Dynamics near heteroclinic networks

Dynamical Systems: 100 years after Poincaré

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Dynamics near heteroclinic networks

Joint work with

- Alexandre A.P. Rodrigues
  Faculdade de Ciências da Universidade do Porto

also based on work with

- Manuela A.D. Aguiar
  Faculdade de Economia da Universidade do Porto
- Sofia B.S.D. Castro
  Faculdade de Economia da Universidade do Porto

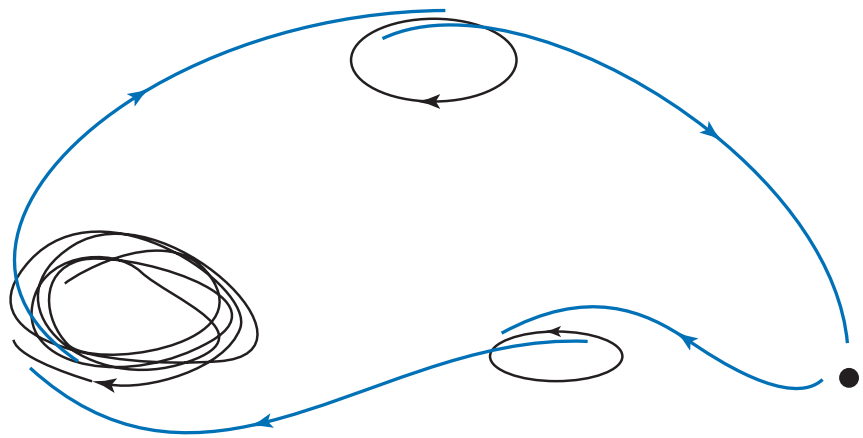
Outline

Context: smooth vector fields on $\mathbf{R}^n$ or on a smooth manifold ordinary differential equations.

Heteroclinic cycle

Finite set of flow-invariant hyperbolic objects (nodes)

trajectories joining them (connections) in a cycle



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Context here:

- 3 dimensional phase-space $\mathbf{S}^3$;
- nodes are equilibria;

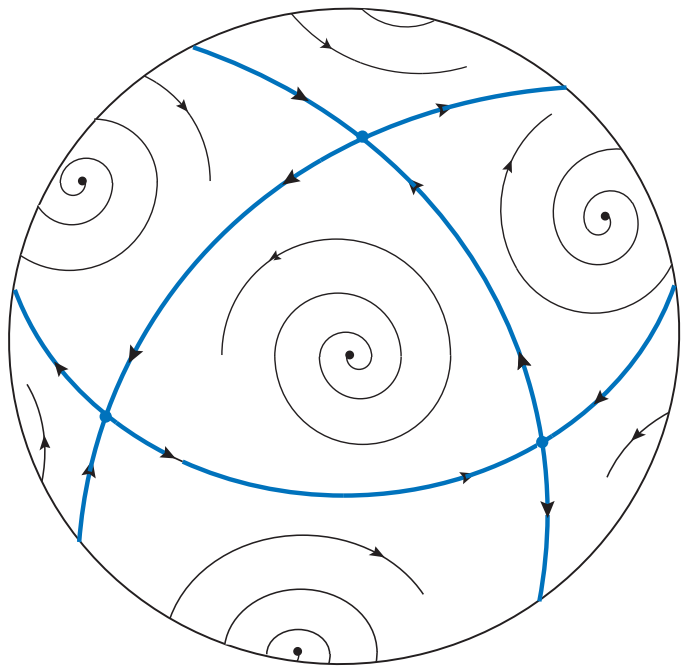
Heteroclinic cycles are not persistent.

Special contexts for persistent heteroclinic cycles:
(flow-invariant subspaces)

- game theory, economics;
- population dynamics;
- coupled cell networks;
- reversible dynamics;
- equations with symmetry.

