On the occasion of the 60th birthdays of
Maria Aparecida Ruas
and
Terence Gaffney
Cidinha

GEORG-CANTOR-HAUS
Cidinha’s work

The kernel of Cidinha’s work lies in the problem of classification of singularities, specially those finitely determined.

From her work in this problem we can distinguish three main directions:

- Topological and Differential Classification of Singularities.
- Study of the triviality and equisingularity of families of maps.
- Applications to the Extrinsic Geometry of Submanifolds (Generic Geometry) and Dynamical Systems.
Problem

*Classification of germs of functions according to the order of determinacy.*
Problem

Classification of germs of functions according to the order of determinacy.

Main Result

Independent determination of Arnol’d series $A_k$ and $D_k$. 
**Problem**

*Classification of germs of functions according to the order of determinacy.*

**Main Result**

*Independent determination of Arnol’d series $A_k$ and $D_k$.*

That is not Cidinha in the picture!
first joint work...

J.G. Ruas Filho and M.A.S. Ruas, Janaína
first joint work...

J.G. Ruas Filho and M.A.S. Ruas, Janaína
First objective:

**Question**

*Does finite determinacy occur in general?*

(C.T.C. Wall).
First objective:

**Question**

*Does finite determinacy occur in general?*  
(C.T.C. Wall). This naturally led to the next

**Question**

*Are the 1-modal families of finitely determined germs appearing in the boundary of Mather’s nice dimensions topologically stable?*  
(J. Damon).
First objective:

**Question**

*Does finite determinacy occur in general?*

(C.T.C. Wall). This naturally led to the next

**Question**

*Are the 1-modal families of finitely determined germs appearing in the boundary of Mather’s nice dimensions topologically stable?*

(J. Damon). Then — Study of problems in finite determinacy.
In parallel, Cidinha attends a seminar with T. Banchoff, T. Gaffney and C. Mc Crory on Porteous’ work on the generic singularities of distance squared functions. She proposes to study the analogous problem for projections of surfaces in $\mathbb{R}^3$ into planes.
Brown University period: initial work on Generic Geometry

In parallel, Cidinha attends a seminar with T. Banchoff, T. Gaffney and C. Mc Crory on Porteous’ work on the generic singularities of distance squared functions. She proposes to study the analogous problem for projections of surfaces in $\mathbb{R}^3$ into planes.

In collaboration with Gaffney she starts to study this problem. They get the classification of all the germs of codimension $\leq 2$ of projections of a surface onto the plane. They show that, for a residual subset of surfaces in $\mathbb{R}^3$, the family of projections onto planes is locally versal and provide the geometrical interpretation of the singularities (unpublished work).
Cidinha uses infinitesimal methods of classification of singularities and controlled vector fields to address the following problems:

1. **Estimates of the order of determinacy** (participation of L. Wilson)

An example of result is the following:

**Theorem**

*If* $f : (k^n, 0) \rightarrow (k^p, 0)$ *satisfies convenient Lojasiewicz conditions (G-condition), and the coordinate functions are homogeneous, then* $f$ *is $(r - l + 1) - C^l - G$-determined, $1 \leq l \leq \infty$.**

M.A.S. Ruas On The Degree Of $C^l$-Determinacy, *Mathematica Scandinavica* 1986
Cidinha uses infinitesimal methods of classification of singularities and controlled vector fields to address the following problems:

1. **Estimates of the order of determinacy** (participation of L. Wilson)

These results were extended for quasihomogeneous germs.

2. Topological triviality results for the group $\mathcal{A}$.

3. Motivated by A. du Plessis work on the determinacy of smooth maps (Invent. Math. 1980), Cidinha proposes a method for the $\mathcal{A}$-classification of singularities, that relies into the consideration of the structure of the $\mathcal{A}$-orbits inside the $\mathcal{K}$-orbits:
She obtains sufficient conditions for the existence of open $\mathcal{A}$-orbits in a given contact class.
The existence of open $\mathcal{A}$-orbits is a necessary condition for the existence of $\mathcal{A}$-simple germs in a given $\mathcal{K}$-orbit.
Doctoral thesis, 1983

2. Topological triviality results for the group $\mathcal{A}$.

3. Motivated by A. du Plessis work on the determinacy of smooth maps (Invent. Math. 1980), Cidinha proposes a method for the $\mathcal{A}$-classification of singularities, that relies into the consideration of the structure of the $\mathcal{A}$-orbits inside the $\mathcal{K}$-orbits:

This has been used in later works on the classification of $\mathcal{A}$-simple germs as, for instance:
2. Topological triviality results for the group \( \mathcal{A} \).

