Dynamics near heteroclinic networks

ICMC Summer Meeting in Differential Equations celebrating the 70th birthday of Plácido Zoega Táboas

Isabel S. Labouriau

Centro de Matemática da Universidade do Porto
Portugal

São Carlos, 8–10 February 2010

Dynamics near heteroclinic networks

Joint work with

- Manuela A.D. Aguiar
  Faculdade de Economia da Universidade do Porto
- Sofia B.S.D. Castro
  Faculdade de Economia da Universidade do Porto

and

- Alexandre A.P. Rodrigues
  Faculdade de Ciências da Universidade do Porto

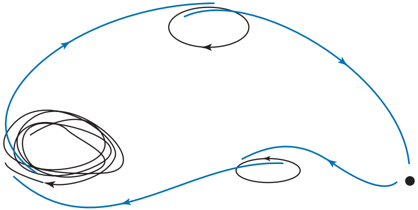
Outline

Context: smooth vector fields on $\mathbf{R}^n$ or on a smooth manifold ordinary differential equations.

Heteroclinic cycle

Finite set of flow-invariant hyperbolic objects (nodes)

trajectories joining them (connections) in a cycle



Context: smooth vector fields on $\mathbf{R}^n$ or on a smooth manifold ordinary differential equations.

Heteroclinic cycle

Finite set of flow-invariant hyperbolic objects (nodes)

trajectories joining them (connections) in a cycle

Special context here:

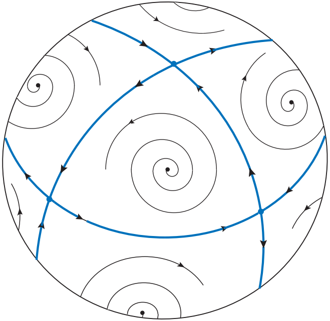
- 3 dimensional phase-space;
- nodes are either equilibria or closed trajectories;

Special contexts for persistent heteroclinic cycles:
(flow-invariant subspaces)

- game theory, economics;
- population dynamics;
- coupled cell networks;
- equations with symmetry.

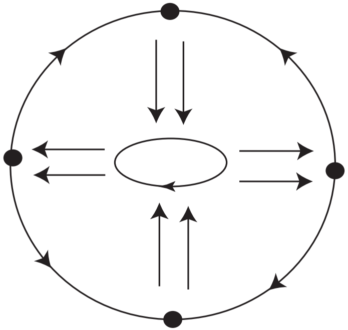
Heteroclinic network

Connected set, finite union of heteroclinic cycles



Heteroclinic network

Connected set, finite union of heteroclinic cycles



Network questions

Dynamics near a network

- switching;
- cycling;

Persistence of dynamics when the network is broken.

Network questions

Dynamics near a network

- switching;
- cycling;

Persistence of dynamics when the network is broken.

A rotating node is:

either

a hyperbolic non-trivial closed trajectory of saddle type

or

an equilibrium of saddle type, with eigenvalues

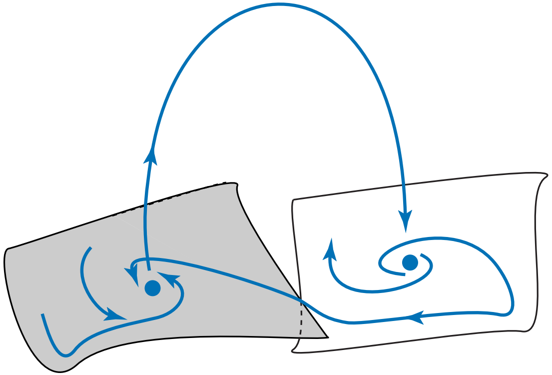
$$A \in \mathbf{R} \quad \text{and} \quad B \pm iC$$

where

$$AB < 0 \qquad AB \neq -1 \quad \text{and} \quad C \neq 0$$

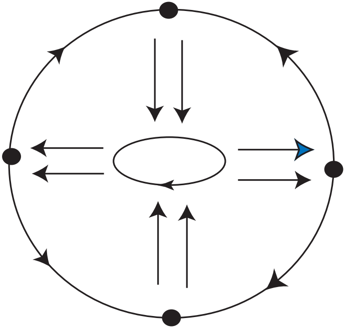
A network of rotating nodes satisfies:

- all the nodes are rotating
- all connections that take place in 2-dimensional invariant manifolds occur as transverse intersections.



