Recognition of symmetries in reversible maps

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Recognition of symmetries in reversible maps

Joint work with

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Part of the project

Singularities and Algebraic Theory in Dynamical Systems

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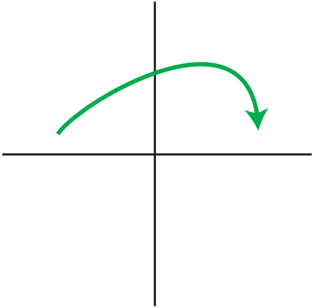
Outline

Definitions

An involution in $\mathbb{R}^n$ is (the germ of) a diffeomorphism $\varphi : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ satisfying $\varphi \circ \varphi = I_n$.

A vector field v in $\mathbb{R}^n$ is φ-reversible if $D\varphi(x)v(x) = -v(\varphi(x))$
i.e.
if $x(t)$ is a solution of $\dot{x} = v(x)$ then
$y(t) = \varphi(x(-t))$ is also a solution.



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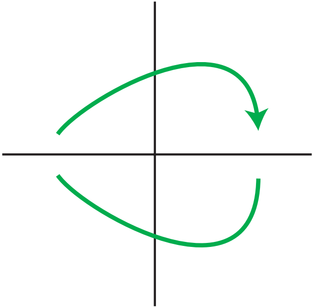

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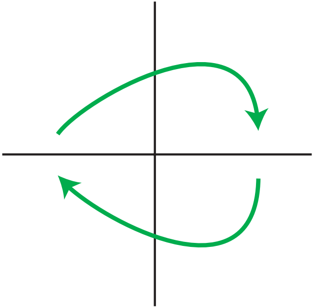
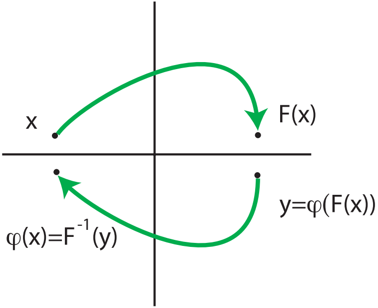


Definitions

A germ of a diffeomorphism $F : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is φ-reversible if $\varphi \circ F = F^{-1} \circ \varphi$.
In this case, φ is a reversing symmetry of F.

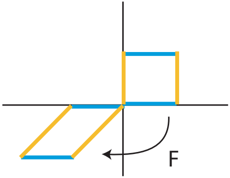


A germ of a diffeomorphism $F : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is φ-equivariant if $F \circ \varphi = \varphi \circ F$.
In this case, φ is a symmetry of F .

Example

$$F(x, y) = (-x - y, -y)$$

$$F \sim \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$$



$$F^{-1} \sim \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$\varphi_1(x, y) = (-x, y) \sim \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$ reversing symmetry of F

Definitions

φ_1, φ_2 reversing symmetries of $F \quad \Rightarrow \quad \varphi_1 \circ \varphi_2$ symmetry of F

$\Gamma_+ = \{\varphi \in \Omega : F \circ \varphi = \varphi \circ F\}$ group of symmetries of F

$\Gamma_- = \{\varphi \in \Omega : F \circ \varphi = \varphi \circ F^{-1}\}$ set of reversing symmetries of F

$\Gamma = \Gamma_+ \cup \Gamma_-$ is a group, if $F^2 \neq Id \quad \Rightarrow \quad \Gamma_-$ a coset of Γ_+

Fixed-point sets

The fixed-point set of a map-germ φ is

$$\mathrm{Fix}(\varphi) = \{x \in (\mathbb{R}^n, 0) : \ \varphi(x) = x\}$$

and t

Lemma (folclore)

Let $F : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a germ of an equivariant diffeomorphism with symmetry group Γ_+.
If $\Sigma \leq \Gamma_+$ is a subgroup, then $\mathrm{Fix}(\Sigma)$ is F-invariant.

