Counting Real Singularities

Workshop Teoria Algébrica de Singularidades

Isabel S. Labouriau
CMUP
Porto, Portugal

Maria Aparecida S.Ruas
ICMC — USP
S. Carlos, SP Brasil

UFF, Niterói, RJ 22 a 24 de Fevereiro de 2010

Set-up

$f : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}^n, 0$ smooth map-germ

finitely determined corank 1 under $\mathcal{A}$-equivalence

general form $f(x,y) = (x, g(x,y)))$ $x \in \mathbf{R}^{n-1}$

Stabilisation of f

a map-germ $F : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}^{n+1}$ of the form

$F(t,x,y) = (t, x, G(x,y,t))$ with $G(x,y,0) = g(x,y)$

such that for $t \neq 0$ the only singularities of $F^t(x,y) = F(t,x,y)$ are of type A_j, $j \leq n$.

Aim

Obtain the number of A_n-singularities in a stabilisation of f.

Simple example

$f : \mathbf{R}^2, 0 \longrightarrow \mathbf{R}^2, 0$

$$f(x, y) = (x, g(x, y)) \quad \text{where} \quad g(x, y) = y^4 + xy$$

Stabilisation

$$F(t, x, y) = (t, x, y^4 + xy - ty^2)$$

Path formulation — Example $(x,y) \mapsto (x,y^4)$

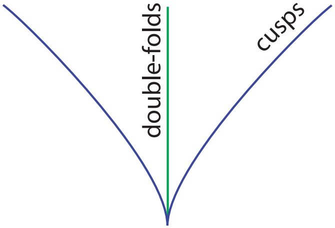
Versal unfolding

$(x,y) \mapsto (x, y^4 + u_1 y - u_2 y^2)$

Bifurcation set

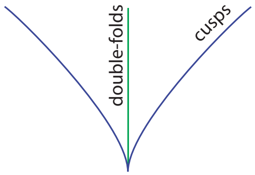
$27u_1^2 - 8u_2^3 = 0$ $\qquad$ $u_1 = 0 \quad u_2 > 0$

cusps A_2 $\qquad$ double-fold (2 A_1, same value)



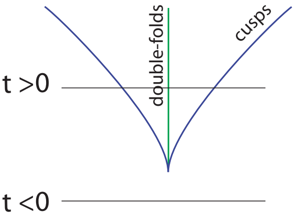
Path formulation — Example $(x, y) \mapsto (x, y^4)$

map-germ		path
$(x, y^4 + xy)$	$\longleftrightarrow$	$p(x) = (u_1, u_2) = (x, 0)$
stabilisation		
$F^t = (x, y^4 + xy - ty^2)$	$\longleftrightarrow$	$P(x) = (u_1, u_2) = (x, t)$



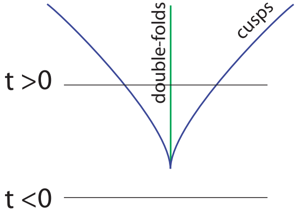
Path formulation — Example $(x, y) \mapsto (x, y^4)$

map-germ		path
$(x, y^4 + xy)$	$\longleftrightarrow$	$p(x) = (u_1, u_2) = (x, 0)$
stabilisation		
$F^t = (x, y^4 + xy - ty^2)$	$\longleftrightarrow$	$P(x) = (u_1, u_2) = (x, t)$



Path formulation — Example $(x, y) \mapsto (x, y^4)$

map-germ		path
$(x, y^4 + xy)$	$\longleftrightarrow$	$p(x) = (u_1, u_2) = (x, 0)$
stabilisation		
$F^t = (x, y^4 + xy - ty^2)$	$\longleftrightarrow$	$P(x) = (u_1, u_2) = (x, t)$



For $t < 0$ two cusps and one double-fold

For $t > 0$ no cusps, no folds.

Upper bounds — well known results:

Given $f(x,y) = (x, g(x,y))$, consider the ideal

$$\mathcal{S}(f) = \left\langle \frac{\partial g}{\partial y}(x,y) \quad \ldots \quad \frac{\partial^n g}{\partial y^n}(x,y) \right\rangle_{\mathcal{E}_{x,y}} \subset \mathcal{E}_{x,y}$$

Lemma

For a finitely determined map-germ $f : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}^n, 0$ *of corank 1,* $\mathcal{S}(f)$ *has finite codimension.*

Upper bounds — well known results:

$$\mathcal{S}(f) = \left\langle \frac{\partial g}{\partial y}(x,y) \quad \dots \quad \frac{\partial^n g}{\partial y^n}(x,y) \right\rangle_{\mathcal{E}_{x,y}} \subset \mathcal{E}_{x,y}$$

Algebraic number of A_n singularities

$$\sigma(f) = \mathrm{cod}_{\mathcal{E}_{x,y}}\left(\mathcal{S}(f)\right)$$

Theorem

For a finitely determined map-germ $f : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}^n, 0$ of corank 1, the number $\sigma(f)$ is an invariant for $\mathcal{A}$-equivalence giving the number of A_n-singularities appearing in a stable deformation of the complexification of f.

