho^*(\mathcal{P}) > 1$ for all $\mathcal{P}$ listed in the table. (These inequalities hold true, e.g., if $|\lambda_{42}|$, $|\lambda_{43}|$ and $|\lambda_{45}|$ are significantly larger than other eigenvalues and in addition for the cycle $[\xi_1 \to \xi_2 \to \xi_5 \to \xi_1]$ the contracting eigenvalues are larger than the expanding ones, in absolute value.)

Example 3. The network shown in Figure 2a was discussed in [3] (Subsection 4.1 *ibid*) as an example of a clean heteroclinic network in $\mathbb{R}^4$. The equilibria $\xi_1, \xi_2, \xi_3, \xi_4$ belong to the respective coordinate axes, while ξ_5 belongs to the coordinate plane P_{34}. Since it is clean, we can apply Theorem 4.15 to derive conditions for symptotic stability of this network. The values of ρ_j are:

$$\rho_1 = \frac{-\max(\lambda_{13}, \lambda_{14})}{\lambda_{12}}, \quad \rho_2 = \frac{-\lambda_{21}}{\max(\lambda_{23}, \lambda_{24})}, \quad \rho_3 = \frac{-\max(\lambda_{32}, \lambda_{34})}{\lambda_{31}},$$
$$\rho_4 = \frac{-\max(\lambda_{42}, \lambda_{43})}{\lambda_{41}}, \quad \rho_5 = 1.$$

Applying the theorem to the four cycles the network is comprised of, we obtain that the conditions for asymptotic stability are:

$$\rho_1\rho_2 \min(\rho_3, \rho_4) > 1.$$

The network in Figure 2b can be obtained from the one in Figure 2a by removing ξ_5 and the connections $[\xi_2 \to \xi_5]$, $[\xi_5 \to \xi_3]$ and $[\xi_5 \to \xi_4]$. Under the condition that the constants of the global maps near the network are uniformly bounded, the above condition is also applicable to this network.

Example 4. The network shown in Figure 9c, the stability and bifurcations of which were studied in [16], can be obtained from the one in Figure 2a by adding a two-dimensional connection $[\xi_1 \to \xi_5]$. Below we apply apply Theorem 4.15 to this network. The values of ρ_j are:

$$\rho_1 = \frac{-\max(\lambda_{13}, \lambda_{14})}{\lambda_{12}}, \quad \rho_2 = \frac{-\lambda_{21}}{\max(\lambda_{23}, \lambda_{24})}, \quad \rho_3 = \min\left(\frac{-\lambda_{32}}{\lambda_{31}}, 1\right),$$
$$\rho_3 = \min\left(\frac{-\lambda_{42}}{\lambda_{41}}, 1\right), \quad \rho_5 = \frac{\lambda_{52}}{\lambda_{52} - \lambda_{51}}.$$

The network has five subcycles, which implies that by Theorem 4.15 the conditions for stability are:

$$\rho_1\rho_2\rho_3 > 1, \quad \rho_1\rho_2\rho_4 > 1, \quad \rho_1\rho_2\rho_5 > 1, \quad \rho_1\rho_2\rho_5\rho_3 > 1, \quad \rho_1\rho_2\rho_5\rho_4 > 1,$$

which can be combined into

$$\rho_1\rho_2\rho_5 \min(\rho_3, \rho_4, 1) > 1.$$

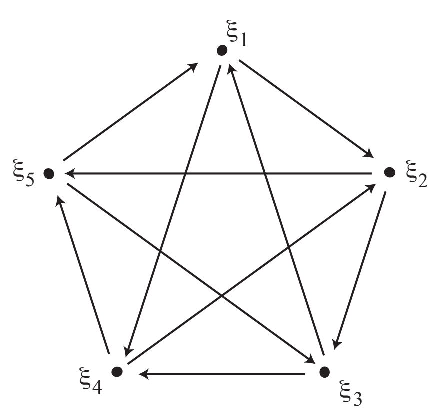
Example 5. The networks $Y_5 \subset \mathbb{R}^5$ and $Y_6 \subset \mathbb{R}^6$ shown in Figure 10 are the ones discussed in Remark 4.18. They have subcycles where all equilibria have f-long vectors as contracting eigenvectors and s-long vectors of the same Δ-clique as expanding eigenvectors. The subcycles are

$$Y_5 \supset X_5 = [\xi_1 \to \xi_4 \to \xi_2 \to \xi_5 \to \xi_3 \to \xi_1]$$

and

$$Y_6 \supset X_6 = [\xi_1 \to \xi_4 \to \xi_2 \to \xi_5 \to \xi_3 \to \xi_6 \to \xi_1],$$

which implies that $\rho^*(X_5) = \rho^*(X_6) = 1$. As we noted, according to [1], for such cycles in the Lotka-Volterra system the sufficient conditions for asymptotic stability are that $|c_j/e_j| > 1$ for any $\xi_j \in Y$.



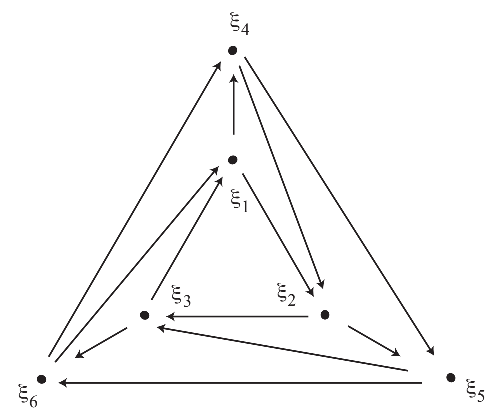


Figure 10: The networks of Example 5.

6 Concluding remarks

In this paper we introduce ac-networks and prove sufficient conditions for their asymptotic stability, which are applicable also in more general cases. Using these results, one can construct various kinds of dynamical systems that have asymptotically stable heteroclinic networks, for example, using a generalisation of the methods proposed in [4] or [6, 28]. Alternatively, given any graph of the structure described in Theorem 3.5, there exists a Lotka-Volterra system that has an associated network. Conditions on the coefficients of the system that ensure the existence of heteroclinic trajectories can be found, e.g., in [22]. Note also that an ac-network can be realised as a network on a simplex or in coupled identical cell system [9].

The conditions for asymptotic stability that we derive are general and have a simple form. This is related to the fact that they are not accurate. In [25] we obtained necessary and sufficient conditions for fragmentary asymptotic stability of a particular heteroclinic network in $\mathbb{R}^6$, which involve rather cumbersome expressions. The study indicated that derivation of conditions for stability of a network in $\mathbb{R}^n$ is a highly non-trivial task, unless in some special cases or if n small. Here simplicity is achieved by compromising in accuracy. It would be of interest to compare the sufficient conditions that we obtained with precise conditions for asymptotic stability that could be derived for some ac-networks.

The present study is also a first step towards considering more general types of heteroclinic networks, such as networks involving unstable manifolds of dimension greater than two, with equilibria not just on coordinate axes, with more than one equilibrium per coordinate axis, networks that are not clean, as well as networks in systems equivariant under the action of symmetry groups other than $\mathbf{Z}^n$.

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