

# Global Dynamics for Symmetric Planar Maps

Isabel S. Labouriau

Centro de Matemática da Universidade do Porto  
Portugal

ICMC - USP, São Carlos, SP, Brasil 23 October 2012



# Global Dynamics for Symmetric Planar Maps

Joint work with

- ▶ Begoña Alarcón  
Department of Mathematics, University of Oviedo, Spain
- ▶ Sofia Castro  
Faculty of Economics, University of Porto, Portugal

# Context

## Planar Maps

- ▶  $f : \mathbf{R}^2 \rightarrow \mathbf{R}^2$
- ▶  $f \in \text{Emb}(\mathbf{R}^2)$ , that is, continuous and injective

## Symmetric Maps

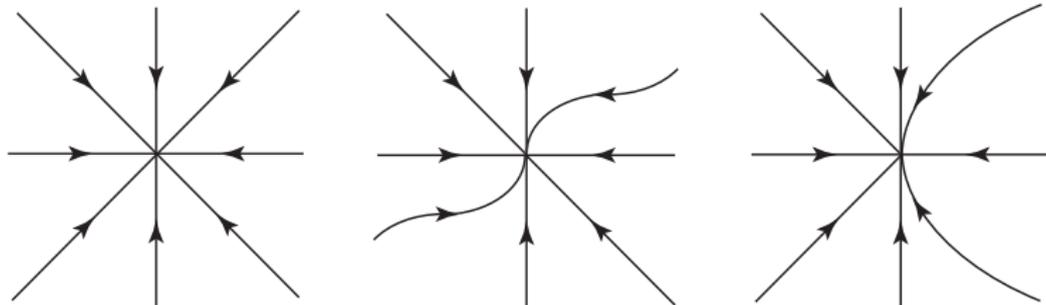
- ▶  $\gamma \in O(2) \subset GL(2)$ , is a **symmetry** of  $f$  if

$$f(\gamma \cdot p) = \gamma \cdot f(p) \quad \text{for all } p \in \mathbf{R}^2, \gamma \in \Gamma$$

- ▶  $\Gamma \subset O(2) \subset GL(2)$  **symmetry group** of  $f$ :  
closed subgroup of  $O(2)$  of all symmetries of  $f$ .
- ▶  $\Gamma \subset O(2)$  compact Lie group acting on  $\mathbf{R}^2$
- ▶ also say  $f$  is  **$\Gamma$ -equivariant**
- ▶ action of  $\Gamma \implies f(0) = 0$  unless  $\Gamma = \mathbf{Z}_2 \langle \kappa \rangle$

# Dynamics and Symmetries

- ▶ attractor/repellor



- ▶ symmetries

- ▶ **Left:** two reflections, one on each coordinate axis

$$\Gamma = \mathbf{Z}_2 \oplus \mathbf{Z}_2$$

- ▶ **Middle:** no reflection

$$\Gamma = \mathbf{Z}_2 = \langle -Id \rangle$$

- ▶ **Right:** one reflection on the horizontal axis

$$\Gamma = \mathbf{Z}_2 \langle \kappa \rangle$$

# Dynamics

- ▶  $f \in \text{Emb}(\mathbf{R}^2)$  has *trivial dynamics* if for all  $p \in \mathbf{R}^2$ , either  $\omega(p) \subset \text{Fix}(f)$  or  $\omega(p) = \infty$ .
- ▶  $\gamma : [0, \infty) \rightarrow \mathbf{R}^2$  continuous is an *f-invariant ray* if
  - ▶  $\gamma(0) = (0, 0)$
  - ▶  $f(\gamma) \subset \gamma \equiv \gamma([0, \infty))$
  - ▶  $\lim_{t \rightarrow \infty} |\gamma(t)| = \infty$

## Proposition (Alarcón, Guíñez, Gutierrez (2008))

Let  $f \in \text{Emb}^+(\mathbf{R}^2)$  be such that  $\text{Fix}(f) = \{0\}$ . If there exists an  $f$ -invariant ray, then  $f$  has trivial dynamics.