New reversing symmetries for free

φ_1 an involution and a reversing symmetry of F

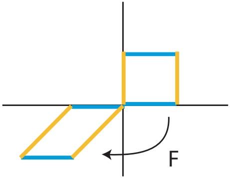
$\Rightarrow \quad \varphi_2 = \varphi_1 \circ F$ is an involution

and $F = \varphi_1 \circ \varphi_2$.
Also F is φ_2-reversible since

$$F \circ \varphi_2 = \varphi_1 = \varphi_1^{-1} = \varphi_2 \circ \varphi_2^{-1} \circ \varphi_1^{-1} = \varphi_2 \circ F^{-1}.$$

we say F corresponds to the pair of reversing symmetries (φ_1, φ_2)

Example



$$F \sim \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$$

$$F^{-1} \sim \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

$\varphi_1 \sim \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ reversing symmetry of F

$\varphi_2 = \varphi_1 \circ F \sim \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ reversing symmetry of F

Algebra — New reversing symmetries for free

Proposition

$\varphi_1 : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ *involution*
$F : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ *diffeomorphism* φ_1*-reversible* $\quad F^m \neq I_n$
Then F has an infinite group of symmetries and an infinite set of reversing symmetries.

Proof.

Symmetries: F is a symmetry of itself $\Rightarrow$

$$\langle F \rangle = \{F^k : k \in \mathbb{Z}\} \subset \Gamma_+.$$

Reversing symmetries: $\varphi_k, \varphi_k' \in \Gamma_- \quad k \in \mathbb{N}$:

$$\varphi_k = \varphi_1 \circ F^{k-1}, \qquad \varphi_k' = F^{k-1} \circ \varphi_1, \quad k \in \mathbb{N}, \quad k \geq 1$$

(φ_1 and φ_2 consistent $\qquad$ for $k = 1$ get $\varphi_1' = \varphi_1$) $\square$

Example

$$F(x, y) = (-x - y, -y)$$

$$F \sim \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$$

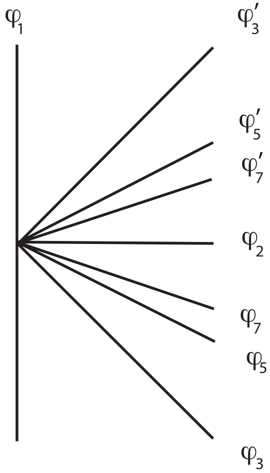
$$\varphi_1 \sim \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\varphi_2 \sim \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$\varphi_k \sim (-1)^k \begin{pmatrix} 1 & k-1 \\ 0 & -1 \end{pmatrix}$$

$$\varphi'_k \sim (-1)^k \begin{pmatrix} 1 & 1-k \\ 0 & -1 \end{pmatrix}$$

Fixed-point subspaces



Fixed-point sets

Theorem

$\varphi_1 : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ *involution*
$F : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ *diffeomorphism* φ_1*-reversible*
Then F *maps:*

$$\cdots \to \mathrm{Fix}(\varphi_{2k}) \to \cdots \to \mathrm{Fix}(\varphi_2) \to \mathrm{Fix}(\varphi_2') \to \cdots \to \mathrm{Fix}(\varphi_{2k}') \to \cdots$$

$$\cdots \to \mathrm{Fix}(\varphi_{2k+1}) \to \cdots \to \mathrm{Fix}(\varphi_1) \to \mathrm{Fix}(\varphi_3') \to \cdots \to \mathrm{Fix}(\varphi_{2k+1}') \to \cdots$$

Dynamics

Proposition

φ reversing symmetry of F, $x \in \mathrm{Fix}(\varphi)$ and $F^\ell(x) \in \mathrm{Fix}(\varphi)$, then $F^k(x) = x$ for some k that divides 2ℓ.

Proof.

If $x, F^\ell(x) \in \mathrm{Fix}(\varphi)$ then

$$F^{-\ell}(x) = F^{-\ell}(\varphi(x)) = \varphi(F^\ell(x)) = F^\ell(x).$$

□

Example

$$F(x, y) = (-x - y, -y)$$

$$F \sim \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$$

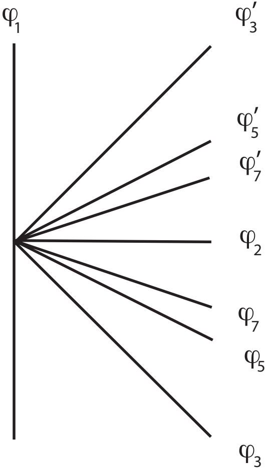
$$\varphi_2 \sim \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$\varphi_k \sim (-1)^k \begin{pmatrix} 1 & k-1 \\ 0 & -1 \end{pmatrix}$$

$$\varphi'_k \sim (-1)^k \begin{pmatrix} 1 & 1-k \\ 0 & -1 \end{pmatrix}$$

$$\mathrm{Fix}(\varphi_{2k}) = \{(x, 0)\}$$
$$\mathrm{Fix}(\varphi'_{2k}) = \{(x, 0)\}$$
$$\Rightarrow \quad F^2(x, 0) = (x, 0)$$

Fixed-point subspaces



Dynamics

Proposition

φ_k, φ_ℓ reversing symmetries of F.