3. Motivated by A. du Plessis work on the determinacy of smooth maps (Invent. Math. 1980), Cidinha proposes a method for the \( \mathcal{A} \)-classification of singularities, that relies into the consideration of the structure of the \( \mathcal{A} \)-orbits inside the \( \mathcal{K} \)-orbits:

\[ \text{J.H. Rieger, M.A.S. Ruas, Classification of } \mathcal{A} \text{-simple germs from } K^n \text{ to } K^2, \ n \geq p, \ Compositio Mathematica 1991 \]
other joint work at the time

J.G. Ruas Filho and M.A.S. Ruas, José

J.G. Ruas Filho and M.A.S. Ruas, Juliana
other joint work at the time

J.G. Ruas Filho and M.A.S. Ruas, José
J.G. Ruas Filho and M.A.S. Ruas, Juliana
Topological and Differential Classification of Singularities

Marar and Mond studied the topology of finitely determined germs $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, and showed that for a stable perturbation $f_t$ of $f$, the number of Whitney umbrellas of $f_t$, the number of triple points of $f_t$ and the Milnor number of the curve of double points of $f$ are analytical invariants of the germ $f$. These invariants may be computed as the dimension of algebras associated to the singularities.
Topological and Differential Classification of Singularities

Marar and Mond studied the topology of finitely determined germs $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, and showed that for a stable perturbation $f_t$ of $f$, the number of Whitney umbrellas of $f_t$, the number of triple points of $f_t$ and the Milnor number of the curve of double points of $f$ are analytical invariants of the germ $f$. These invariants may be computed as the dimension of algebras associated to the singularities.

Question (Mond)

*Given a family of $A$-finitely determined germs $f_t : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ such that the invariants above are constant, is the family $f_t$ topologically trivial?*
Topological and Differential Classification of Singularities

First result, for a special class of germs from $\mathbb{C}^2$ to $\mathbb{C}^3$, in

M.A.S. Ruas, On equisingularity of families of corank 1 generic germs, Contemporary Maths. 1994

Based on Gaffney’s results on equisingularity of families of $\mathcal{A}$-finite germs, Cidinha has shown that for a generic family of germs with isomorphic local algebras:

i) D. Mond’s invariants are constant along the family,

ii) the family is Whitney equisingular.
Topological and Differential Classification of Singularities

With R. Bedregal and K. Houston, Cidinha answers Mond’s question in the affirmative, as a consequence of a characterization of the equisingularity of families in two and three dimensions:

**Theorem**

*Topological triviality and Whitney equisingularity are equivalent notions for families of finitely determined germs in dimensions 2 and 3. Moreover the Milnor number of the curve of double points in the source is complete invariant for equisingularity.*

---

0-stable invariants for weighted homogeneous, $\mathcal{A}$-finite germs of corank 1, $f : \mathbb{C}^n \to \mathbb{C}^n$, were studied by W.L. Marar, J. Montaldi and Cidinha, who found formulæ for the number of isolated multiple points, in terms of the weights and quasi-homogeneity degree of the germ.

Maximal Deformations

A related problem in the real case is:
Existence of stable perturbations realizing the 0-stable invariants appearing in the complex case ($M$-deformations).

Previous results:

- Gusein-Zade (1975) and A’Campo (1975) (independently) treated the case of germs of plane curves, proving that $M$-deformations always exist.
- Arnol’d (1992) and Entov (1993) showed the existence of $M$-deformations for singularities of type $A_k$ and $D_k$. 
Maximal Deformations
With J. Rieger and R.G. Wik Atique, Cidinha proved the existence of $M$-deformations for $\mathcal{A}$-simple germs of corank 1, in the dimensions $(n, p)$, $n \geq p$, and $(n, n+1)$, for all $n \neq 4$.