$F(t,x,y) = (t,x,G(x,y,t))$ a stabilisation of f.

$$\frac{\partial G}{\partial y}(x,y,t) = 0 \quad \dots \quad \frac{\partial^n G}{\partial y^n}(x,y,t) = 0 \tag{1}$$

Algebraic number of A_n singularities

$\sigma(f) = \mathrm{cod}_{\mathcal{E}_{x,y}}(\mathcal{S}(f))$
counts simple solutions $(x,y) \in \mathbf{C}^n$ of (1) for $t \neq 0$

Effective number of A_n singularities

$\mathbf{s}_+(F)$ / $\mathbf{s}_-(F)$ counts simple solutions $(x,y) \in \mathbf{R}^n$ of (1) for $t > 0$ / $t < 0$.

$\mathbf{s}_+(F)$ and $\mathbf{s}_-(F)$ usually depend on the choice of F.
$\mathbf{s}_+(F)$ and $\mathbf{s}_-(F) \leq \sigma(f)$.

Simple example

$f:\mathbf{R}^2,0 \longrightarrow \mathbf{R}^2,0$

$f(x,y)=(x,g(x,y))$ where $g(x,y)=y^4+xy$

Algebraic number of A_n singularities

$$\mathcal{S}(f)=\langle 4y^3+x, y^2\rangle_{\mathcal{E}_{x,y}} \subset \mathcal{E}_{x,y}$$

$$\sigma(f)=\mathrm{cod}_{\mathcal{E}_{x,y}}(\mathcal{S}(f))=2$$

Effective number of A_n singularities

Stabilisation $F(t,x,y)=(t,x,y^4+xy-ty^2)$

From path formulation:

$\mathbf{s}_+(F)=2$ $\mathbf{s}_-(F)=0$

Questions

Given $f(x, y)$ smooth, finitely determined, corank 1.

Computability

Given $F(t, x, y)$ a stabilisation,
obtain a general method for computing $\mathbf{s}_+(F)$.

Maximality

Show that there is a stabilisation $F(t, x, y)$ for which $\mathbf{s}_+(F) = \sigma(f)$.

or, failing this,

Geometric number of A_n singularities

compute $\mathbf{s}_{max}(f)$ the maximum value of $\mathbf{s}_+(F)$ for any stabilisation F of f.

This is work in progress!

Questions

Given $f(x, y)$ smooth, finitely determined, corank 1.

Computability

Given $F(t, x, y)$ a stabilisation,
obtain a general method for computing $\mathbf{s}_+(F)$.

Maximality

Show that there is a stabilisation $F(t, x, y)$ for which $\mathbf{s}_+(F) = \sigma(f)$.

or, failing this,

Geometric number of A_n singularities

compute $\mathbf{s}_{max}(f)$ the maximum value of $\mathbf{s}_+(F)$ for any stabilisation F of f.

This is work in progress!

Questions

Given $f(x, y)$ smooth, finitely determined, corank 1.

Computability

Given $F(t, x, y)$ a stabilisation,
obtain a general method for computing $\mathbf{s}_+(F)$.

Maximality

Show that there is a stabilisation $F(t, x, y)$ for which $\mathbf{s}_+(F) = \sigma(f)$.

or, failing this,

Geometric number of A_n singularities

compute $\mathbf{s}_{max}(f)$ the maximum value of $\mathbf{s}_+(F)$ for any stabilisation F of f.

This is work in progress!

Questions

Given $f(x, y)$ smooth, finitely determined, corank 1.

Computability

Given $F(t, x, y)$ a stabilisation,
obtain a general method for computing $\mathbf{s}_+(F)$.

Maximality

Show that there is a stabilisation $F(t, x, y)$ for which $\mathbf{s}_+(F) = \sigma(f)$.

or, failing this,

Geometric number of A_n singularities

compute $\mathbf{s}_{max}(f)$ the maximum value of $\mathbf{s}_+(F)$ for any stabilisation F of f.