Heteroclinic network

Connected set, finite union of heteroclinic cycles



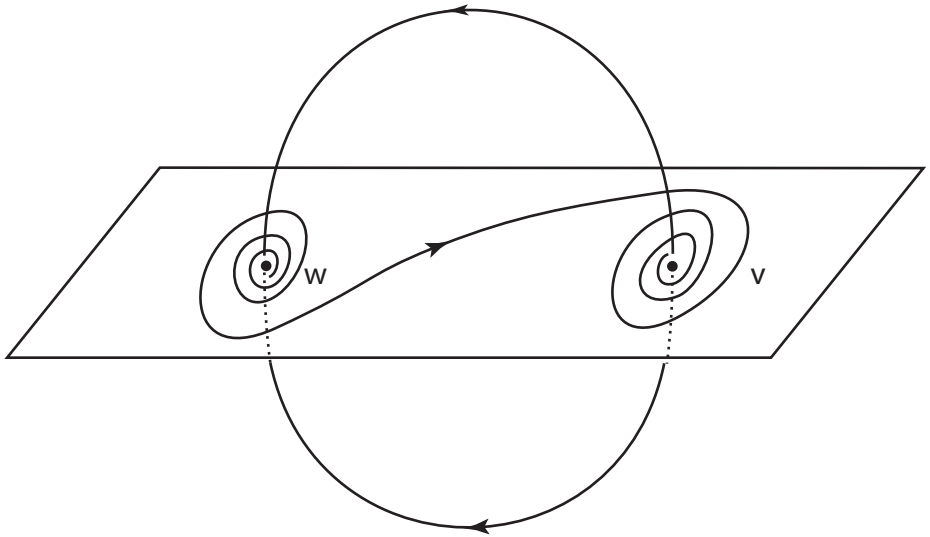
Network questions

Dynamics around a network

- switching;
- cycling;
- nearby periodic solutions;
- geometric structure nearby;

- Persistence of dynamics when the network is broken.

A specific type of network on $\mathbf{S}^3$



Two equilibria v and w

2-D connection

$$W^u(w) = W^s(v)$$

1-D connection

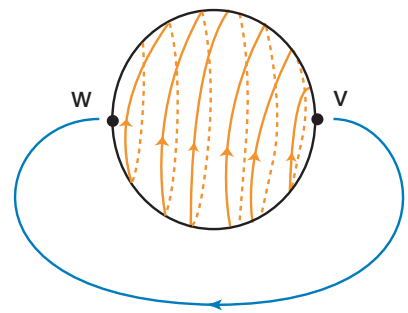
$$W^s(w) = W^u(v)$$

Network is asymptotically stable

From symmetry:

$$\overline{W^u(w)} = \overline{W^s(v)} = \mathbf{S}^2$$

$$\overline{W^s(w)} = \overline{W^u(v)} = \mathbf{S}^1$$



Breaking the symmetry, get:

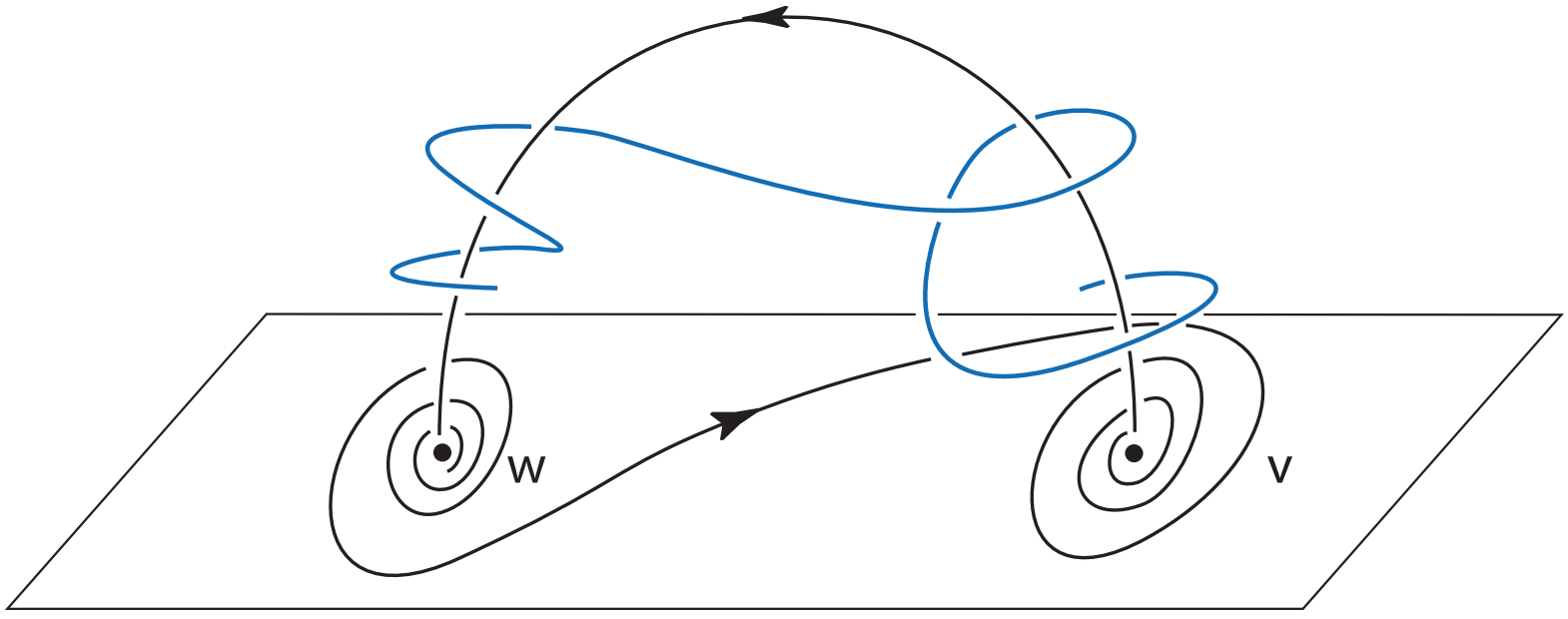
$$W^s(v) \pitchfork W^u(w)$$

$$W^u(v) \cap W^s(w) = \emptyset$$

Explicit examples constructed in Aguiar, Castro, Labouriau (2006) and in Labouriau, Rodrigues (2011)

