Dynamics near heteroclinic networks

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Dynamics near heteroclinic networks

Joint work with

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  Faculdade de Ciências da Universidade do Porto

also based on work with

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  Faculdade de Economia da Universidade do Porto
- Sofia B.S.D. Castro
  Faculdade de Economia da Universidade do Porto

Outline

Introduction
- Heteroclinic cycles
- Heteroclinic networks
- Network questions

A specific type of network with symmetry
- Symmetry
- Chirality

Breaking the symmetries — same chirality

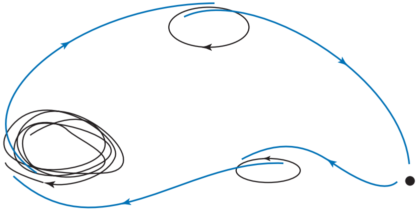
Breaking the symmetries — different chirality

Context: smooth vector fields on $\mathbf{R}^n$ or on a smooth manifold
ordinary differential equations.

Heteroclinic cycle

Finite set of flow-invariant hyperbolic objects (nodes)

trajectories joining them (connections) in a cycle



Context: smooth vector fields on $\mathbf{R}^n$ or on a smooth manifold ordinary differential equations.

Heteroclinic cycle

Finite set of flow-invariant hyperbolic objects (nodes)

trajectories joining them (connections) in a cycle

Context here:

- 3 dimensional phase-space $\mathbf{S}^3$;
- nodes are equilibria;

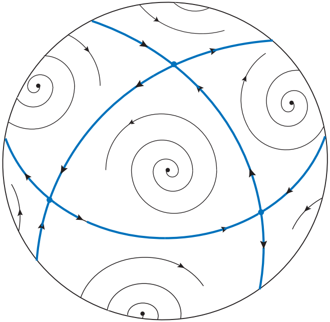
Heteroclinic cycles are not persistent.

Special contexts for persistent heteroclinic cycles:
(flow-invariant subspaces)

- game theory, economics;
- population dynamics;
- coupled cell networks;
- reversible dynamics;
- equations with symmetry.

Heteroclinic network

Connected set, finite union of heteroclinic cycles



Network questions

Dynamics around a network

- switching;
- cycling;
- nearby periodic solutions;
- geometric structure;

- Persistence of dynamics when the network is broken.

A specific type of network

Symmetries in $\mathbf{S}^3$ — inherited from $\mathbf{R}^4$

$$\gamma_1(x_1, x_2, x_3, x_4) = (-x_1, -x_2, x_3, x_4)$$
$$\gamma_2(x_1, x_2, x_3, x_4) = (x_1, x_2, -x_3, x_4)$$

In $\mathbf{S}^3$, invariant sets:

circle $\mathbf{S}^1 = \{(0, 0, x_3, x_4)\} \cap \mathbf{S}^3$

2-sphere $\mathbf{S}^2 = \{(x_1, x_2, 0, x_4)\} \cap \mathbf{S}^3$

two equilibria $\mathbf{S}^1 \cap \mathbf{S}^2 = \{(0, 0, 0, \pm 1)\}$

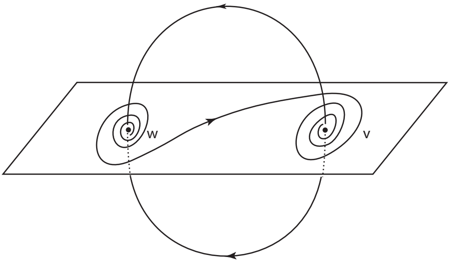
Compatible with heteroclinic network

that persists under symmetry-preserving perturbations.