This is work in progress!

Questions

Given $f(x, y)$ smooth, finitely determined, corank 1.

Computability

Given $F(t, x, y)$ a stabilisation,
obtain a general method for computing $\mathbf{s}_+(F)$.

Maximality

Show that there is a stabilisation $F(t, x, y)$ for which $\mathbf{s}_+(F) = \sigma(f)$.

or, failing this,

Geometric number of A_n singularities

compute $\mathbf{s}_{max}(f)$ the maximum value of $\mathbf{s}_+(F)$ for any stabilisation F of f.

This is work in progress!

Previous results on maximality

- *M*-deformations of plane curve germs (S.M. Gusein-Zade, 1974; N. ACampo, 1975);
- Deformations of $\mathcal{K}$-simple map-germs (J. Damon & A. Galligo, 1976);
- *M*-morsifications of real function-germs (V. I. Arnold, 1992; M.R. Entov, 1993);
- Good real perturbations (T. Cooper, D. Mond & R.W. Atique, 2002; W.L. Marar & D. Mond, 1996);
- *M*- deformations of $\mathcal{A}$-simple map-germs of corank 1 (M.A.S. Ruas & J.H. Rieger, 2005).

Counter example — maximality

M- deformations of $\mathcal{A}$-simple map-germs of corank 2 (M.A.S. Ruas & J.H. Rieger, 2005)

$$f(x,y) = (x^2 - y^2 + x^3, xy)$$

Algebraic number of singularities

Any stabilisation has
3 complex cusps
2 complex double-folds

Effective number of singularities

Any stabilisation has
3 real cusps
0 real double-folds

All known examples have the maximum number of cusps!

Theorem

If $F(t,x,y) = (t,x,G(x,y,t))$ is a stabilisation of $f(x,y)$, a smooth, finitely determined, corank 1, map-germ $f : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}^n, 0$, then

- $\mathbf{s}_+(F) + \mathbf{s}_-(F) = 2\,\mathrm{degree}(\Phi_1)$;
- $\mathbf{s}_+(F) - \mathbf{s}_-(F) = 2\,\mathrm{degree}(\Phi_2)$.

where Φ_1 and $\Phi_2 : \mathbf{R}^{n+1}, 0 \longrightarrow \mathbf{R}^{n+1}, 0$ are

$$\Phi_1(x,y,t) = \left(\frac{\partial G}{\partial y}, \ldots, \frac{\partial^n G}{\partial y^n}, tJ_{(x,y)}H\right)$$

$$\Phi_2(x,y,t) = \left(\frac{\partial G}{\partial y}, \ldots, \frac{\partial^n G}{\partial y^n}, J_{(x,y)}H\right)$$

and $H : \mathbf{R}^{n+1}, 0 \longrightarrow \mathbf{R}^n, 0$ $\qquad J_{(x,y)}H = \det(D_{(x,y)}H)$

$$H(x,y,t) = \left(\frac{\partial G}{\partial y}, \ldots, \frac{\partial^n G}{\partial y^n}\right) .$$

Corollary

If F is a stabilisation of a finitely determined map-germ $f : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}^n, 0$ of corank 1, then

$$\mathbf{s}_+(F) \equiv \mathbf{s}_-(F) \pmod 2 .$$

Similar methods count:

- Branches in bifurcation problems (T. Nishimura, T. Fukuda & K. Aoki, 1989);
- Branches in symmetric bifurcations (J. Damon, 1991);
- Pitchforks in symmetric bifurcations (S.D.S. Castro & L. 2000);
- Folds in stabilisations of bifurcation diagrams (L. & R. 2006).

Simple example

$f : \mathbf{R}^2, 0 \longrightarrow \mathbf{R}^2, 0$

$$f(x,y) = (x, g(x,y)) \quad \text{where} \quad g(x,y) = y^4 + xy$$

Stabilisation: $F(t,x,y) = (t, x, y^4 + xy - ty^2) = (t, x, G)$

$H(x,y,t) = \left(\partial G/\partial y, \partial^2 G/\partial y^2\right)$

$$H(x,y,t) = \left(4y^3 + x - 2yt, 12y^2 - 2t\right) \qquad J_{xy}H = 24y$$

$$\Phi_1 = \begin{pmatrix} 4y^3 + x - 2yt \\ 12y^2 - 2t \\ 24yt \end{pmatrix}^T \qquad \Phi_2 = \begin{pmatrix} 4y^3 + x - 2yt \\ 12y^2 - 2t \\ 24y \end{pmatrix}^T$$

