Periodic perturbation of cyclic dynamics

Workshop on Long-Term Dynamics: Attraction and Stability Properties of Invariant Sets

SIAM Conference on Applications of Dynamical Systems

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Periodic perturbation of cyclic dynamics

Joint work with

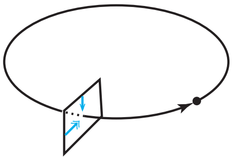
Alexandre A. P. Rodrigues — ISEG and CMUP — Portugal

Outline

General setting for forced cyclic dynamics

Example: heteroclinic network in $\mathbb{R}^3$

Setting — cyclic dynamics and periodic forcing



$\dot{X} = F(X) \quad X \in \mathbb{R}^3$

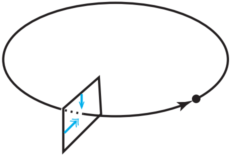
With an attracting flow-invariant set $\sim \mathbf{S}^1$

First return map on a section.

Contracting real eigenvalues $\lambda_1 \ll \lambda_2$.

Reduce to 1-dimensional map $g(y)$.

Setting — cyclic dynamics and periodic forcing



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Periodic forcing

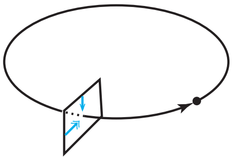
$$(*) \begin{cases} \dot{X} = F(X) + f(s, \gamma) \\ \dot{s} = \omega \pmod{2\pi}. \end{cases}$$

$f(s, \gamma) \quad 2\pi$-periodic in s.

First return map $G(s, y)$ on a cylinder

$\mathcal{C} = \{(s, y) : \ s \in \mathbb{R} \pmod{2\pi}, \quad y > 0\}$

Setting — cyclic dynamics and periodic forcing



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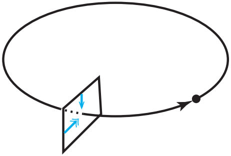
Bifurcation diagrams

Fixed points of $\mathcal{G}(s, y) = G(s, y) - (T, 0)$ $\qquad 0 \leq T$

Setting — cyclic dynamics and periodic forcing



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Contracting real eigenvalues $\lambda_1 \ll \lambda_2$.

Reduce to 1-dimensional map $g(y)$.

Periodic forcing

$$(*) \begin{cases} \dot{X} = F(X) + f(s,\gamma) \\ \dot{s} = \omega \pmod{2\pi}. \end{cases} \qquad f(s,\gamma) \quad 2\pi\text{-periodic in } s.$$

First return map $G(s,y)$ on a cylinder

$\mathcal{C} = \{(s,y): \ s \in \mathbb{R} \pmod{2\pi}, \quad y > 0\}$

Bifurcation diagrams

Fixed points of $\mathcal{G}(s,y) = G(s,y) - (T,0)$ $\qquad 0 \leq T$

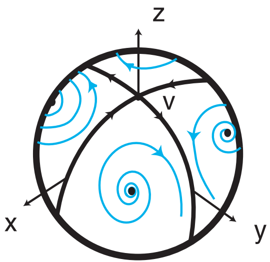
Frequency locking

Fixed points of $\mathcal{G}$ — periodic solutions of $(*)$ if $T = 2n\pi/\omega$, $n \in \mathbb{N}$

Example: heteroclinic network in $\mathbb{R}^3$

$\dot{X} = F(X) \quad X \in \mathbb{R}^3$

With an attracting invariant sphere[1] containing a heteroclinic network



[1] Aguiar *et al* Int. J. Bif. Chaos, **16** No. **2**, 369–381, 2006

Example: heteroclinic network in $\mathbb{R}^3$

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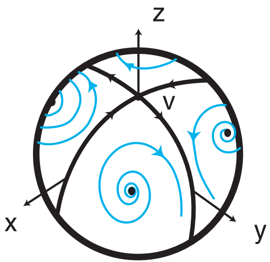


With an attracting invariant sphere[1] containing a heteroclinic network

First return map on a section on one of the connections.

Radial attraction is stronger, reduce to 1-dimensional section.

$\delta > 1$ contraction/expansion rate at the equilibria.