Explicit examples constructed in: Aguiar, Castro, Labouriau (Int. J. Bif. Chaos 2006),

Rodrigues, Labouriau (Physica D 2014), Labouriau, Rodrigues (preprint 2014)

A specific type of network on $\mathbf{S}^3$



Two equilibria v and w

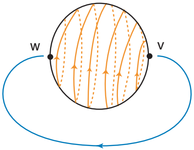
2-D connection
$$W^u(w) = W^s(v)$$

1-D connection
$$W^s(w) = W^u(v)$$

Network is asymptotically stable

From symmetry:

$$\overline{W^u(w)} = \overline{W^s(v)} = \mathbf{S}^2$$
$$\overline{W^s(w)} = \overline{W^u(v)} = \mathbf{S}^1$$



Breaking the symmetry, get:

$W^s(v) \pitchfork W^u(w)$

$W^u(v) \cap W^s(w) = \emptyset$

Two different kinds of networks of this type

The nodes have the same chirality

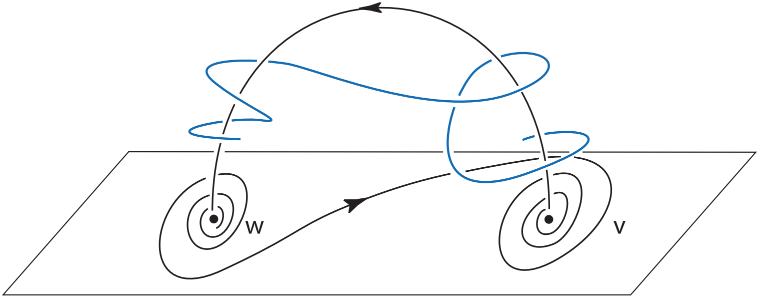
Flow turns with the same orientation around equilibria.

The nodes have different chirality

Flow turns with the opposite orientation around equilibria.

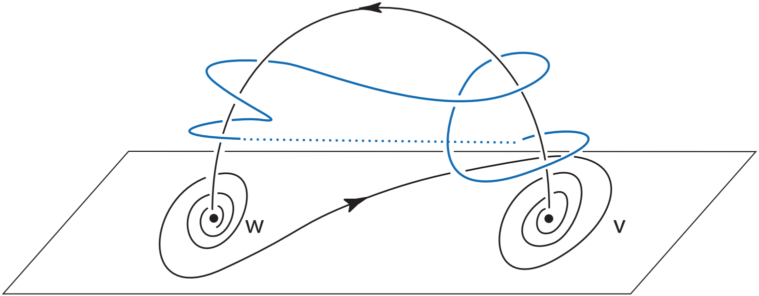
Two different kinds of networks of this type

Turn with the same orientation around equilibria



Two different kinds of networks of this type

Turn with the same orientation around equilibria

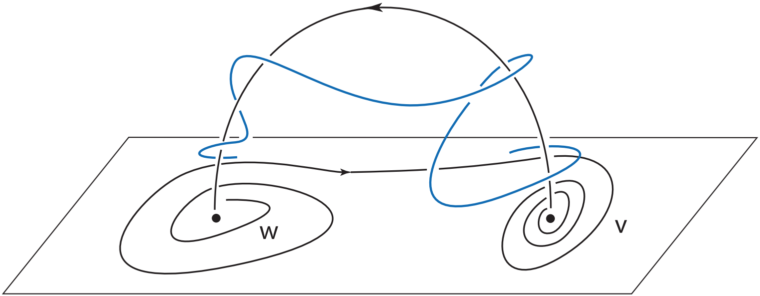


Joining the end points of nearby trajectories links them to the network.

The nodes have the same chirality.

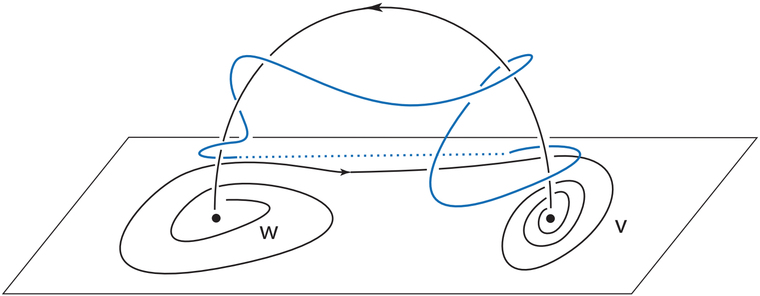
Two different kinds of networks of this type