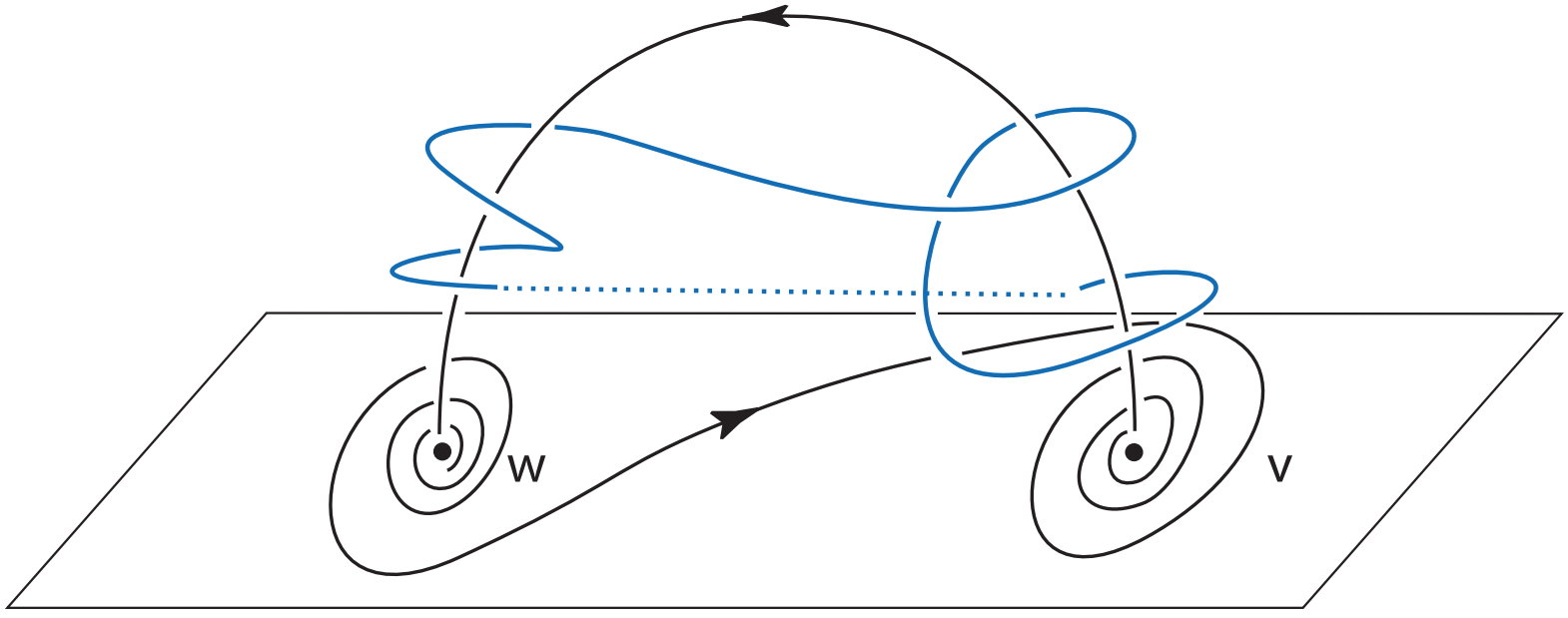
Two different kinds of networks of this type