These results are the consequence of new classification techniques that refine those of Cidinha’s thesis. They provide necessary conditions for the simplicity of map germs and the analysis of the existence of real $M$-deformations of these singularities.

J.H. Rieger and M.A.S. Ruas, $M$-deformations of $\mathcal{A}$-simple $\Sigma^{n-p+1}$-germs from $\mathbb{R}^n$ to $\mathbb{R}^p$, $n \geq p$, Math. Proc. Cambridge Phil. Soc. 2005

J.H. Rieger, M.A.S. Ruas and R.G. Wik-Atique, $M$-deformations of $\mathcal{A}$-simple germs from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$, Math. Proc. Cambridge Phil. Soc. 2008

Singularities of Real Analytic Varieties

Results on topological triviality and equisingularity of families of sections of analytical varieties were obtained with J. Tomazella.

M.A.S. Ruas and J.N. Tomazella, An infinitesimal criterion for topological triviality of families of sections of analytic varieties, Proc. of MSJ-IRI Symposium ”Singularity theory and its applications”

Singularities of Real Analytic Varieties
With J. Seade, A. Verjovsky and R.N.A. dos Santos, Cidinha obtains
results connected to those of King on the study of the topology of Milnor
fibrations associated to germs of analytic maps with isolated singularities.

M.A.S. Ruas, J. Seade and A. Verjovsky, On real singularities
with a Milnor fibration, Trends in singularities, Birkhäuser 2002

M.A.S. Ruas and R.N.A. dos Santos, Real Milnor Fibrations
and (C)-Regularity, Manuscripta Mathematica 2005
Bi-Lipschitz Triviality

With A.C.G. Fernandes, L. Birbrair and J.C.F. Costa Cidinha obtained:

- A sufficient condition for bi-Lipschitz triviality of deformations of quasihomogeneous map-germs with an isolated singularity at the origin.

- Invariants for bi-Lipschitz contact equivalence of germs.


**Euler Obstruction**

A project with J.P. Brasselet (I.M. Luminy) and N.G. Grulha Junior (post-doc) is the study of the Euler obstruction associated to singular varieties that are discriminant sets of finitely determined germs $f : \mathbb{C}^n, 0 \rightarrow \mathbb{C}^p, 0$. 
**Generic Geometry**

- With D.K. Hayashida Mochida and M.C. Romero-Fuster, Cidinha gets a geometrical description of the singularities of the family of height functions on surfaces immersed in \( \mathbb{R}^4 \). This leads to the introduction of binormal directions on these surfaces and to a new characterization of asymptotic directions and inflection points. The convexity of a surface in 4-space is also characterized in terms of existence of asymptotic directions.

Generic Geometry

With D.K. Hayashida Mochida and M.C. Romero-Fuster, Cidinha gets a geometrical description of the singularities of the family of height functions on surfaces immersed in $\mathbb{R}^4$. This leads to the introduction of binormal directions on these surfaces and to a new characterization of asymptotic directions and inflection points. The convexity of a surface in 4-space is also characterized in terms of existence of asymptotic directions.

This starts the work on Generic Geometry of submanifolds of codimension higher than one in Euclidean space. It is the first of a series of works extending these methods to higher codimensions and to submanifolds of Hyperbolic and Lorentz-Minkowski spaces by several authors.
Generic Geometry

With R. Garcia, D.K. Hayashida Mochida and M.C. Romero-Fuster, Cidinha studies the isolated singularities of the differential implicit equation of the asymptotic lines on a generic surface in $\mathbb{R}^4$. This leads to the obtaining lower bounds for the number of inflection points in terms of the topology of closed surfaces.

**Generic Geometry**

- With R. Garcia, D.K. Hayashida Mochida and M.C. Romero-Fuster, Cidinha studies the isolated singularities of the differential implicit equation of the asymptotic lines on a generic surface in $\mathbb{R}^4$. This leads to the obtaining lower bounds for the number of inflection points in terms of the topology of closed surfaces.