[1] Aguiar *et al* Int. J. Bif. Chaos, **16** No. **2**, 369–381, 2006

Periodic forcing

$$\begin{cases} \dot{X} = F(X) + (\gamma(1-x)\sin(s), 0, 0) \\ \dot{s} = 2\omega \pmod{2\pi}. \end{cases}$$

First return map
$G(s, y) = \left(s - \frac{\ln y}{K},\ y^{p_1(\delta)} + \gamma y^{p_2(\delta)}(1 + k \sin s)\right)$
defined on a cylinder $\mathcal{C} = \{(s, y) :\ s \in \mathbb{R} \pmod{2\pi}, \quad y > 0\}$

Attracting heteroclinic network

$\delta > 1$ contraction/expansion rate

$p_1(\delta) = \delta^2, \quad p_2(\delta) = \delta^2 - \delta$

γ controls the amplitude of forcing

The cylinder map — periodic points

Fixed points of $\mathcal{G}(s,y) = G(s,y) + \left(\frac{\ln\tau}{K}, 0\right)$ $\qquad 0 < \tau \le 1$

$$\mathcal{G}(s,y) = \left(s - \frac{\ln y}{K} + \frac{\ln\tau}{K}, y^{p_1(\delta)} + \gamma y^{p_2(\delta)}(1 + k\sin s)\right) = (s,y)$$

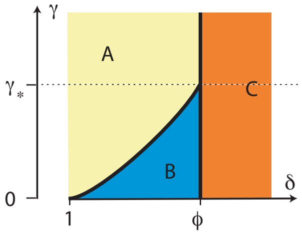
$\tau = 1$ — constant solutions

$\lim_{\tau\to 0}$ — solutions whose period tends to ∞.

$p_1(\delta) = \delta^2, \quad p_2(\delta) = \delta^2 - \delta$

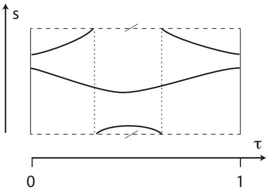
Solutions: $\tau = y$ $\qquad \tau^{-\delta^2+\delta+1} - \tau^{\delta} = \gamma(1 + k\sin s)$

Fixed points of $\mathcal{G}(s, y)$ $\quad k > 1 \quad \phi = (1 + \sqrt{5})/2$

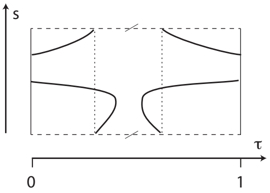


transition diagram

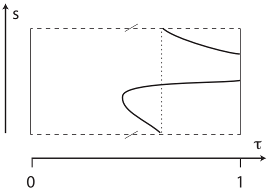
$\gamma_* = 1/(1 + k)$



region A

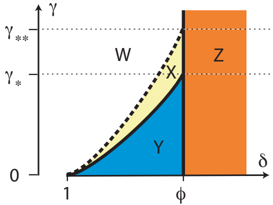


region B



region C

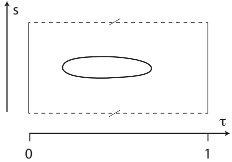
Fixed points of $\mathcal{G}(s, y)$ $\quad 0 < k < 1 \quad \phi = (1 + \sqrt{5})/2$



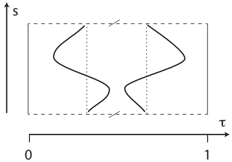
transition diagram

$\gamma_* = 1/(1 + k)$

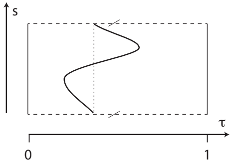
$\gamma_{**} = 1/(1 - k)$



bifurcation diagram
region X



bifurcation diagram
region Y



bifurcation diagram
region Z

Back to the original problem — frequency locking

Recall

$$(*)\begin{cases} \dot{X} = F(X) + (\gamma(1-x)\sin(s), 0, 0) \\ \dot{s} = 2\omega \pmod{2\pi}. \end{cases}$$

First return map reduced to

$$T = -\frac{\ln \tau}{K}$$

$$\mathcal{G}(s,y) = \left(s - \frac{\ln y}{K}, y^{p_1(\delta)} + \gamma y^{p_2(\delta)}(1 + k\sin s)\right) - (T, 0) = (s, y)$$

Fixed points of $\mathcal{G}$ corresponds to two solutions of $(*)$ when

$$T = n\pi/\omega,\ n \in \mathbb{N}$$

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$$(*)\begin{cases} \dot{X} = F(X) + (\gamma(1-x)\sin(s), 0, 0) \\ \dot{s} = 2\omega \pmod{2\pi}. \end{cases}$$