## Symmetry and $f$ -invariant rays

- ▶  $\kappa \in \Gamma$  acts a reflection

$$\kappa.(x, y) = (x, -y)$$

- ▶  $\text{Fix}\langle\kappa\rangle = \{p \in \mathbf{R}^2 : \kappa.p = p\} = \{(x, 0) : x \in \mathbf{R}\}$

### Lemma

Let  $f : \mathbf{R}^2 \rightarrow \mathbf{R}^2$  be a map with symmetry group  $\Gamma$  and  $\text{Fix}(f) = \{0\}$ . If  $\kappa \in \Gamma$ , then  $\text{Fix}\langle\kappa\rangle$  is an  $f$ -invariant line. Moreover,  $\text{Fix}\langle\kappa\rangle$  contains an  $f^2$ -invariant ray.

## $\omega$ -limits of second iterates

$$\omega_2 = \{q \in \mathbf{R}^2 : \lim_{n_k \rightarrow \infty} f^{2n_k}(p) = q \text{ for some sequence } n_k \rightarrow \infty\}$$

### Lemma

Let  $f \in \text{Emb}(\mathbf{R}^n)$  be such that  $f(0) = 0$ . For  $p \in \mathbf{R}^n$ ,

- a) if  $\omega_2(p) = \{0\}$ , then  $\omega(p) = \{0\}$ ;
- b) if  $\omega_2(p) = \infty$ , then  $\omega(p) = \infty$ .

### Proof.

Let  $p \in \mathbf{R}^n$ . Suppose there exists  $r \in \omega(p) \neq \omega_2(p)$  and show that

$$f(r) = \lim_{n_k \rightarrow \infty} f^{n_k+1}(p) \text{ for } n_k + 1 \text{ even, that is, } f(r) \in \omega_2(p)$$

Contradictions in:

- (a)  $\omega_2(p) = \{0\}$  and  $r \neq 0$
- (b)  $\omega_2(p) = \infty$  and  $r < \infty$



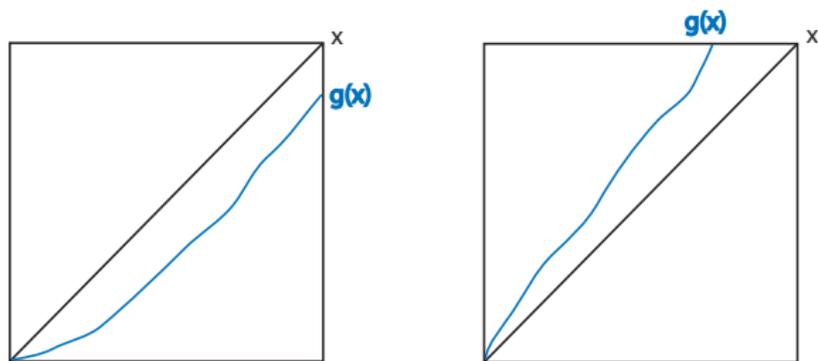
# Dynamics on rays

## Lemma

Let  $g : [0, 1) \rightarrow [0, 1)$  be a continuous and injective map such that  $\text{Fix}(g) = \{0\}$ . The following holds:

- a) If 0 is a local attractor for  $g$ , then 0 is a global attractor for  $g$ .
- b) If 0 is a local repeller for  $g$ , then 0 is a global repeller for  $g$ .

Proof.



## Admissible $\omega$ -limit sets ( $\kappa \in \Gamma$ )

### Proposition

Let  $f \in Emb(\mathbf{R}^2)$  have symmetry group  $\Gamma$  with  $\kappa \in \Gamma$ , such that  $Fix(f) = \{0\}$ . Suppose one of the followings holds:

- a)  $f \in Emb^+(\mathbf{R}^2)$  and  $f$  does not interchange connected components of  $\mathbf{R}^2 \setminus Fix\langle \kappa \rangle$ .
- b)  $Fix(f^2) = \{0\}$ .

Then for each  $p \in \mathbf{R}^2$  either  $\omega(p) = \{0\}$  or  $\omega(p) = \infty$ .

## Admissible $\omega$ -limit sets ( $\kappa \in \Gamma$ )

### Proposition

Let  $f \in \text{Emb}(\mathbf{R}^2)$  have symmetry group  $\Gamma$  with  $\kappa \in \Gamma$ , such that  $\text{Fix}(f) = \{0\}$ . Suppose the followings holds:

b)  $\text{Fix}(f^2) = \{0\}$ .