The results obtained in this work extend Feldman’s results on the number of umbilics of generic surfaces in 3-space and lead to the formulation of a generalized form for Caratheodory’s and Loewner’s Conjectures. These results may be seen as a generic proof of these generalized conjectures. Extensions to certain non necessarily generic cases have been later obtained by C. Gutierrez and Ruas, and J.J. Nuño-Ballesteros.
**Generic Geometry**

- With D.K. Hayashida Mochida and M.C. Romero-Fuster, Cidinha extends the concept of binormal directions to submanifolds immersed with codimension 2 in Euclidean spaces, providing also a characterization of the asymptotic directions in terms of height functions. Necessary conditions for convexity and sphericity of these immersions in terms of the global behaviour of the asymptotic directions are also obtained.

**Generic Geometry**

With D.K. Hayashida Mochida and M.C. Romero-Fuster, Cidinha extends the concept of binormal directions to submanifolds immersed with codimension 2 in Euclidean spaces, providing also a characterization of the asymptotic directions in terms of height functions. Necessary conditions for convexity and sphericity of these immersions in terms of the global behaviour of the asymptotic directions are also obtained. This has been the basis for further works on global extrinsic properties of submanifolds immersed with codimension two, (concerning the characterization of semiumblicility and of flatness of the normal bundle) by M.C. Romero-Fuster with F. Sánchez-Bringas and with J.J. Nuño-Ballesteros.
**Generic Geometry**

- With D.K. Hayashida Mochida and M.C. Romero-Fuster, Cidinha introduces the study of asymptotic direction fields on surfaces immersed in $\mathbb{R}^5$, analyzing their generic existence and showing that the singularities of these fields are 2-singular points in the sense of Feldman.

- With M.C. Romero-Fuster and F. Tari, Cidinha determines the equation of the asymptotic curves in terms of the coefficients of the second fundamental form and studies their generic local configurations.

These techniques provide a new tool in the open problem of the existence of 2-singular points on generic surfaces in $\mathbb{R}^5$. 
**Generic Geometry**

- With D.K. Hayashida Mochida and M.C. Romero-Fuster, Cidinha introduces the study of asymptotic direction fields on surfaces immersed in $\mathbb{R}^5$, analyzing their generic existence and showing that the singularities of these fields are 2-singular points in the sense of Feldman.

- With M.C. Romero-Fuster and F. Tari, Cidinha determines the equation of the asymptotic curves in terms of the coefficients of the second fundamental form and studies their generic local configurations.

These techniques provide a new tool in the open problem of the existence of 2-singular points on generic surfaces in $\mathbb{R}^5$.


Applications to Dynamical Systems

With C. Gutierrez, Cidinha obtains a determinacy condition for the index of a vector field on the plane at an isolated singularity, in terms of Newton polyhedra for the vector field. This was used to study generically embedded surfaces in $R^4$.

With A.C. Nabarro, Cidinha continues this study. They obtain sufficient conditions ensuring that the maximal absolute value of the degree of a planar vector field is attained.


A.C. Nabarro and M.A.S. Ruas, Vector fields in $R^2$ with maximal index, Quart. J. Mathematics 2008
Applications - Bifurcation

- Bifurcation problems with symmetry in parameter space - with A.M. Sitta and J.E. Furter
Applications - Bifurcation

- Bifurcation problems with symmetry in parameter space - with A.M. Sitta and J.E. Furter
- Singularities in nerve impulse equations - with I.S. Labouriau

The general formulation of the problem has started a new line in *Mathematical Biology*. The work continues, by I.S. Labouriau with other collaborators - C.M. Rito, H.M. Rodrigues, C.M. Alves-Pinto, P.R.F. Pinto.