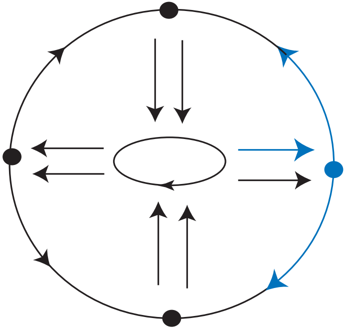
Switching at a node

Given a connection on the network arriving at the node
for every connection coming out of the node
there is a trajectory that shadows each one of the of connections coming out.



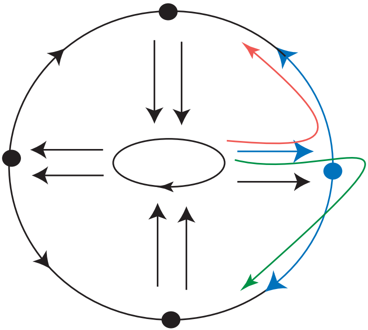
Switching at a node

Given a connection on the network arriving at the node
for every connection coming out of the node
there is a trajectory that shadows each one of the of connections
coming out.



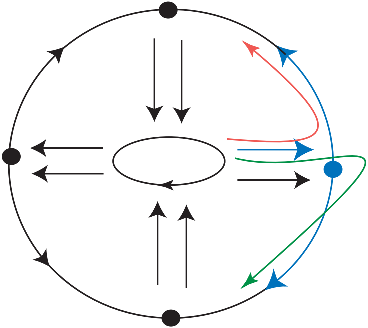
Switching at a node

Given a connection on the network arriving at the node
for every connection coming out of the node
there is a trajectory that shadows each one of the of connections coming out.



Switching at a node

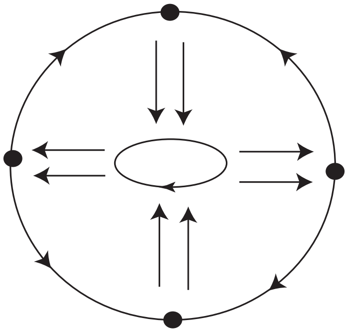
Given a connection on the network arriving at the node
for every connection coming out of the node
there is a trajectory that shadows each one of the of connections coming out.



For hyperbolic nodes, follows from λ-lemma.

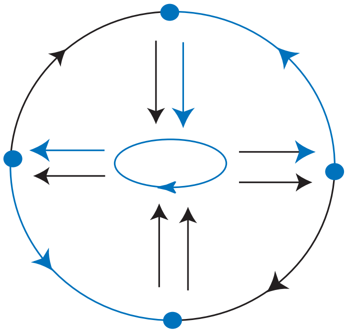
Finite switching near the network

Given any finite path on the network
there is a trajectory that shadows it.



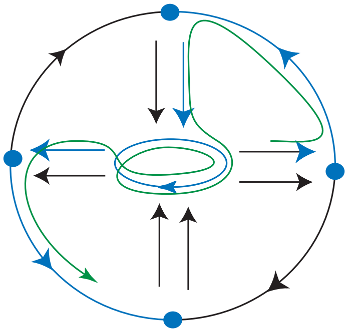
Finite switching near the network

Given any finite path on the network
there is a trajectory that shadows it.



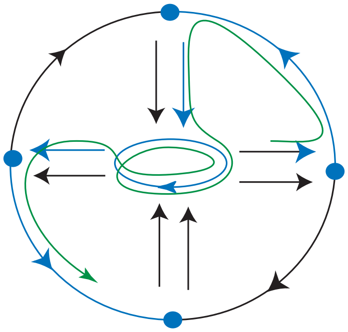
Finite switching near the network

Given any finite path on the network
there is a trajectory that shadows it.



Finite switching near the network

Given any finite path on the network
there is a trajectory that shadows it.



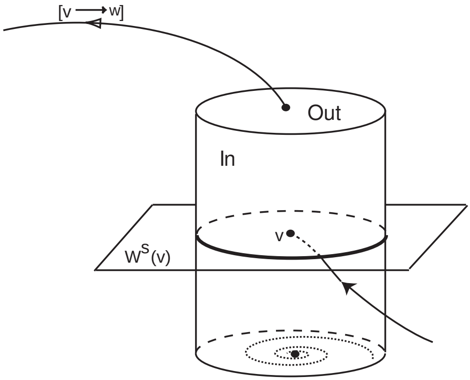
Infinite switching near the network: every infinite path is shadowed.