Turn with the same orientation around equilibria



Two different kinds of networks of this type

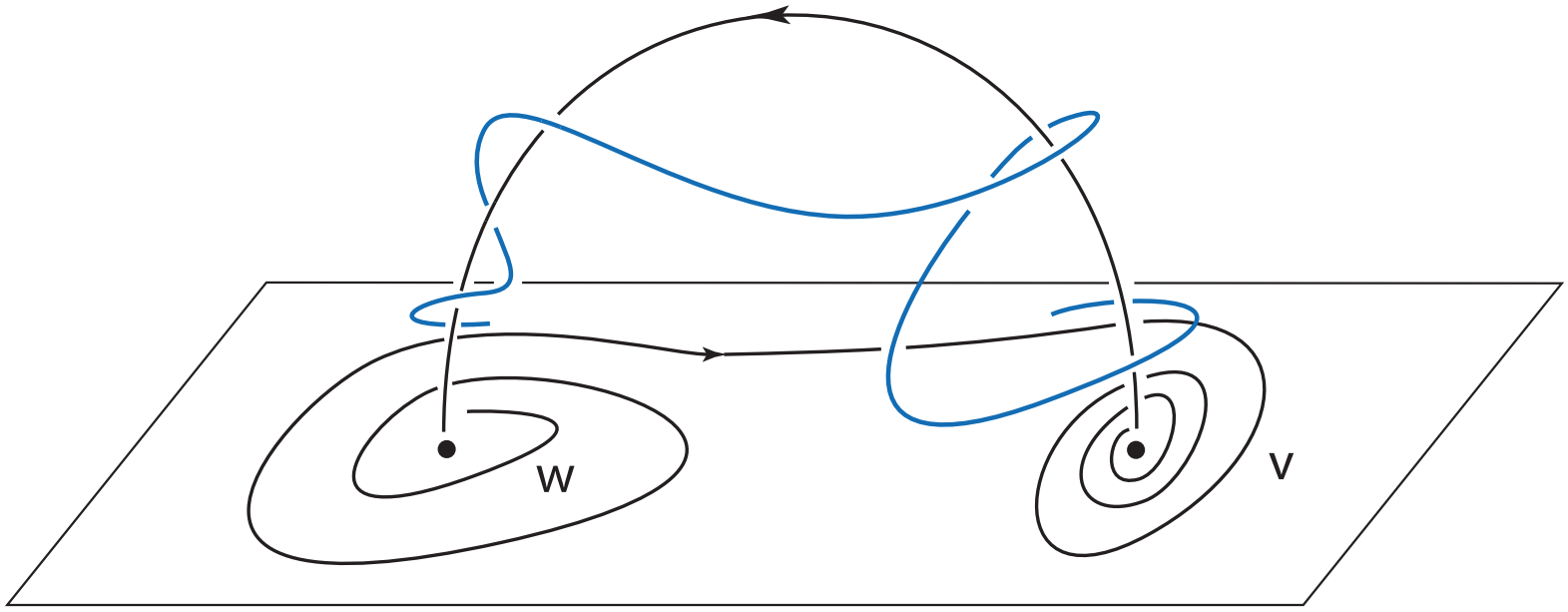
Turn with the same orientation around equilibria



Joining the end points of nearby trajectories
links them to the network.

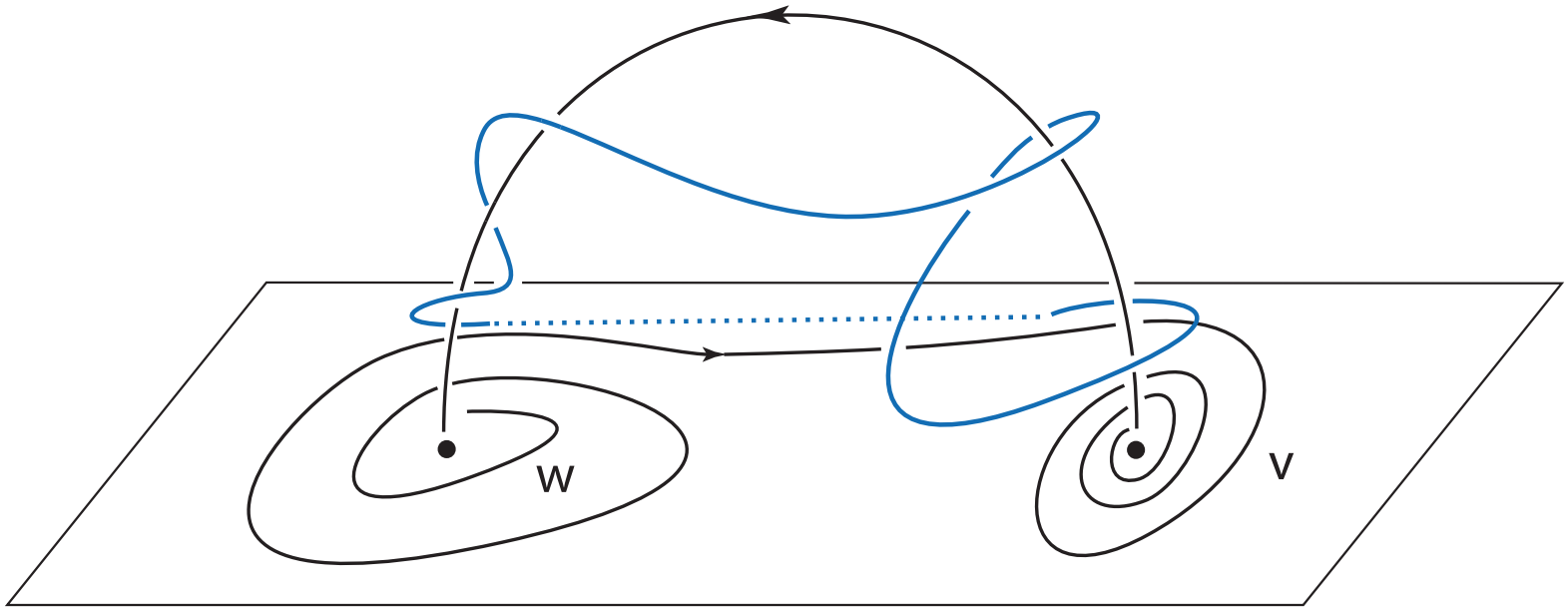
Two different kinds of networks of this type