Then for each  $p \in \mathbf{R}^2$  either  $\omega(p) = \{0\}$  or  $\omega(p) = \infty$ .

### Proof.

- ▶  $\text{Fix}\langle \kappa \rangle$  contains  $f^2$ -invariant ray,  $f^2 \in \text{Emb}^+(\mathbf{R}^2)$ .
- ▶ Alarcón et al (2008)  $\implies \omega_2(p) = \{0\}$  or  $\omega_2(p) = \infty$ . □

## Admissible $\omega$ -limit sets ( $\kappa \in \Gamma$ )

### Proposition

Let  $f \in \text{Emb}(\mathbf{R}^2)$  have symmetry group  $\Gamma$  with  $\kappa \in \Gamma$ , such that  $\text{Fix}(f) = \{0\}$ . Suppose the followings holds:

- a)  $f \in \text{Emb}^+(\mathbf{R}^2)$  and  $f$  does not interchange connected components of  $\mathbf{R}^2 \setminus \text{Fix}\langle\kappa\rangle$ .

Then for each  $p \in \mathbf{R}^2$  either  $\omega(p) = \{0\}$  or  $\omega(p) = \infty$ .

### Proof outline.

- ▶  $\Omega(f) \subset \text{Fix}\langle\kappa\rangle$
- ▶  $p \in \text{Fix}\langle\kappa\rangle \implies \omega(p) = 0$  or  $\omega(p) = \infty$
- ▶  $p \in \mathbf{R}^2 \setminus \text{Fix}\langle\kappa\rangle \implies \omega(p) = 0$  or  $\omega(p) = \infty$



# Admissible $\omega$ -limit sets ( $\kappa \in \Gamma$ )

## Proposition

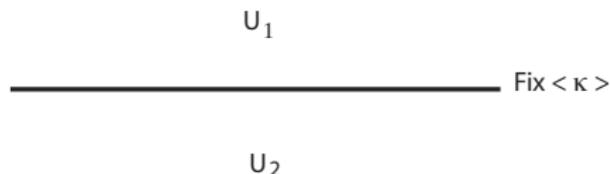
Let  $f \in \text{Emb}(\mathbf{R}^2)$  have symmetry group  $\Gamma$  with  $\kappa \in \Gamma$ , such that  $\text{Fix}(f) = \{0\}$ . Suppose the followings holds:

- a)  $f \in \text{Emb}^+(\mathbf{R}^2)$  and  $f$  does not interchange connected components of  $\mathbf{R}^2 \setminus \text{Fix}\langle \kappa \rangle$ .

Then for each  $p \in \mathbf{R}^2$  either  $\omega(p) = \{0\}$  or  $\omega(p) = \infty$ .

## Proof.

- $\Omega(f) \subset \text{Fix}\langle \kappa \rangle$



$$f_i = |_{U_i} : U_i \rightarrow U_i \text{ with } U_i \sim \mathbf{R}^2$$

Murthy (1998)  $g \in \text{Emb}^+(\mathbf{R}^2)$  and  $\text{Fix}(g) = \emptyset \implies \Omega(g) = \emptyset$



# Admissible $\omega$ -limit sets ( $\kappa \in \Gamma$ )

## Proposition

Let  $f \in Emb(\mathbf{R}^2)$  have symmetry group  $\Gamma$  with  $\kappa \in \Gamma$ , such that  $Fix(f) = \{0\}$ . Suppose the followings holds:

- a)  $f \in Emb^+(\mathbf{R}^2)$  and  $f$  does not interchange connected components of  $\mathbf{R}^2 \setminus Fix\langle \kappa \rangle$ .

Then for each  $p \in \mathbf{R}^2$  either  $\omega(p) = \{0\}$  or  $\omega(p) = \infty$ .

## Proof.

- $p \in Fix\langle \kappa \rangle \implies \omega(p) = 0$  or  $\omega(p) = \infty$



$f^2: \gamma_i \rightarrow \gamma_i$  and  $f^2(0) = 0 \implies \omega_2(p) = \{0\}$  or  $\omega_2(p) = \infty$  for all  $p \in \gamma_i$



# Admissible $\omega$ -limit sets ( $\kappa \in \Gamma$ )

## Proposition

Let  $f \in Emb(\mathbf{R}^2)$  have symmetry group  $\Gamma$  with  $\kappa \in \Gamma$ , such that  $Fix(f) = \{0\}$ . Suppose the following holds:

- a)  $f \in Emb^+(\mathbf{R}^2)$  and  $f$  does not interchange connected components of  $\mathbf{R}^2 \setminus Fix\langle \kappa \rangle$ .