First return map reduced to $\qquad T = -\dfrac{\ln\tau}{K}$

$$\mathcal{G}(s,y) = \left(s - \frac{\ln y}{K}, y^{p_1(\delta)} + \gamma y^{p_2(\delta)}(1 + k\sin s)\right) - (T, 0) = (s, y)$$

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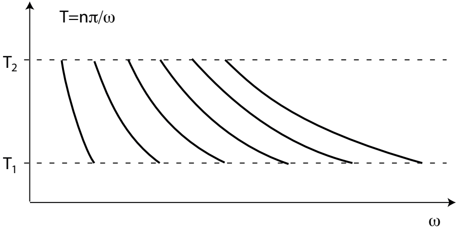


region (X)

Frequency locking

Theorem

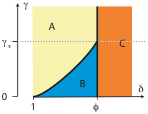
In the forced heteroclinic network, there are two values $0 < T_1 < T_2$ such that there are two frequency locked solutions of $(*)$ with period $n\pi/\omega$, $n \in \mathbb{N}$ in the following cases:

If $k > 1$

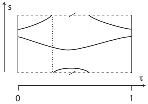
(A) *for all ω if $\gamma > 1/(1+k)$ and $1 < \delta < \phi$;*

(B) *for $\omega \notin [n\pi/T_2, n\pi/T_1]$ if $\gamma < 1/(1+k)$ and $1 < \delta < \phi$;*

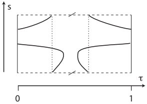
(C) *for $\omega > n\pi/T_1$ if $\delta > \phi$.*



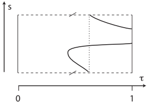
$\gamma_* = 1/(1+k)$



region A



region B



region C

Frequency locking

Theorem

In the forced heteroclinic network, there are two values $0 < T_1 < T_2$ such that there are two frequency locked solutions of $(*)$ with period $n\pi/\omega$, $n \in \mathbb{N}$ in the following cases:

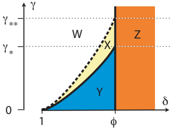
If $0 < k < 1$ and $1 < \delta < \phi$

(X) *for $\omega \in (n\pi/T_2, n\pi/T_1)$ if $1/(1+k) < \gamma < 1/(1-k)$;*

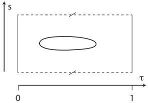
(Y) *for $\omega \notin [n\pi/T_2, n\pi/T_1]$ if $\gamma < 1/(1+k)$;*

and if $\delta > \phi$

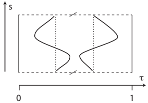
(Z) *for $\omega \in (n\pi/T_2, n\pi/T_1)$.*



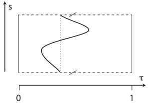
$\gamma_* = 1/(1+k)$
$\gamma_{**} = 1/(1-k)$



region X

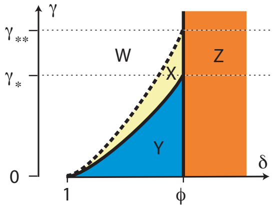


region Y

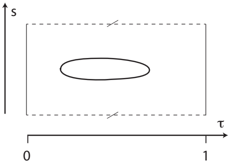


region Z

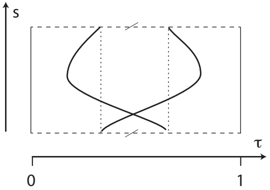
Bifurcation of the cylinder map $0 < k < 1$ $\delta < \phi$