A) $x \in \mathrm{Fix}(\varphi_k) \cap \mathrm{Fix}(\varphi_\ell)$, $\ell > k$ and $F^\ell(x) \in \mathrm{Fix}(\varphi_k) \cap \mathrm{Fix}(\varphi_\ell)$, then $F^j(x) = x$ and j divides $\ell - k$.
If $x \in \mathrm{Fix}(\varphi_k)$, $F^k(x) = x$ and k divides $\ell - k$ then $x \in \mathrm{Fix}(\varphi_\ell)$.

B) $x \in \mathrm{Fix}(\varphi'_k) \cap \mathrm{Fix}(\varphi'_\ell)$, $\ell > k$ and $F^\ell(x) \in \mathrm{Fix}(\varphi'_k) \cap \mathrm{Fix}(\varphi'_\ell)$, then $F^j(x) = x$ and j divides $\ell - k$.
If $x \in \mathrm{Fix}(\varphi'_k)$, $F^k(x) = x$ and k divides $\ell - k$ then $x \in \mathrm{Fix}(\varphi'_\ell)$.

C) $x \in \mathrm{Fix}(\varphi'_k) \cap \mathrm{Fix}(\varphi_\ell)$, $\ell > k$ and $F^\ell(x) \in \mathrm{Fix}(\varphi'_k) \cap \mathrm{Fix}(\varphi_\ell)$, then $F^j(x) = x$ and j divides $\ell - k$.
If $x \in \mathrm{Fix}(\varphi'_k)$, $F^k(x) = x$ and k divides $\ell - k$ then $x \in \mathrm{Fix}(\varphi_\ell)$.

Particular case $F : (\mathbb{R}^{2n}, 0) \longrightarrow (\mathbb{R}^{2n}, 0)$ and $\dim \mathrm{Fix}(\varphi_j) = n$
in Devaney (1976) Trans. AMS

Example

$$F(x,y) = (-x-y, -y)$$

$$F \sim \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$$

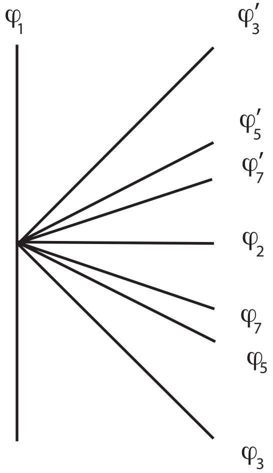
$$\varphi_2 \sim \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$\varphi_k \sim (-1)^k \begin{pmatrix} 1 & k-1 \\ 0 & -1 \end{pmatrix}$$

$$\varphi'_k \sim (-1)^k \begin{pmatrix} 1 & 1-k \\ 0 & -1 \end{pmatrix}$$

$\mathrm{Fix}(\varphi_{2k}) = \{(x,0)\}$
$\mathrm{Fix}(\varphi'_{2k}) = \{(x,0)\}$

Fixed-point subspaces



Dynamics

Theorem

Let $\varphi_1, \varphi_2 : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ *be two involutions with* $\dim \mathrm{Fix}(\varphi_1) = \dim \mathrm{Fix}(\varphi_2) = n - 1$*. Let* φ_k*,* $\varphi'_k \in \Gamma_-$*,* $k \in \mathbb{N}$*,* $k \geq 1$ *be as before.*
Then $F = \varphi_1 \circ \varphi_2$ *interchanges the connected components of the germ at the origin of*

$$\mathcal{C} = \mathbb{R}^n \setminus \bigcup_{k=1}^{\infty} \left(\mathrm{Fix}(\varphi_k) \cup \mathrm{Fix}(\varphi'_k)\right).$$

Examples — preliminaries

Two involutions φ_1, φ_2 on $(\mathbb{R}^n, 0)$, $n \geq 2$, are transversal if $\mathrm{Fix}(\varphi_1) \pitchfork_0 \mathrm{Fix}(\varphi_2)$

ψ_1 and ψ_2 transversal linear involutions

classified in Mancini *et al* (2005) Discr. Cont. Dyn. Sys.