Turn with the opposite orientation around equilibria



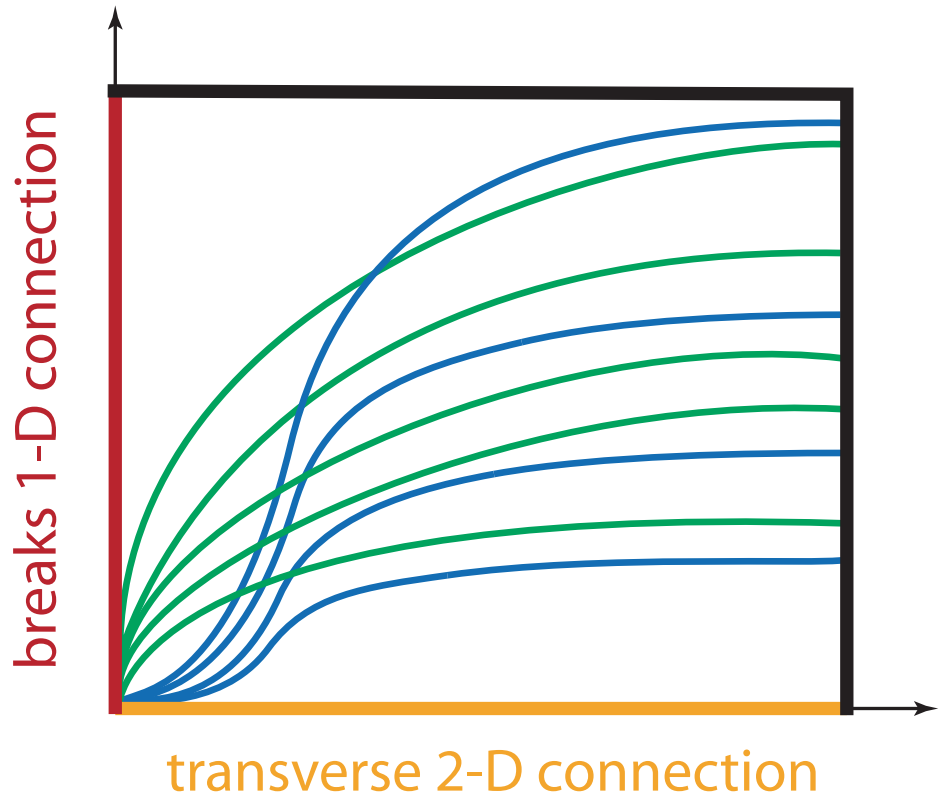
Two different kinds of networks of this type

Turn with the opposite orientation around equilibria

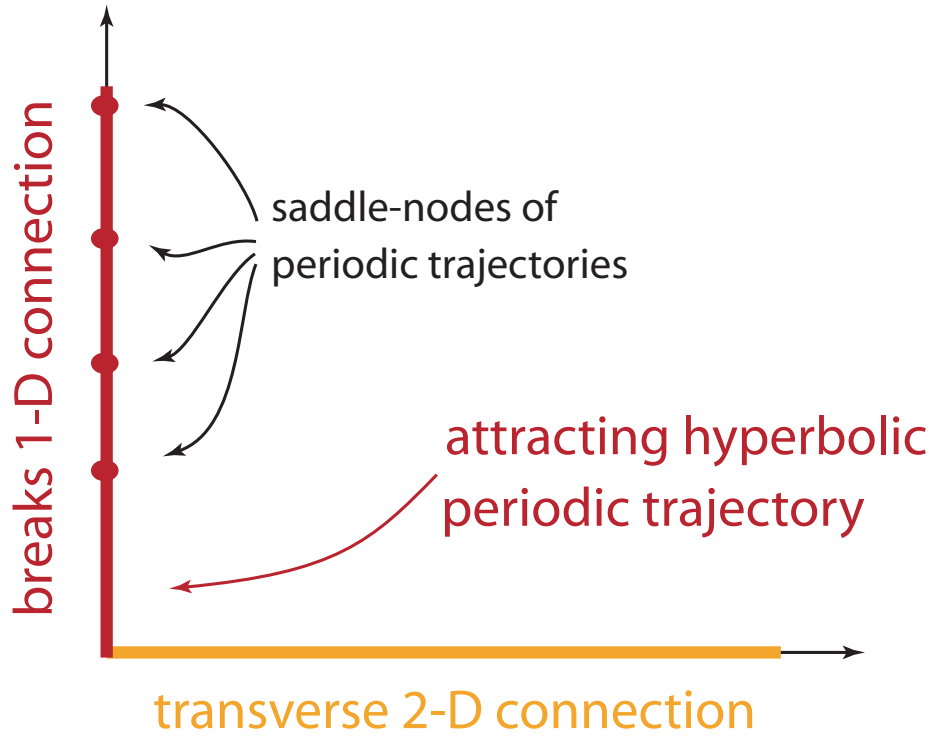


Joining the end points of nearby trajectories may not yield a link.