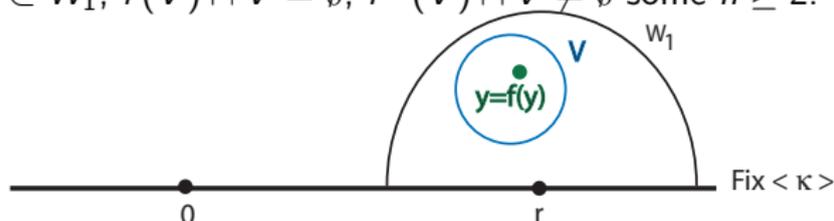
Then for each  $p \in \mathbf{R}^2$  either  $\omega(p) = \{0\}$  or  $\omega(p) = \infty$ .

## Proof.

- $p \in \mathbf{R}^2 \setminus Fix\langle \kappa \rangle \implies \omega(p) = 0$  or  $\omega(p) = \infty$

If  $0 \neq r \in \omega(p)$  get  $p \in W_1$  open,  $r \in \overline{W_1}$ ,  $0 \notin \overline{W_1}$ ,  $f(r) \neq r$ , then

$\exists$  open  $V \subset W_1$ ,  $f(V) \cap V = \emptyset$ ,  $f^m(V) \cap V \neq \emptyset$  some  $n \geq 2$ .



Murthy (1998): there exists  $y \in V$  such that  $f(y) = y \neq 0$



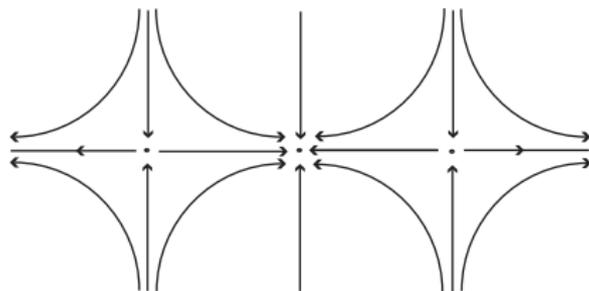
## Necessity of $\text{Fix}(f^2) = \{0\}$

$\Gamma = D_2 = \langle \kappa_1, \kappa_2 \rangle$  where

$\kappa_1(x, y) = (x, -y)$  and  $\kappa_2(x, y) = (-x, y)$

$$f(x, y) = \left( -ax^3 + (a-1)x, -\frac{y}{2} \right), \quad 0 < a < 1$$

- ▶  $f \in \text{Emb}^+(\mathbf{R}^2)$
- ▶  $\{-1, 0, 1\} \subset \text{Fix}(f^2)$ .



# Local to Global

## Theorem

Let  $f \in \text{Emb}(\mathbf{R}^2)$  be dissipative with symmetry group  $\Gamma$  with  $\kappa \in \Gamma$  such that  $\text{Fix}(f) = \{0\}$ . Suppose in addition that one of the following holds:

- a)  $f \in \text{Emb}^+(\mathbf{R}^2)$  and  $f$  does not interchange connected components of  $\mathbf{R}^2 \setminus \text{Fix}\langle \kappa \rangle$ .
- b)  $\text{Fix}(f^2) = \{0\}$ .

Then  $0$  is a global attractor.

## Corollary

Suppose the assumptions above are verified and  $f$  is differentiable at  $0$ . If every eigenvalue of  $Df(0)$  has norm strictly less than one, then  $0$  is a global asymptotic attractor.

# Local to Global

## Theorem

Let  $f \in \text{Emb}(\mathbf{R}^2)$  be a map with symmetry group  $\Gamma$  with  $\kappa \in \Gamma$  such that  $\text{Fix}(f) = \{0\}$ . Suppose in addition that one of the following holds:

- a)  $f \in \text{Emb}^+(\mathbf{R}^2)$  and  $f$  does not interchange connected components of  $\mathbf{R}^2 \setminus \text{Fix}\langle \kappa \rangle$ .
- b)  $\text{Fix}(f^2) = \{0\}$ .