Theorem

For a vector field on the 3-sphere $\mathbf{S}^3$ *with a network of rotating nodes there is infinite switching on the network.*

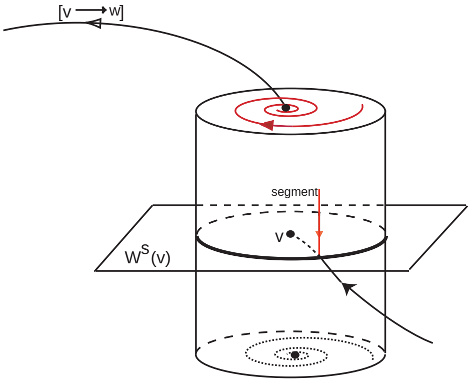
Sketch proof — linearised dynamics near v

path $[v \to w \to C]$



Sketch proof — linearised dynamics near v

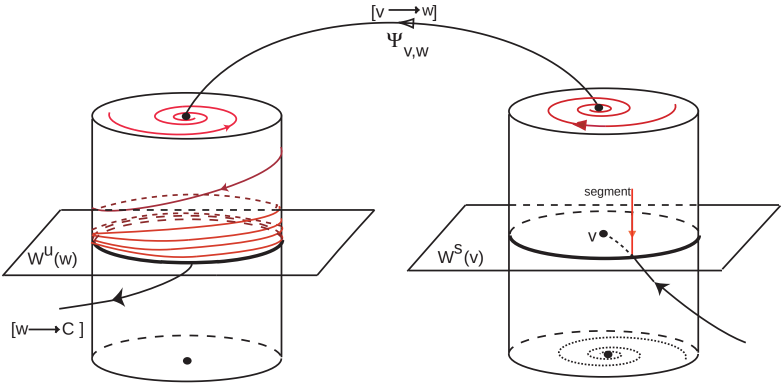
path $[v \to w \to C]$



the red line is **not** a trajectory!

Sketch proof — first connection $[v \to w]$

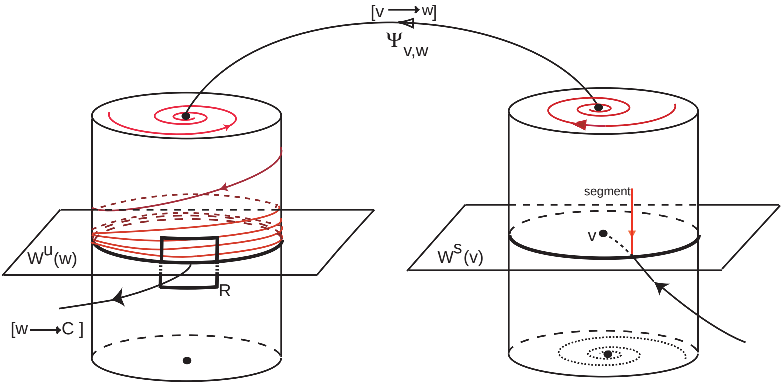
path $[v \to w \to C]$



the red line is **not** a trajectory!

Sketch proof — first connection $[v \to w]$

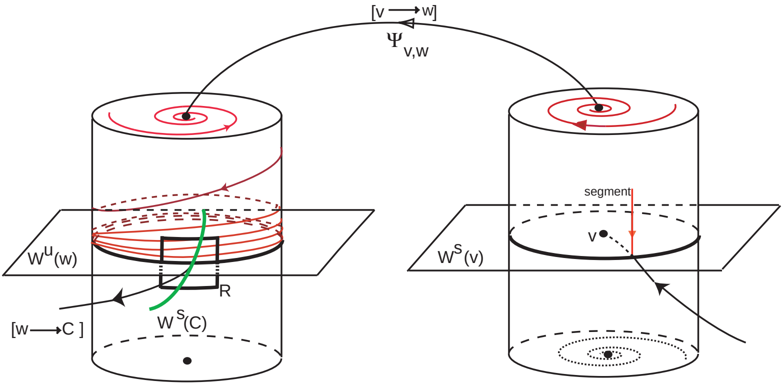
path $[v \to w \to C]$



the red line is **not** a trajectory!