$\Phi : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}^{n+1}$ η a regular value of Φ

$$\text{degree}(\Phi) = \sum_{\xi \in \Phi^{-1}(\eta)} \text{sign} J\Phi(\xi) \ .$$

$$\Phi_2 = \begin{pmatrix} 4y^3 + x - 2yt \\ 12y^2 - 2t \\ 24y \end{pmatrix}^T \qquad J_{(x,y,t)}\Phi_2 = 48$$

$\text{degree}(\Phi_2) = 1$

$$\Phi_1 = \begin{pmatrix} 4y^3 + x - 2yt \\ 12y^2 - 2t \\ 24yt \end{pmatrix}^T \qquad J_{(x,y,t)}\Phi_1 = 24(24y^2 + 2t)$$

regular value $(0, -2, 0)$ at $(x, y, t) = (0, -2, 0)$
$\text{degree}(\Phi_1) = 1$
$\mathbf{s}_+(F) = 2$ $\quad \mathbf{s}_-(F) = 0$ as in path formulation.

Theorem

If $F(t,x,y) = (t,x,G(x,y,t))$ is a stabilisation of $f(x,y)$, a smooth, finitely determined, corank 1, map-germ $f : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}^n, 0$, then

- $\mathbf{s}_+(F) + \mathbf{s}_-(F) = 2\,\mathrm{degree}(\Phi_1)$;
- $\mathbf{s}_+(F) - \mathbf{s}_-(F) = 2\,\mathrm{degree}(\Phi_2)$.

where Φ_1 and $\Phi_2 : \mathbf{R}^{n+1}, 0 \longrightarrow \mathbf{R}^{n+1}, 0$ are

$$\Phi_1(x,y,t) = \left(\frac{\partial G}{\partial y}, \dots, \frac{\partial^n G}{\partial y^n}, tJ_{(x,y)}H\right)$$

$$\Phi_2(x,y,t) = \left(\frac{\partial G}{\partial y}, \dots, \frac{\partial^n G}{\partial y^n}, J_{(x,y)}H\right)$$

and $H : \mathbf{R}^{n+1}, 0 \longrightarrow \mathbf{R}^n, 0$ $\qquad J_{(x,y)}H = \det(D_{(x,y)}H)$

$$H(x,y,t) = \left(\frac{\partial G}{\partial y}, \dots, \frac{\partial^n G}{\partial y^n}\right) .$$

Idea of proof

$F(t,x,y) = (t,x,G(x,y,t))$ a stabilisation of f.

$H : \mathbf{R}^{n+1},0 \longrightarrow \mathbf{R}^n$ be

$$H(x,y,t) = \left(\frac{\partial G}{\partial y}(x,y,t), \ldots, \frac{\partial^n G}{\partial y^n}(x,y,t) \right) \ .$$

A_n-singularities in F are branches of

$$\{(x,y,t) : \quad H(x,y,t) = 0 \quad (x,y,t) \neq (0,0,0)\} \ .$$

Counting branches (for proof)

$$\begin{array}{rccc} H: & \mathbf{R}^n \times \mathbf{R}, (0,0) & \longrightarrow & \mathbf{R}^n, 0 \\ & (z,t) & \longrightarrow & H(z,t) \end{array}$$

Branches of germ of $H^{-1}(0) - \{(0,0)\}$

$\mathbf{r}_+(H) =$ branches $t > 0$ $\mathbf{r}_-(H) =$ branches $t < 0$

Theorem (Nishimura, Fukuda & Aoki, 1989)

$H : \mathbf{R}^{n+1}, 0 \longrightarrow \mathbf{R}^n$ *a smooth germ with*

$$\dim_{\mathbf{R}} \left(\frac{\mathcal{E}_{(z,t)}}{\langle H_1, \ldots, H_n, J_z H \rangle} \right) < \infty$$

Then

- $\mathbf{r}_+(H) + \mathbf{r}_-(H) = 2\,\mathrm{degree}(H_1, \ldots, H_n, tJ_z H);$
- $\mathbf{r}_+(H) - \mathbf{r}_-(H) = 2\,\mathrm{degree}(H_1, \ldots, H_n, J_z H).$

where $J_z H(z,t) = \det D_z H(z,t)$

Computing the degreee

$\Phi_j : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}^{n+1} \qquad \Phi_j = (\Phi_{j,1}, \dots, \Phi_{j,n+1})$

ring

$$Q(\Phi_j) = \frac{\mathcal{E}_{(x,y,t)}}{\langle \Phi_{j,1}, \dots, \Phi_{j,n+1} \rangle_{\mathcal{E}_{(x,y,t)}}}$$