Bifurcation diagram — same orientation around equilibria

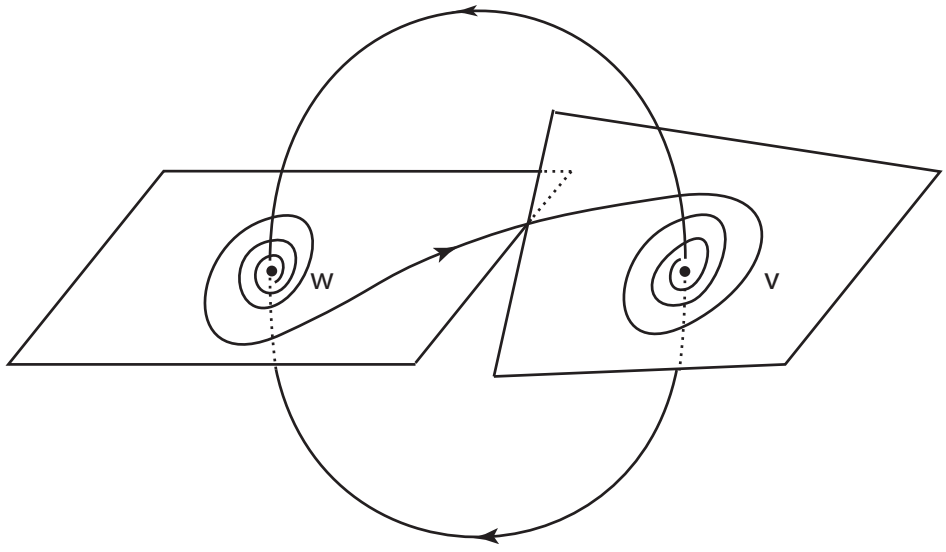


Breaking only the 1-D connection



Attracting hyperbolic periodic trajectories remain when 2-D connection is made transverse

Breaking only the 2-D connection — same orientation around equilibria

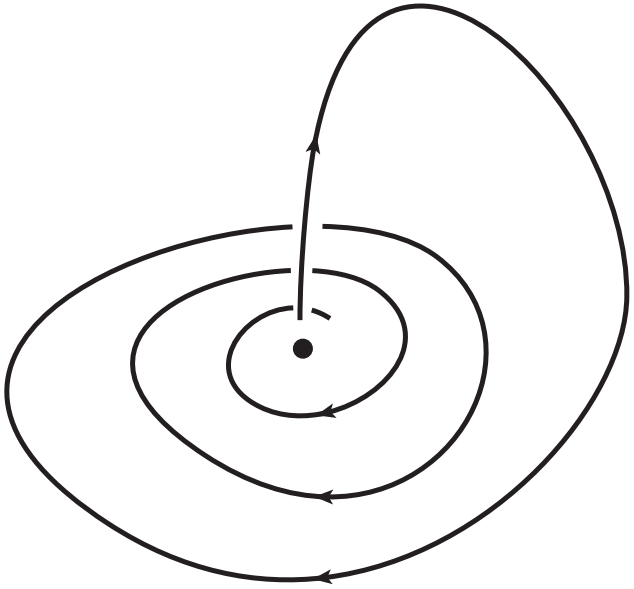


Suspended horseshoe around the network.

Hyperbolic Poincaré first return map.