Turn with the opposite orientation around equilibria



Two different kinds of networks of this type

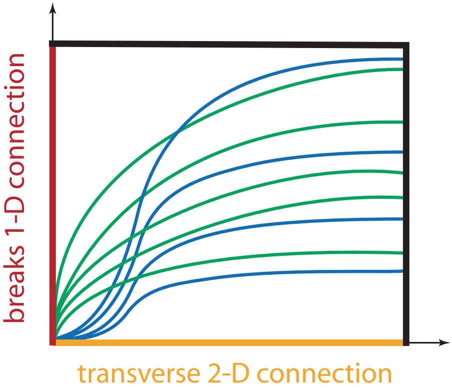
Turn with the opposite orientation around equilibria



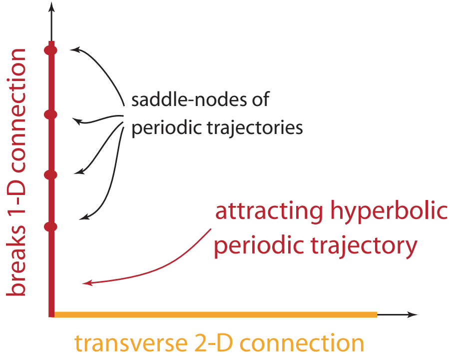
Joining the end points of nearby trajectories may not yield a link.

The nodes have different chirality.

Bifurcation diagram — same chirality

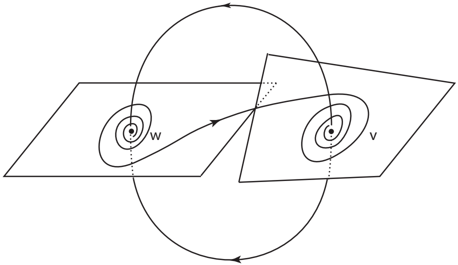


Breaking only the 1-D connection — same chirality



Attracting hyperbolic periodic trajectories remain when 2-D connection is made transverse

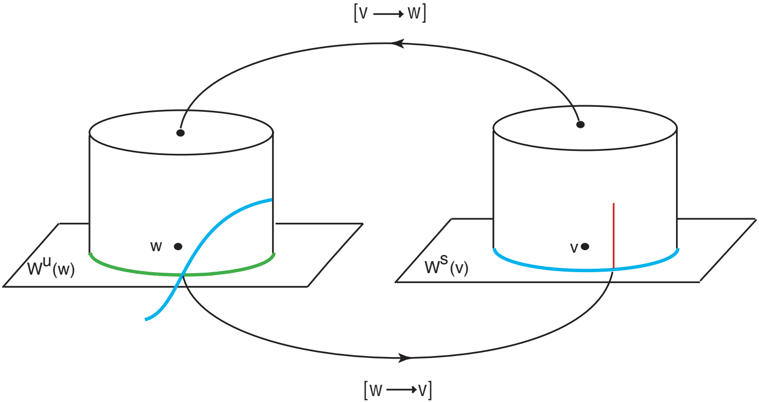
Transverse 2-D connection — same chirality



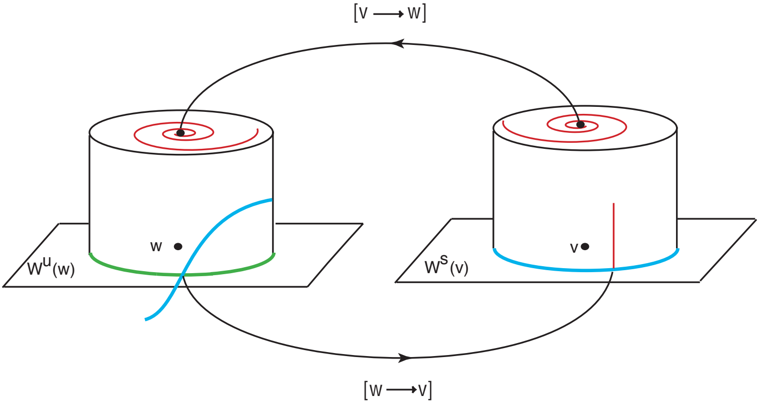
Suspended horseshoe around the network.

Hyperbolic Poincaré first return map.