Then, if  $0$  is a local repellor, then  $0$  is a global repellor.

## Corollary

Suppose the assumptions above are verified and  $f$  is differentiable at  $0$ . If every eigenvalue of  $Df(0)$  has norm strictly greater than one, then  $0$  is a global asymptotic repellor.

# Symmetric Local Dynamics

Symmetry group	$Df(0)$	hyperbolic local dynamics
$O(2)$	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \alpha \in \mathbf{R}$	attractor / repellor
$SO(2)$	$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \alpha, \beta \in \mathbf{R}$	attractor / repellor
$D_n, n \geq 3$	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \alpha \in \mathbf{R}$	attractor / repellor
$\mathbf{Z}_n, n \geq 3$	$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \alpha, \beta \in \mathbf{R}$	attractor / repellor
$\mathbf{Z}_2$	any matrix	saddle / attractor / repellor
$\mathbf{Z}_2\langle\kappa\rangle$	$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \alpha, \beta \in \mathbf{R}$	saddle / attractor / repellor
$D_2$	$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \alpha, \beta \in \mathbf{R}$	saddle / attractor / repellor

Conventions:

$$\mathbf{Z}_2 = \langle -Id \rangle$$

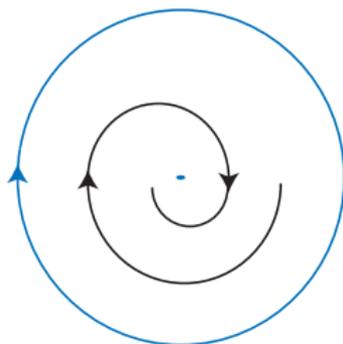
$$D_2 = \mathbf{Z}_2\langle -\kappa \rangle \oplus \mathbf{Z}_2\langle \kappa \rangle$$

## Dynamics when $\kappa \notin \Gamma$ — $SO(2)$

In polar coordinates

$$f(\rho, \theta) = (R(\rho, \theta), T(\rho, \theta)) \equiv (R(\rho), T(\rho, \theta))$$

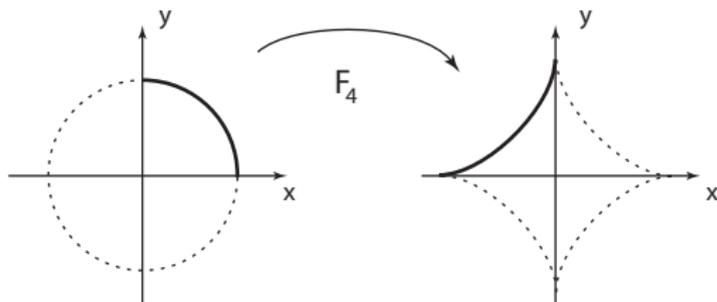
- ▶  $\{0\} = \text{Fix}(R) \implies$  local to global OK
- ▶  $\{0\} \subsetneq \text{Fix}(R) \implies$  radial dynamics



## Dynamics when $\kappa \notin \Gamma$ — $\mathbf{Z}_4$

Cima, Gasull, Mañosas (1999) — Szlenk's example —  
 $\mathbf{Z}_4$ -equivariant:

$$F_4(x, y) = \left( -\frac{ky^3}{1+x^2+y^2}, \frac{kx^3}{1+x^2+y^2} \right) \quad \text{for } 1 < k < \frac{2}{\sqrt{3}}$$



- ▶  $\text{Fix}(F_4) = \{0\}$ , a local attractor;
- ▶  $F_4^4(P) = P$  for  $P = ((k-1)^{-1/2}, 0)$  with  $F_4^j(P) \neq P$  for  $j = 1, 2, 3$ ;
- ▶  $F_4$  maps a quarter of the unit circle into a quarter of the astroid  $\frac{k}{2}(-\sin^3 \theta, \cos^3 \theta)$ ;
- ▶  $F_4$  maps line rays  $t(\alpha, \beta)$ ,  $t \in \mathbf{R}$  into line rays.

## Dynamics when $\kappa \notin \Gamma$ — $\mathbf{Z}_4$

### Proposition

A  $\mathbf{Z}_4$  universal unfolding under contact equivalence of the germ at the origin of the singularity  $F_4$  is given by

$$G_4(x, y, \alpha, \beta, \delta) = F_4(x, y) + \alpha(x, y) + [\beta + \delta(x^2 + y^2)](-y, x),$$

where parameters  $\alpha$ ,  $\beta$  and  $\delta$  are real.