Theorem

If F is a stabilisation of a finitely determined map-germ $f : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}^n, 0$ of corank 1, then $\mathbf{s}_+(F)$ and $\mathbf{s}_-(F)$ are

$$\dim_{\mathbf{R}} Q(\Phi_1) - 2\dim_{\mathbf{R}} I_1 \pm (\dim_{\mathbf{R}} Q(\Phi_2) - 2\dim_{\mathbf{R}} I_2))$$

where each $I_j \subset Q(\Phi_j)$ is an ideal that is maximal with respect to the property $I_j^2 = 0$.

Computing the degreee (for proof)

$\Phi : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}^{n+1} \qquad \Phi = (\Phi_1, \ldots, \Phi_{n+1})$

ring

$$Q(\Phi) = \frac{\mathcal{E}_{(x,y,t)}}{\langle \Phi_1, \ldots, \Phi_{n+1} \rangle_{\mathcal{E}_{(x,y,t)}}}$$

Theorem (Eisenbud & Levine 1977)

$\Phi : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}^{n+1}$ a finite map-germ with local ring $Q(\Phi)$. If $I \subset Q(\Phi)$ is an ideal that is maximal with respect to the property $I^2 = 0$ then

$$|\mathrm{degree}(\Phi)| = \dim_{\mathbf{R}}(Q(\Phi)) - 2\dim_{\mathbf{R}}(I) \ .$$

Computing the degreee

$\Phi_j : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}^{n+1} \qquad \Phi_j = (\Phi_{j,1}, \ldots, \Phi_{j,n+1})$

ring

$$Q(\Phi_j) = \frac{\mathcal{E}_{(x,y,t)}}{\langle \Phi_{j,1}, \ldots, \Phi_{j,n+1} \rangle_{\mathcal{E}_{(x,y,t)}}}$$

socle $s_j = J_{(x,y,t)} \Phi_j$

$\lambda_j : Q(\Phi_j) \longrightarrow \mathbf{R}$ linear $l_j(s_j) > 0$.

$L_j : Q(\Phi_j) \times Q(\Phi_j) \longrightarrow \mathbf{R}$ bilinear $L_j(p, q) = \lambda_j(pq)$.

Theorem

If F is a stabilisation of a finitely determined map-germ $f : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}^n, 0$ of corank 1, then

$$\mathbf{s}_+(F) = signature(L_1) + signature(L_2) \ .$$

Computing the degreee (for proof)

$\Phi : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}^{n+1} \qquad \Phi = (\Phi_1, \ldots, \Phi_{n+1})$

ring

$$Q(\Phi) = \frac{\mathcal{E}_{(x,y,t)}}{\langle \Phi_1, \ldots, \Phi_{n+1} \rangle_{\mathcal{E}_{(x,y,t)}}}$$

socle $s = J_{(x,y,t)}\Phi$

$\lambda : Q(\Phi) \longrightarrow \mathbf{R}$ linear $l(s) > 0$.

$L : Q(\Phi) \times Q(\Phi) \longrightarrow \mathbf{R}$ bilinear $L(p,q) = \lambda(pq)$.

Theorem (Eisenbud & Levine 1977)

$\Phi : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}^{n+1}$ *a finite map-germ with local ring* $Q(\Phi)$ *then*

$$\mathrm{degree}(\Phi) = \mathit{signature}(L) \ .$$

Estimates

We had, for F a stabilisation of f:

$$\mathbf{s}_+(F), \mathbf{s}_-(F) \leq \sigma(f)$$

$$\mathbf{s}_+(F) = \mathrm{degree}(\Phi_1) + \mathrm{degree}(\Phi_2)$$

$$\mathbf{s}_-(F) = \mathrm{degree}(\Phi_1) - \mathrm{degree}(\Phi_2)$$

$$\mathrm{degree}(\Phi_j) = \dim_{\mathbf{R}} Q(\Phi_j) - 2\dim_{\mathbf{R}} I_j < \dim_{\mathbf{R}} Q(\Phi_j)$$