Transverse 2-D connection — same chirality



Transverse 2-D connection — same chirality



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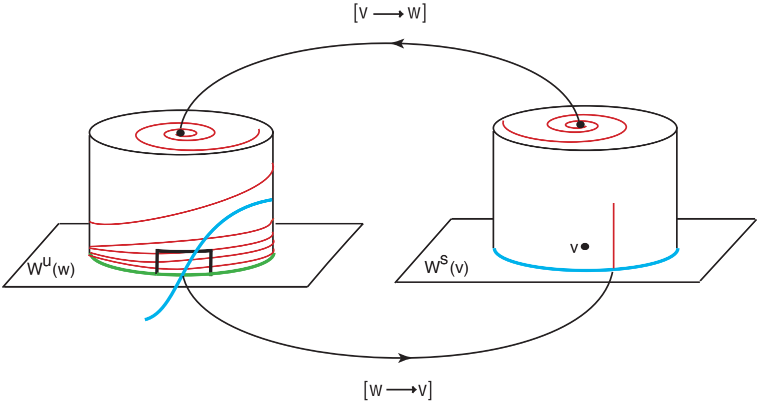
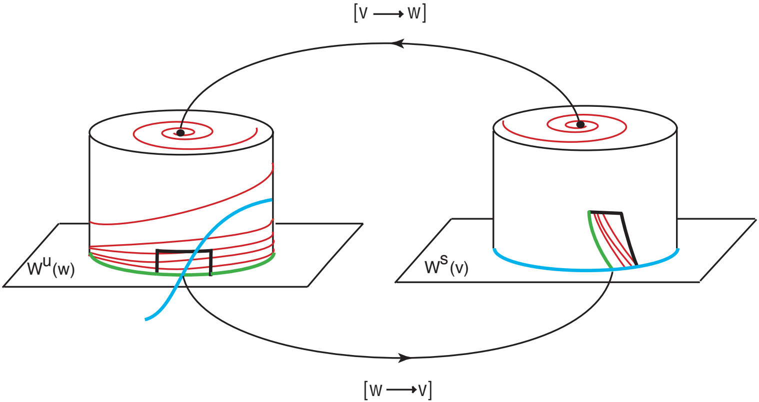
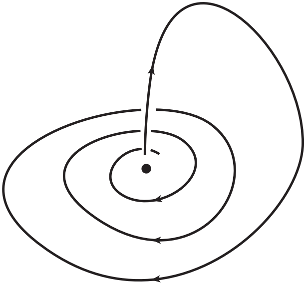