Sketch proof — first connection $[v \to w]$

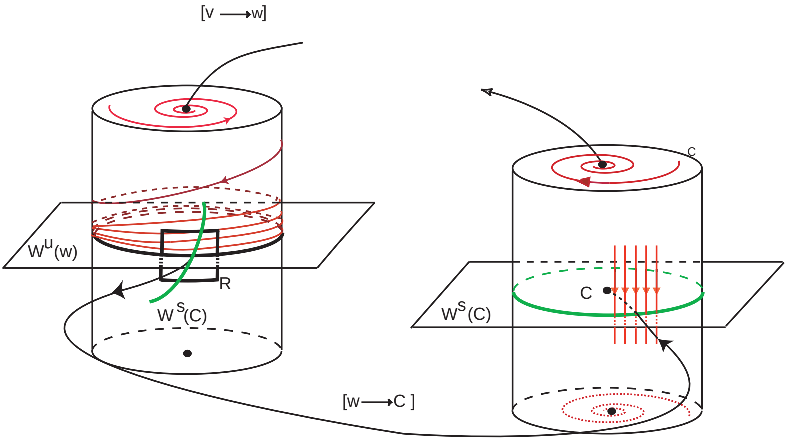
path $[v \to w \to C]$



the red line is **not** a trajectory!

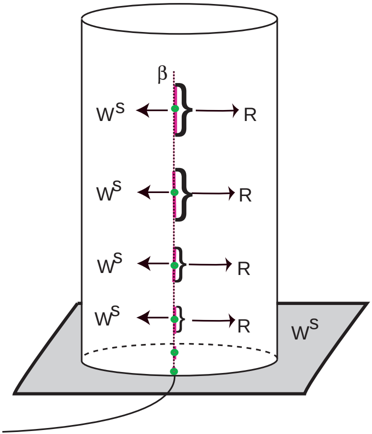
Sketch proof — second connection $[w \to C]$

path $[v \to w \to C]$



the red line is **not** a trajectory!

Sketch proof



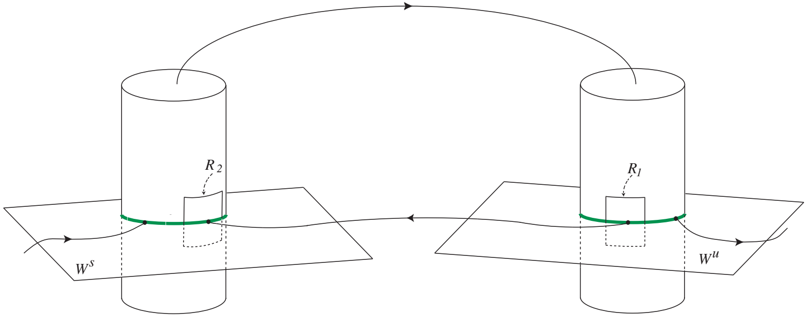
Iterate the process to get nested sequences of intervals. □

Dynamics near a network

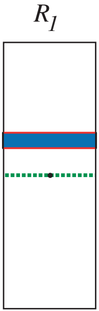
Theorem

For a vector field on the 3-sphere $\mathbf{S}^3$ *with a network of rotating nodes there is a suspended horseshoe following each cycle in the network.*