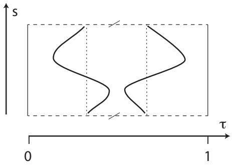
transition diagram



bifurcation diagram
region X

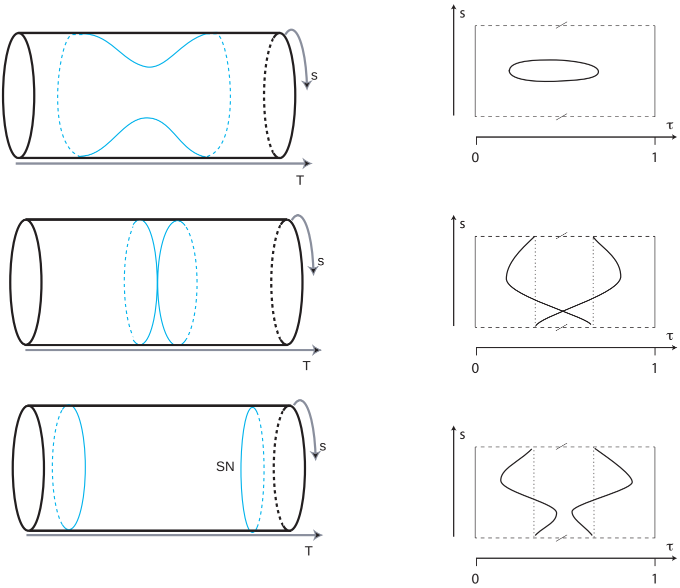


bifurcation diagram
transition X $\leftrightarrow$ Y

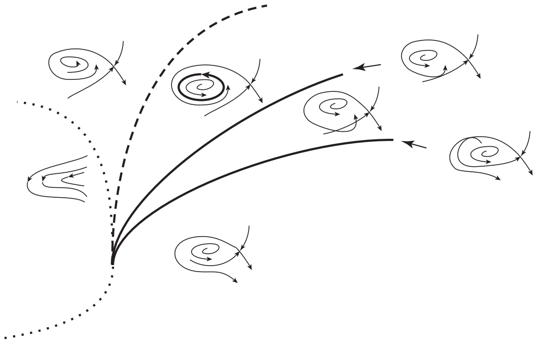


bifurcation diagram
region Y

Bifurcation of the cylinder map



Discrete-time Bogdanov-Takens bifurcation[2]

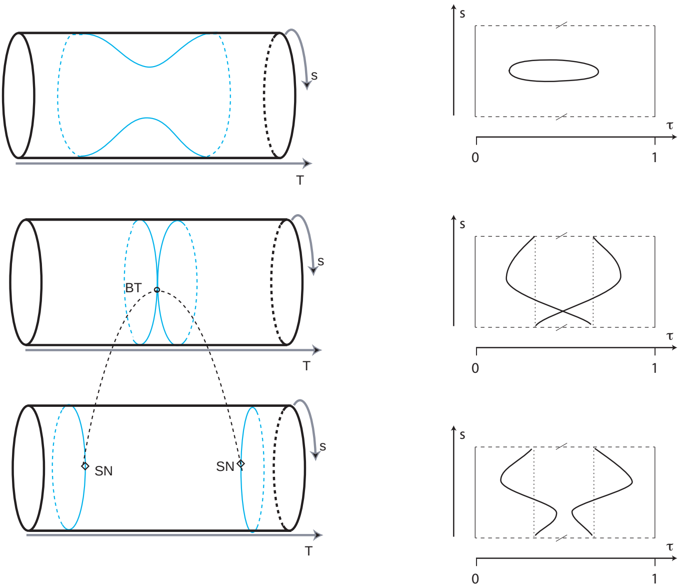


Features

- Invariant circle
- Homoclinic tangencies
- Horseshoe dynamics

[2] H. Broer, R. Roussarie, C. Simó (1996)

Bifurcation of the cylinder map



Frequency locking

Theorem

In the forced heteroclinic network, if $0 < k < 1$ *and if* $1 < \delta < \phi$ *there are values of* $\gamma < 1/(1+k)$ *such that*

- *for* $T \in (T_{H_1}, T_{H_2})$
  where there is a closed, $\mathcal{G}(s, y)$*-invariant curve,*
  for $\omega \in (n\pi/T_{H_2}, n\pi/T_{H_1})$, $n \in \mathbb{N}$,
  there is a frequency locked invariant torus for $(*)$.
- *for* $T \in (T_{h_1}, T_{h_2})$
  where the dynamics of $\mathcal{G}(s, y)$ *is conjugate to a shift*
  for $\omega \in (n\pi/T_{h_2}, n\pi/T_{h_1})$, $n \in \mathbb{N}$,
  there is a frequency locked suspended horseshoe for $(*)$.

References

- Model with heteroclinic network taken from:
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- Weakly attracting case in:
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  T.-L. Tsai, J.H.P. Dawes, Physica D, **262** 14–34, 2013
- Homoclinic loops in:
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- Periodic orbit in:
  Q. Wang, L.-S. Young, Comm. Math. Phyis. **2**25 275–304, 2002

Thank you for your attention

THE END