Dynamics near heteroclinic networks
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Plácido Zoega Táboas

Isabel S. Labouriau

Centro de Matemática da Universidade do Porto
Portugal

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Dynamics near heteroclinic networks

Joint work with

- Manuela A.D. Aguiar
  Faculdade de Economia da Universidade do Porto

- Sofia B.S.D. Castro
  Faculdade de Economia da Universidade do Porto

and

- Alexandre A.P. Rodrigues
  Faculdade de Ciências da Universidade do Porto
Outline

Introduction
  Heteroclinic cycles
  Heteroclinic networks
  Network questions

Dynamics near the network

A network of rotating nodes

Switching

Horseshoes

Cycling
Context: smooth vector fields on $\mathbb{R}^n$ or on a smooth manifold ordinary differential equations.

**Heteroclinic cycle**
Finite set of flow-invariant hyperbolic objects (**nodes**) trajectories joining them (**connections**) in a cycle
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**Heteroclinic cycle**
Finite set of flow-invariant hyperbolic objects (nodes) trajectories joining them (connections) in a cycle

Special context here:
- 3 dimensional phase-space;
- nodes are either equilibria or closed trajectories;
Special contexts for persistent heteroclinic cycles: (flow-invariant subspaces)

- game theory, economics;
- population dynamics;
- coupled cell networks;
- equations with symmetry.
Heteroclinic network
Connected set, finite union of heteroclinic cycles
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Connected set, finite union of heteroclinic cycles
Network questions

Dynamics near a network

- switching;
- cycling;

Persistence of dynamics when the network is broken.
Network questions

Dynamics near a network

- switching;
- cycling;

Persistence of dynamics when the network is broken.
A rotating node is:

either

a hyperbolic non-trivial closed trajectory of saddle type

or

an equilibrium of saddle type, with eigenvalues

\[ A \in \mathbb{R} \quad \text{and} \quad B \pm iC \]

where

\[ AB < 0 \quad AB \neq -1 \quad \text{and} \quad C \neq 0 \]
A network of rotating nodes satisfies:

- all the nodes are rotating
- all connections that take place in 2-dimensional invariant manifolds occur as transverse intersections.
Switching at a node

Given a connection on the network arriving at the node for every connection coming out of the node there is a trajectory that shadows each one of the connections coming out.
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Given a connection on the network arriving at the node for every connection coming out of the node there is a trajectory that shadows each one of the of connections coming out.

For hyperbolic nodes, follows from $\lambda$-lemma.
Finite switching near the network

Given any finite path on the network there is a trajectory that shadows it.
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Infinite switching near the network: every infinite path is shadowed.
Theorem

For a vector field on the 3-sphere $S^3$ with a network of rotating nodes there is infinite switching on the network.
Sketch proof — linearised dynamics near $v$

path $[v \rightarrow w \rightarrow C]$
Sketch proof — linearised dynamics near $v$

path $[v \to w \to C]$

the red line is not a trajectory!
Sketch proof — first connection $[v \to w]$

path $[v \to w \to C]$

the red line is **not** a trajectory!
Sketch proof — first connection $[v \rightarrow w]$

path $[v \rightarrow w \rightarrow C]$

the red line is **not** a trajectory!
Sketch proof — first connection \([v \rightarrow w]\)

path \([v \rightarrow w \rightarrow C]\)

the red line is not a trajectory!
Sketch proof — second connection \([w \rightarrow C]\)

path \([v \rightarrow w \rightarrow C]\)

the red line is not a trajectory!
Sketch proof

Iterate the process to get nested sequences of intervals.
Dynamics near a network

Theorem

For a vector field on the 3-sphere $S^3$ with a network of rotating nodes there is a suspended horseshoe following each cycle in the network.
Sketch proof
Sketch proof

$R_I$
Sketch proof

$R_1$

$R_2$
Sketch proof
Cycling near a heteroclinic network

Given:

- an infinite path on the network through the nodes $n_1, n_2, \ldots$
- a sequence of natural numbers $t_1, t_2, \ldots$

there is a trajectory that makes $t_i$ turns around the node $n_i$. 
Number of turns inside a neighbourhood of a node
number of times the trajectory hits a section $\Pi$
Theorem

For a vector field on the 3-sphere $\mathbb{S}^3$ with a network of rotating nodes there is cycling near the network.
M. A.D. Aguiar, S. B. Castro and I. S. Labouriau, 
*Dynamics near a heteroclinic network*,
Nonlinearity 18, 2005

M. A. D. Aguiar, I. S. Labouriau and A. A. P. Rodrigues, 
*Switching near a heteroclinic network of rotating nodes*,

A. A. P. Rodrigues, I. S. Labouriau and M. A. D. Aguiar, 
*Chaotic double cycling*,
preprint CMUP 2009–41

I. S. Labouriau, A. A. P. Rodrigues and M. A. D. Aguiar, 
*Global generic dynamics close to symmetry*,
in preparation
The End