Transverse 2-D connection — same chirality



Breaking the two symmetries — same chirality

Create Shilnikov homoclinic cycles



Eigenvalues:
$-a \pm i\omega$ and b
$a, b, \omega > 0$

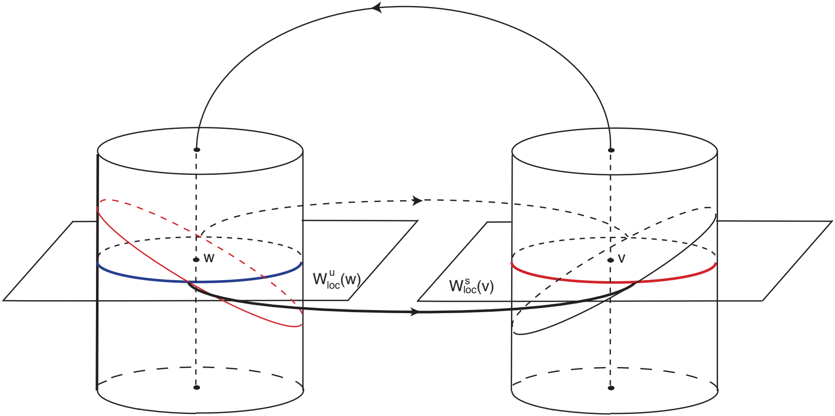
If $a > b$

cycle attracts
bifurcates into attracting periodic orbit

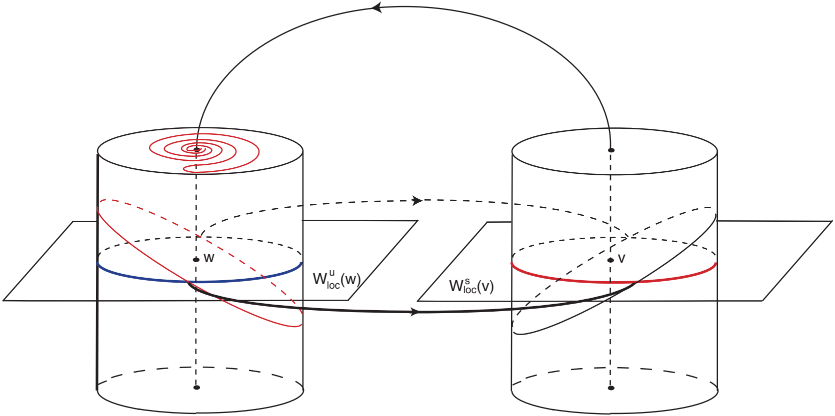
If $a < b$

suspended horseshoes around cycle
some horseshoes persist when cycle is broken

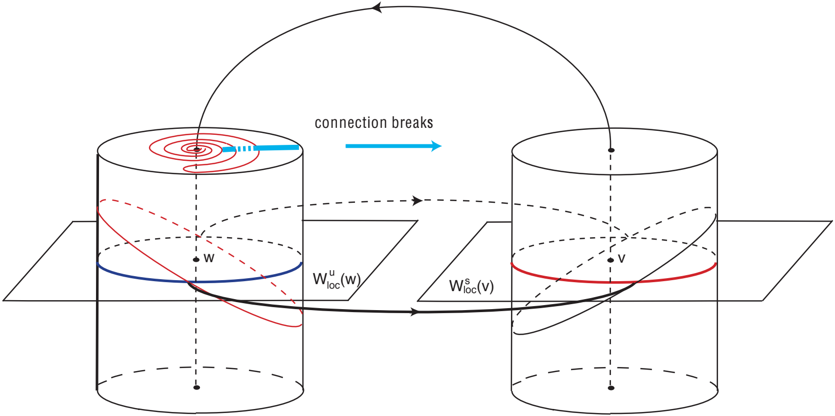
Breaking the two symmetries — Shilnikov cycles



Breaking the two symmetries — Shilnikov cycles

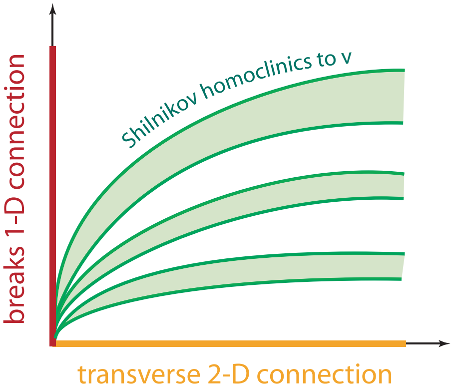


Breaking the two symmetries — Shilnikov cycles



Breaking the two symetries — same chirality

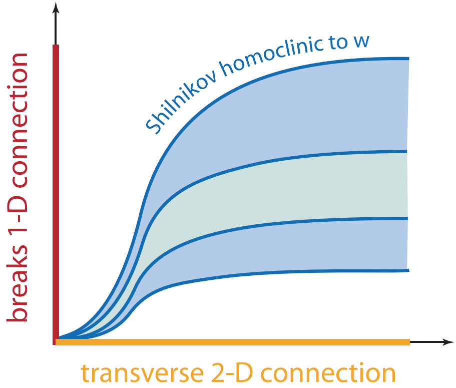
Homoclinic cycles to v



Shaded tongues — periodic trajectories.

Breaking the two symetries — same chirality

Homoclinic cycles to *w*

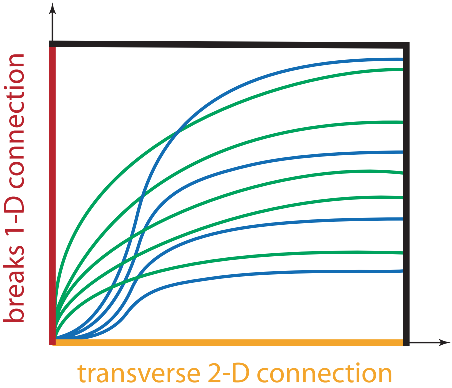


Shaded tongues — horseshoes.