Breaking the two connections — same orientation around equilibria

Create Shilnikov homoclinic cycles



Eigenvalues:
$-a \pm i\omega$ and b
$a, b, \omega > 0$

If $a > b$

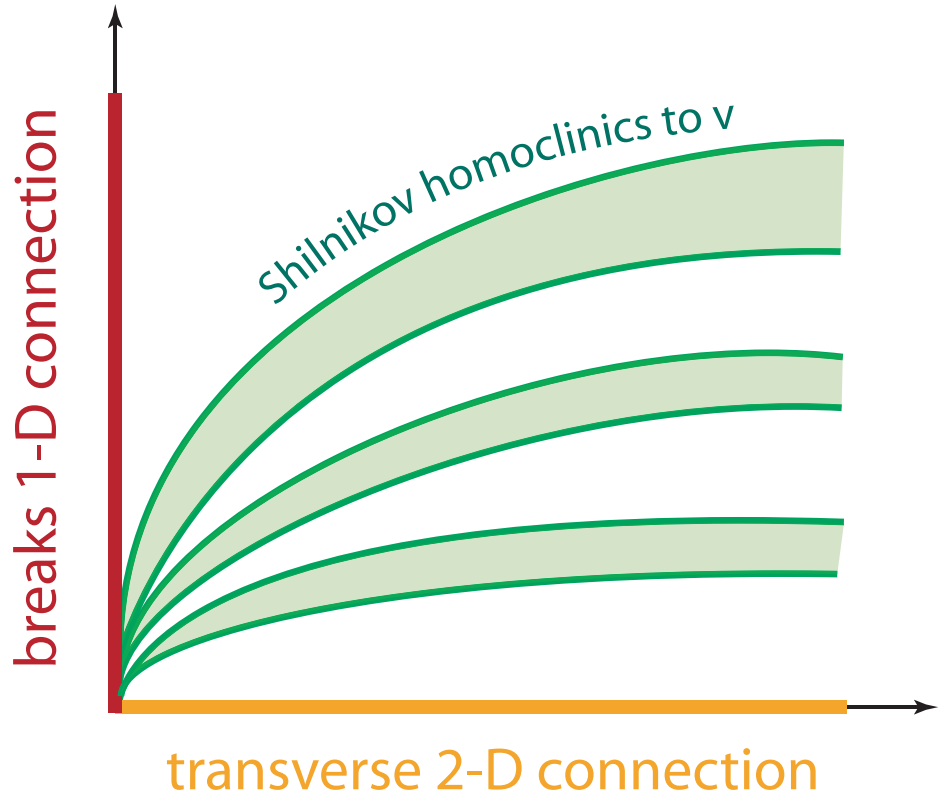
cycle attracts
bifurcates into attracting periodic orbit

If $a < b$

suspended horseshoes around cycle
some horseshoes persist when cycle is broken

Breaking the two connections — same orientation around equilibria

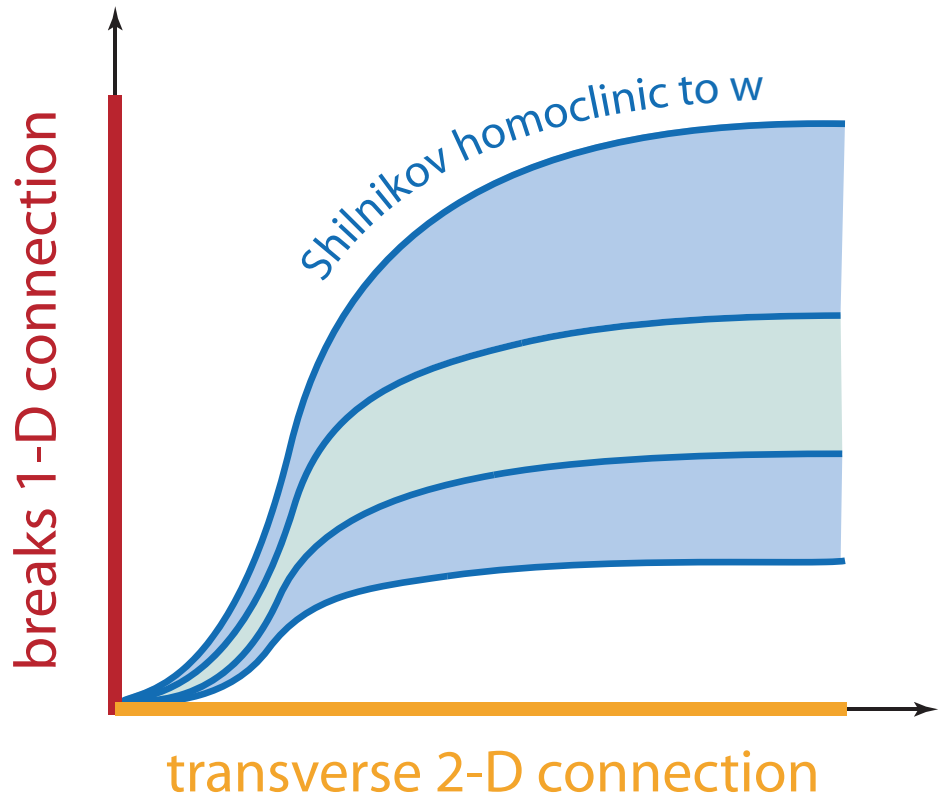
Homoclinic cycles to v



Shaded tongues — periodic trajectories.

