Dynamics near heteroclinic networks

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Dynamics near heteroclinic networks

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Outline

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  Heteroclinic networks
  Network questions

A specific type of network with symmetry
  Symmetry
  Chirality

Breaking the symmetries — same chirality

Breaking the symmetries — different chirality
Context: smooth vector fields on $\mathbb{R}^n$ or on a smooth manifold ordinary differential equations.

**Heteroclinic cycle**
Finite set of flow-invariant hyperbolic objects (nodes) trajectories joining them (connections) in a cycle
Context: smooth vector fields on $\mathbb{R}^n$ or on a smooth manifold ordinary differential equations.

**Heteroclinic cycle**

Finite set of flow-invariant hyperbolic objects (nodes) trajectories joining them (connections) in a cycle

Context here:
- 3 dimensional phase-space $S^3$;
- nodes are equilibria;

Heteroclinic cycles are not persistent.
Special contexts for persistent heteroclinic cycles:
(flow-invariant subspaces)

- game theory, economics;
- population dynamics;
- coupled cell networks;
- reversible dynamics;
- equations with symmetry.
Heteroclinic network
Connected set, finite union of heteroclinic cycles
Network questions

Dynamics around a network

- switching;
- cycling;
- nearby periodic solutions;
- geometric structure;
- Persistence of dynamics when the network is broken.
A specific type of network

Symmetries in $S^3$ — inherited from $\mathbb{R}^4$

\[
\gamma_1(x_1, x_2, x_3, x_4) = (-x_1, -x_2, x_3, x_4)
\]
\[
\gamma_2(x_1, x_2, x_3, x_4) = (x_1, x_2, -x_3, x_4)
\]

In $S^3$, invariant sets:
circle $S^1 = \{(0, 0, x_3, x_4)\} \cap S^3$

2-sphere $S^2 = \{(x_1, x_2, 0, x_4)\} \cap S^3$

two equilibria $S^1 \cap S^2 = \{(0, 0, 0, \pm 1)\}$

Compatible with heteroclinic network
that persists under symmetry-preserving perturbations.

A specific type of network on $S^3$

Two equilibria $v$ and $w$

2-D connection

$$W^u(w) = W^s(v)$$

1-D connection

$$W^s(w) = W^u(v)$$

Network is asymptotically stable

From symmetry:

$$W^u(w) = W^s(v) = S^2$$

$$W^s(w) = W^u(v) = S^1$$

Breaking the symmetry, get:

$$W^s(v) \cap W^u(w)$$

$$W^u(v) \cap W^s(w) = \emptyset$$
Two different kinds of networks of this type

The nodes have the same chirality
Flow turns with the same orientation around equilibria.

The nodes have different chirality
Flow turns with the opposite orientation around equilibria.
Two different kinds of networks of this type turn with the same orientation around equilibria.
Two different kinds of networks of this type

Turn with the same orientation around equilibria

Joining the end points of nearby trajectories links them to the network.

The nodes have the same chirality.
Two different kinds of networks of this type

Turn with the opposite orientation around equilibria
Two different kinds of networks of this type

Turn with the opposite orientation around equilibria

Joining the end points of nearby trajectories may not yield a link.
The nodes have different chirality.
Bifurcation diagram — same chirality

transverse 2-D connection

breaks 1-D connection
Breaking only the 1-D connection — same chirality

Attracting hyperbolic periodic trajectories remain when 2-D connection is made transverse
Transverse 2-D connection — same chirality

Suspended horseshoe around the network.

Hyperbolic Poincaré first return map.
Transverse 2-D connection — same chirality
Transverse 2-D connection — same chirality

\[ W^u(w) \quad \rightarrow \quad W^s(v) \]

\[ [v \rightarrow w] \quad \text{and} \quad [w \rightarrow v] \]
Transverse 2-D connection — same chirality
Transverse 2-D connection — same chirality
Breaking the two symmetries — same chirality

Create Shilnikov homoclinic cycles

Eigenvalues: $-a \pm i\omega$ and $b$
$a, b, \omega > 0$

If $a > b$
cycle attracts
bifurcates into attracting periodic orbit

If $a < b$
suspended horseshoes around cycle
some horseshoes persist when cycle is broken
Breaking the two symmetries — Shilnikov cycles
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Breaking the two symmetries — Shilnikov cycles
Breaking the two symmetries — same chirality

Homoclinic cycles to $v$

Shaded tongues — periodic trajectories.
Breaking the two symetries — same chirality

Homoclinic cycles to $w$

Shaded tongues — horseshoes.

blue: uniformly hyperbolic
green: non uniformly hyperbolic
Bifurcation diagram — same chirality

transverse 2-D connection breaks 1-D connection
Breaking the two symmetries — same chirality

Coexistence of Shilnikov homoclinic cycles in $v$ and $w$, linked.
Different chirality with 1-D connection

With 2-D connection
Like same chirality.

Transverse 2-D connection
A segment is not mapped into a helix.
Different chirality — transverse 2-D connection
Different chirality — transverse 2-D connection

In(w) \rightarrow Out(v)

\[ W^u(w) \rightarrow W^s(v) \]

[v \rightarrow w]

[w \rightarrow v]
Different chirality — transverse 2-D connection

\[ \text{In}(w) \quad \text{Out}(v) \]

\[ W^u(w) \quad W^s(v) \]

\[ [v \rightarrow w] \quad [w \rightarrow v] \]
Different chirality—transverse 2-D connection

For an open set of parameters
Heteroclinic tangencies of 2-D invariant manifolds $W^s(v) W^u(w)$ in dense set of parameters.

Infinitely many sinks nearby (Newhouse).

Coexisting with
suspended hyperbolic horseshoes around the network.

All these structures form a spiralling attractor around the network.
The End

Thank you for your attention
References