### Proposition

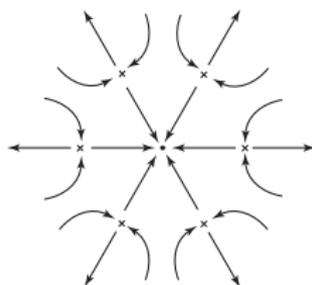
For fixed  $\alpha$  or  $\beta$  positive and small enough, let  $g(x, y)$  be either  $G_4(x, y, \alpha, 0, 0)$  or  $G_4(x, y, 0, \beta, 0)$ . Then

- ▶  $g$  is a global diffeomorphism;
- ▶ at every point in  $\mathbf{R}^2$  the eigenvalues of the jacobian of  $g$  have modulus less than one;
- ▶ there exists  $p \in \mathbf{R}^2$  such that  $g^4(p) = p$ .

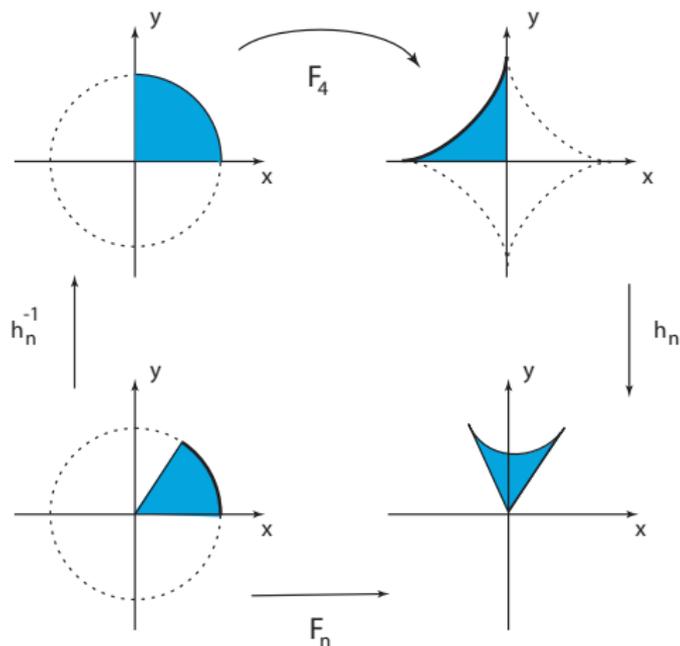
## Dynamics when $\kappa \notin \Gamma$ — $\mathbf{Z}_n$

$F_n$  a  $\mathbf{Z}_n$ -equivariant diffeomorphism

- ▶  $\text{Fix}(F_n) = \{0\}$ , a local attractor;
- ▶  $F_n^j(P) = P$  for  $P = ((k-1)^{-1/2}, 0)$   
with  $F_n^j(P) \neq P$  for  $j \neq 0 \pmod{n}$ .



# Dynamics when $\kappa \notin \Gamma - \mathbf{Z}_n$



Construction of  $F_n$  in a fundamental domain of the  $\mathbf{Z}_n$ -action.

## Dynamics when $\kappa \notin \Gamma$ — $\mathbf{Z}_n$

- $\mathbf{Z}_4$  Cima, Gasull, Mañosas (1999) Szlenk's example — period 4
- $\mathbf{Z}_n$  Alarcón, Castro, Labouriau (2012) “A local but not global attractor for a  $\mathbf{Z}_n$ -symmetric map”, *Journal of Singularities* — period  $n$
- $\mathbf{Z}_n$  Alarcón, “Rotation numbers for planar attractors of equivariant homeomorphisms”, *Preprint CMUP 2012–23* and arXiv:1206.6066 — irrational rotation numbers,  $\mathbf{Z}_n$  Denjoy maps.

## Dynamics when 0 is a saddle

$\Gamma$  is either  $\mathbf{Z}_2$ , or  $\mathbf{Z}_2\langle\kappa\rangle$ , or  $D_2$

Symmetry is not so relevant.

Alarcón, Castro, Labouriau, “Global Saddle for Planar Diffeomorphisms and Equations of Lienard type”, *in preparation*