Breaking the two connections — same orientation around equilibria

Homoclinic cycles to w

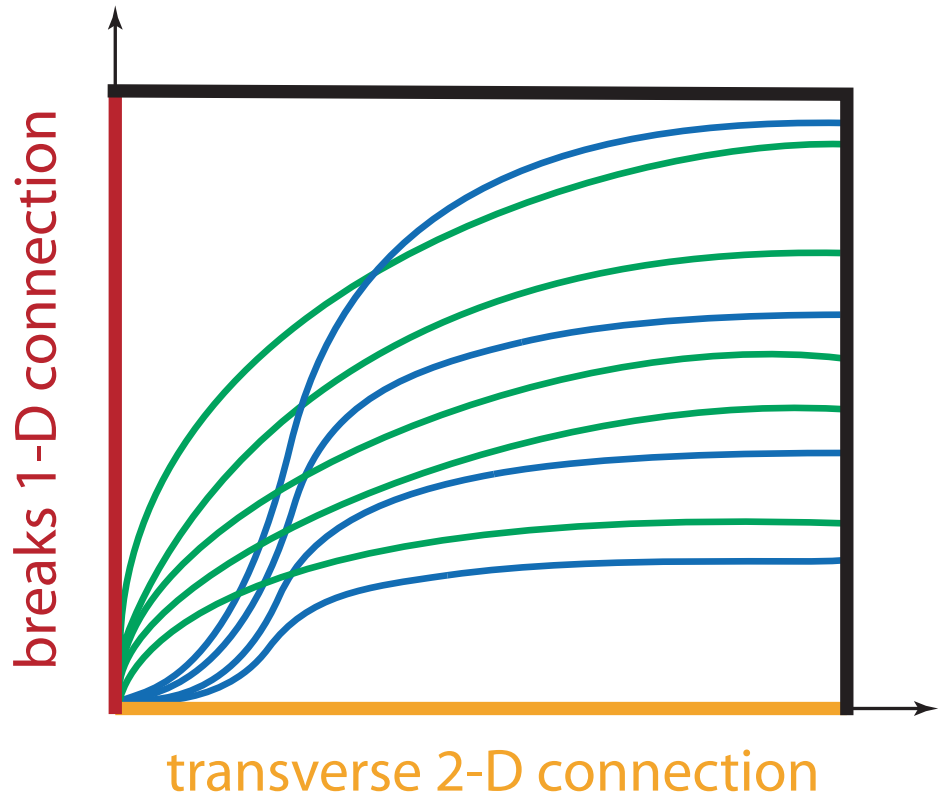


Shaded tongues — horseshoes.

blue: uniformly hyperbolic

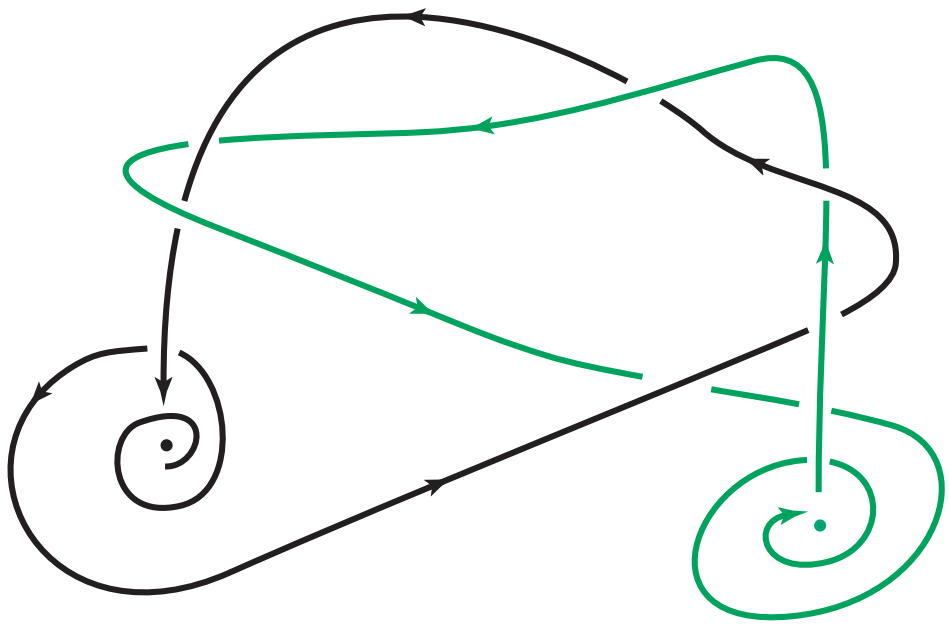
green: non uniformly hyperbolic

Bifurcation diagram — same orientation around equilibria



Breaking the two connections — same orientation around equilibria

Coexistence of Shilnikov homoclinic cycles in v and w, linked.



Different orientation around equilibria

Breaking only the 2-D connection

Suspended geometric horseshoe around the network.
Poincaré first return map not hyperbolic.

Breaking the two connections

Heteroclinic tangencies of 2-D invariant manifolds $W^s(v)$ $W^u(w)$ in dense set of parameters.

The End

Thank you for your attention

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