MSc Students

A. C. Nogueira-UFU-MG
J.C. Souza Jr - UNIFAL - MG
R. Martins - UEM - Maringá-PR
A.C. Nabarro - ICMC/USP
J. N. Tomazella - UFSCar
M. G. Manoel - ICMC/USP
V. Locci - UNESP - Bauru
R. L. Costa - ICMC-USP
G. Zugliani - Petrobras
MSc, Ph.D. Students

- A. C. G. Fernandes-UFC-CE
- A. C. Nogueira-UFU-MG
- J. C. Souza Jr - UNIFAL - MG
- M. Buosi-UFES-ES
- J. C. F. Costa-IBILCE/UNESP
- J. N. Tomazella-UFSCar
- A. M. Sitta-IBILCE/UNESP
- D. K. Hayashida Mochida-UFSCAR/FEI-SP
- M. J. Saia-ICMC/USP
- S. Mancini-IGCE-UNESP
- N. G. Grulha Junior-ICMC-USP
- L. Sanchez-Challapa-ICMC-USP
- R. N. A. Santos-ICMC-USP
- R. G. Wik Atique-ICMC/USP
- A.C. Nabarro - ICMC/USP
- J. N. Tomazella - UFSCar
- M. G. Manoel - ICMC/USP
- V. Locci - UNESP - Bauru
- R. L. Costa - ICMC-USP
- R. Martins - UEM - Maringá-PR
- M. E. R. Hernandes-UEM-Maringá
MSc, Ph.D., Post-Doc Students

- A. C. G. Fernandes-UFC-CE
- A. C. Nogueira-UFU-MG
- J.C. Souza Jr - UNIFAL - MG
- W.S. Motta Jr - UFU/MG
- M. Buosi-UFES-ES
- J. C. F. Costa-IBILCE/UNESP
- J. N. Tomazella-UFSCar
- A. M. Sitta-IBILCE/UNESP
- D. K. Hayashida Mochida-UFSCAR/FEI-SP
- M. J. Saia-ICMC/USP
- S. Mancini-IGCE-UNESP
- N. G. Grulha Junior-ICMC-USP
- L. Sanchez-Challapa-ICMC-USP
- R. N. A. Santos-ICMC-USP
- R. G. Wik Atique-ICMC/USP
- A.C. Nabarro - ICMC/USP
- J. N. Tomazella - UFSCar
- M. G. Manoel - ICMC/USP
- V. Locci - UNESP - Bauru
- R. L. Costa - ICMC-USP
- R. Martins - UEM - Maringá-PR
- M. E. R. Hernandes-UEM-Maringá

- J. Snoussi - UNAM- Cuernavaca - Mexico
- F. Aroca - UNAM- Cuernavaca - Mexico
- Y. Kurokawa – Shibaura Institute of Technology - Japan
- G. Zugliani - Petrobras
- Y. Kurokawa – Shibaura Institute of Technology - Japan
Collaborators

J. Bruce - U. Liverpool
K. Houston - U. Leeds
J. Montaldi - U. Manchester
F. Tari - U. Durham
J. Rieger - U. Hamburg
J.-P. Brasselet - Luminy
S. Izumiya - U. Hokkaido
O. Saeki - U. Kyushu

L. Birbrair - U. Fos
R. Callejas-Bedregal - UFP
V. Carrara Zanetic - USP
J. Costa - UNESP
R. Santos - USP
W. Marar - USP
C. Mendes - USP
D. Hayashida Mochida - FEI
A. Nabarro - USP
M. Buosi - UFES

T. Gaffney - Northwestern
I. Labouriau - U. Porto
C. Gutiérrez - USP
M. Hernandes - UEM
J. Seade - UNAM
A. Fernandes - UFC
R. Garcia - UFG
C. Gutiérrez - USP
M. Hernandes - UEM
J. Seade - UNAM
S. Izumiya - U. Hokkaido
O. Saeki - U. Kyushu

J. Rieger - U. Hamburg
J.-P. Brasselet - Luminy
S. Izumiya - U. Hokkaido
Many people helped preparing this presentation

Míriam Manoel,

Special thanks to

Ana Cláudia Nabarro

José Gaspar Ruas and.....
Many people helped preparing this presentation
Míriam Manoel,

Special thanks to
Ana Cláudia Nabarro

José Gaspar Ruas and..... Cidinha!