Theorem

If F is a stabilisation of a finitely determined map-germ $f : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}^n, 0$ of corank 1, then

$$\dim_{\mathbf{R}} Q(\Phi_1) - \dim_{\mathbf{R}} Q(\Phi_2) = \sigma(f)$$

Estimates — for proof

$H = (\frac{\partial G}{\partial y}, \ldots, \frac{\partial^n G}{\partial y^n}) \qquad Q(\Phi_1) = \frac{\mathcal{E}_{(x,y,t)}}{\langle H, tJ_{(x,y)}H \rangle}$

from intersection multiplicity:

$$\dim_{\mathbf{R}} Q(\Phi_1) = \dim_{\mathbf{R}} \underbrace{\frac{\mathcal{E}_{(x,y,t)}}{\langle H, J_{(x,y)}H \rangle}}_{Q(\Phi_2)} + \underbrace{\dim_{\mathbf{R}} \frac{\mathcal{E}_{(x,y,t)}}{\langle H, t \rangle}}_{\sigma(f)}$$

get

$$\dim_{\mathbf{R}} Q(\Phi_1) - \dim_{\mathbf{R}} Q(\Phi_2) = \sigma(f) \qquad \square$$

because $\frac{\partial^j G}{\partial y^j}(x, y, t) \equiv \frac{\partial^n g}{\partial y^n}(x, y) \pmod{\langle t \rangle}$ implies

$$\frac{\mathcal{E}_{(x,y,t)}}{\left\langle \frac{\partial G}{\partial y}, \ldots, \frac{\partial^n G}{\partial y^n}, t \right\rangle} \sim \frac{\mathcal{E}_{(x,y)}}{\left\langle \frac{\partial g}{\partial y}, \ldots, \frac{\partial^n g}{\partial y^n} \right\rangle} = \frac{\mathcal{E}_{(x,y)}}{\mathcal{S}(f)}$$

Where to, now?

Multisingularities

Results may be extended using formulation of W.L. Marar, J. Montaldi & M.A.S. Ruas, 1999

Special situations

- weighted homogeneous
- symmetric
- dimension $= 2$

Go back to maximality problem!

THE END

References

- N. A'Campo, 1975, *Le groupe de monodromie du déploiement des singularités isolées de courbes planes. I*, Maht. Ann. **213** 1–32
- V.I. Arnol'd, 1992, *Springer numbers and Morsification spaces*, J. Algebraic. Geom. **1** 197–214
- S.B.S.D. Castro, I.S. Labouriau, 2000, *Counting persitent pitchforks*, in *Real and Complex Singularities*CRC Press, 215–222
- T. Cooper, D. Mond and R.W. Atique, 2002, *Vanishing topology of codimension 1 multi-germs over* **R** *and* **C**, Compositio Math. 131 (2002)
- J. Damon, 1991, *Equivariant bifurcations and Morsifications for finite groups* in *Singularity theory and its applications, Part II* , SLNM **1463** 80–106

References (cont.)

- J. Damon, 1991, *On the number of branches for real and complex weighted homogeneous curve singularities*, Topology, **30** 223–229
- J. Damon and A. Galligo, 1976, *A topological invariant for stable map germs*, Inventiones Math. **32** 103–132
- D. Eisenbud and H.I. Levine, 1977, *An algebraic formula for the degree of a* C^∞ *map-germ*, Ann.Math. **106** 19–38
- M.R. Entov, 1993, *On real Morsifications of* D_μ *singularities*, Dokl. Math. **46** 25–29
- S.M. Gusein-Zade, 1974, *Dynkin diagrams of singularities of functions of two variables*, Funct. Anal. Appl. **8**, 295–300
- I.S. Labouriau and M.A.S. Ruas, 2006, *Invariants for bifurcations*, Houston J. of Mathematics **32–2** 445–458

References (cont.)

- W.L. Marar, J.A. Montaldi and M.A.S. Ruas, 1999, *Multiplicities of zero-schemes in quasihomogeneous corank-1 singularities* $\mathbf{C}^n \to \mathbf{C}^n$, 353–367 in: *Singularity theory*, LMS Lec. Notes 263
- W.L. Marar and D. Mond, 1996, *Real topology of map-germs with good perturbations*, Topology **35** 157–165
- T. Nishimura, T. Fukuda and K. Aoki, 1989, *An algebraic formula for the topological types of one parameter bifurcation diagrams*, Arch. Rat. Mech. Anal. **108** 247–265
- J.H. Rieger and M.A.S. Ruas, 2005, *M- deformation of* $\mathcal{A}$*-simple* Σ^{n-p+1}*-germs from* $\mathbf{R}^n$ *to* $\mathbf{R}^p$, $n \geq p$, Math. Proc. Camb. Phil. Soc. **139** 333–349