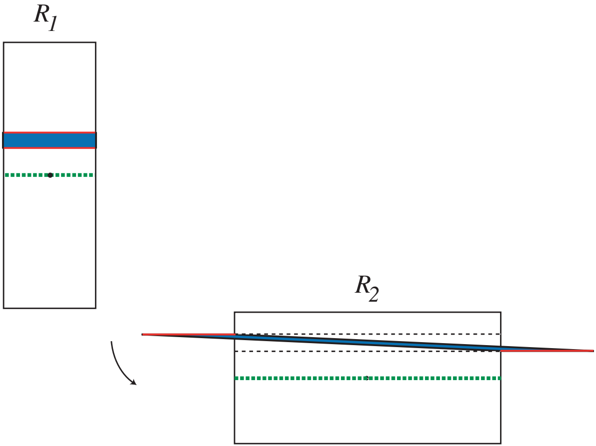
Sketch proof



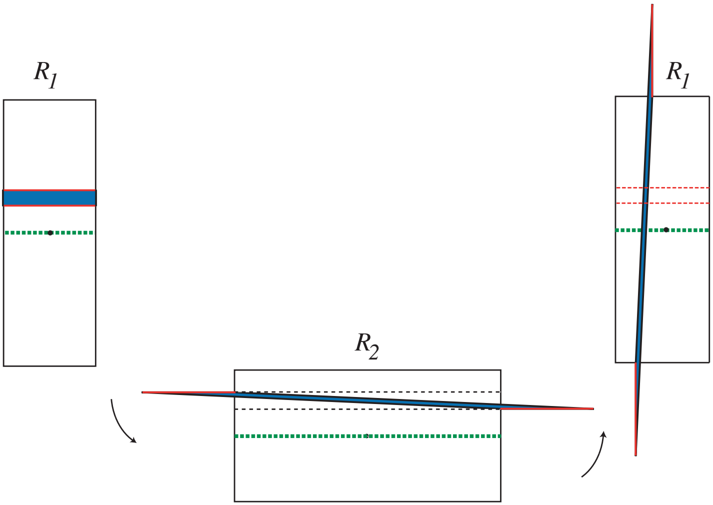
Sketch proof



Sketch proof



Sketch proof



Cycling near a heteroclinic network

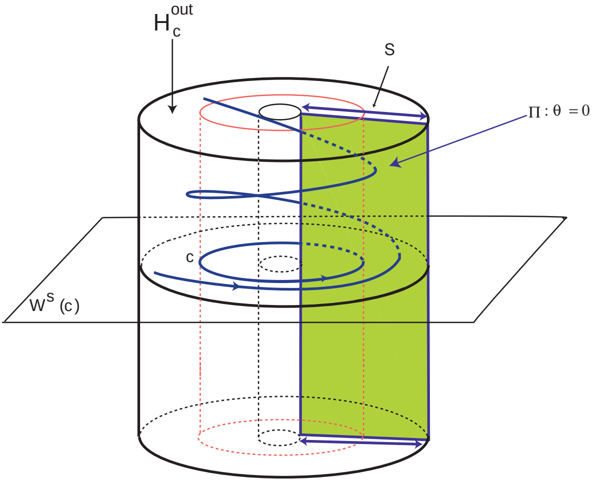
Given:

- an infinite path on the network through the nodes $n_1, n_2, \ldots$
- a sequence of natural numbers $t_1, t_2, \ldots$

there is a trajectory that makes t_i turns around the node n_i.

Number of turns inside a neighbourhood of a node

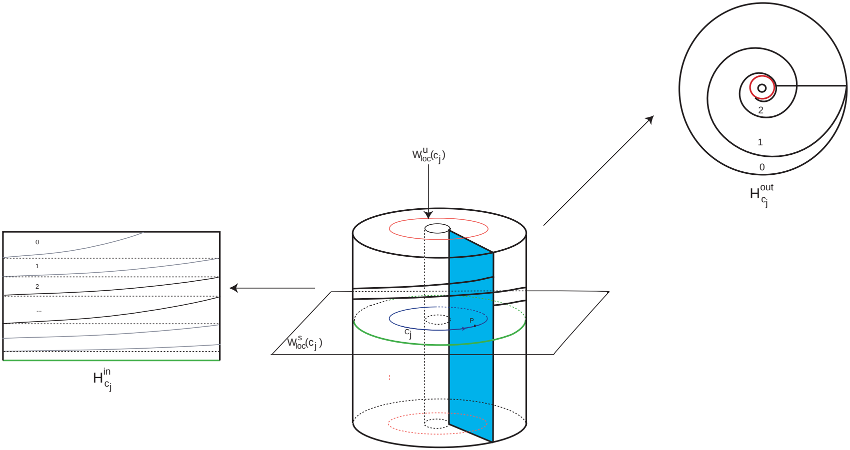
number of times the trajectory hits a section Π



Theorem

For a vector field on the 3-sphere $\mathbf{S}^3$ with a network of rotating nodes there is cycling near the network.

Sketch proof



Bibliography

- M. A.D. Aguiar, S. B. Castro and I. S. Labouriau,
  Dynamics near a heteroclinic network,
  Nonlinearity 18, 2005
- M. A. D. Aguiar, I. S. Labouriau and A. A. P. Rodrigues,
  Switching near a heteroclinic network of rotating nodes,
  Dynamical Systems: an International Journal (to appear), 2010
- A. A. P. Rodrigues, I. S. Labouriau and M. A. D. Aguiar,
  Chaotic double cycling,
  preprint CMUP 2009–41
- I. S. Labouriau, A. A. P. Rodrigues and M. A. D. Aguiar,
  Global generic dynamics close to symmetry,
  in preparation

The End