blue: uniformly hyperbolic

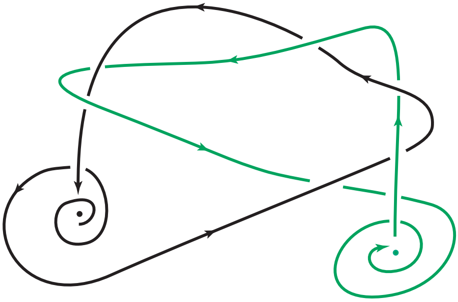
green: non uniformly hyperbolic

Bifurcation diagram — same chirality



Breaking the two symetries — same chirality

Coexistence of Shilnikov homoclinic cycles in v and w, linked.



Different chirality with 1-D connection

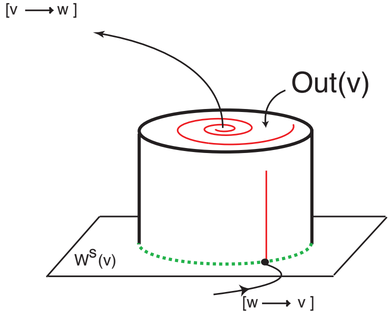
With 2-D connection

Like same chirality.

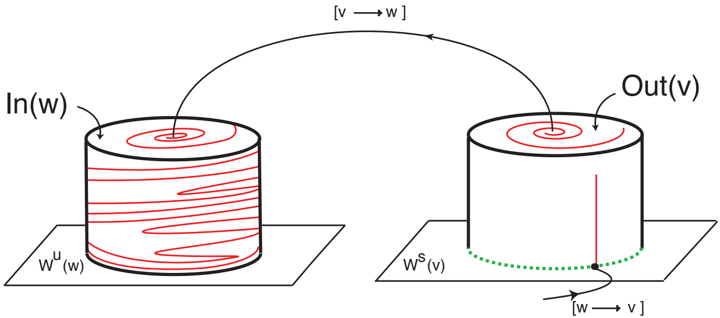
Transverse 2-D connection

A segment is not mapped into a helix.

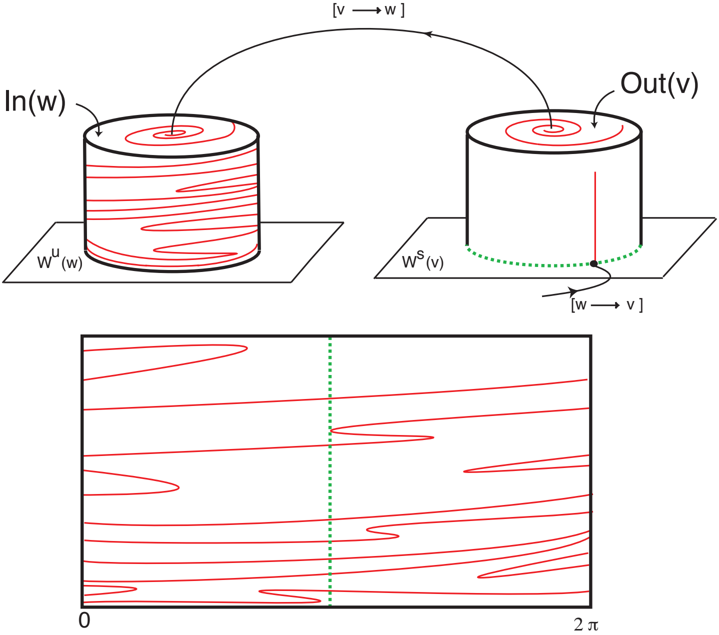
Different chirality — transverse 2-D connection



Different chirality — transverse 2-D connection



Different chirality — transverse 2-D connection



Different chirality— transverse 2-D connection

For an open set of parameters

Heteroclinic tangencies of 2-D invariant manifolds $W^s(v)$ $W^u(w)$ in dense set of parameters.

Infinitely many sinks nearby (Newhouse).

Coexisting with

suspended hyperbolic horseshoes around the network.

All these structures form a spiralling attractor around the network.

The End

Thank you for your attention

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