$\Lambda = [\psi_1, \psi_2]$ the group generated by ψ_1 and ψ_2.

The antipodal subspace of a linear involution φ on $\mathbb{R}^n$ is

$$\mathcal{A}(\varphi) = \{x \in \mathbb{R}^n : \ \varphi(x) = -x\}.$$

Notice that $\mathbb{R}^n = \mathrm{Fix}(\varphi) \oplus \mathcal{A}(\varphi)$.

Cases

i. Λ is Abelian;

ii. Λ is non-Abelian and $\mathcal{A}(\psi_2) = \mathrm{Fix}(\psi_1)$;

iii. Λ is non-Abelian and $\mathcal{A}(\psi_2) \neq \mathrm{Fix}(\psi_1)$.

Example $\Lambda = [\psi_1, \psi_2]$ Abelian

$F = \psi_1 \circ \psi_2$ on $(\mathbb{R}^2, 0)$

Mancini *et al* $\quad \Rightarrow \quad (\psi_1, \psi_2) \sim (\varphi_1, \varphi_2) = ((-x, y), (x, -y))$

$\quad \Rightarrow \quad \Lambda = \mathbb{Z}_2(\varphi_1) \oplus \mathbb{Z}_2(\varphi_2). \qquad F = -I_2$

degenerate dynamics: $F^2 = I_2$

Example $\Lambda = [\psi_1, \psi_2]$ non-Abelian and $\mathcal{A}(\psi_2) = \mathrm{Fix}(\psi_1)$

$F = \psi_1 \circ \psi_2$ on $(\mathbb{R}^2, 0)$

Mancini *et al* $\Rightarrow$ $(\psi_1, \psi_2) \sim (\varphi_1, \varphi_2) = ((-x, x+y), (x, -y))$

$\Rightarrow$ $F(x,y) = (-x, x-y)$

$$F = -I_2 + N \quad N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad F^k = (-1)^k (I_2 - kN) \neq I_2 \quad k \in \mathbb{Z}$$

$$\varphi_k(x,y) = (-1)^k (x, (k-2)x - y) \quad \varphi'_k(x,y) = (-1)^k (x, -kx - y)$$

Fixed-point subspaces

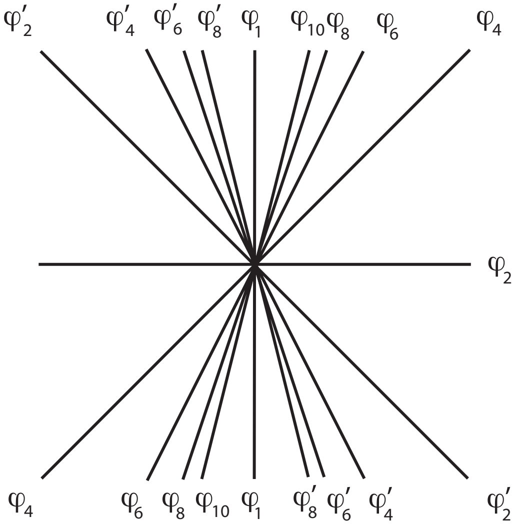
$\mathrm{Fix}(F) = \{(0,0)\} = \mathrm{Fix}(F^{2k+1})$ $k \geq 1$

$\mathrm{Fix}(\varphi_{2k}) = \langle (1, k-1) \rangle$ $\mathrm{Fix}(\varphi'_{2k}) = \langle (1, -k) \rangle$

$$\mathrm{Fix}(F^{2k}) = \mathrm{Fix}(\varphi_{2k+1}) = \mathrm{Fix}(\varphi'_{2k+1}) = \langle (0,1) \rangle$$

$\mathrm{Fix}(\varphi_{2k})$ and $\mathrm{Fix}(\varphi'_{2k}) \to \langle (0,1) \rangle$ as $k \to \infty$

Example $\Lambda = [\varphi_1, \varphi_2]$ non-Abelian and $\mathcal{A}(\varphi_2) = \mathrm{Fix}(\varphi_1)$



Fixed-point subspaces

Example $\Lambda = [\varphi_1, \varphi_2]$ non-Abelian and $\mathcal{A}(\varphi_2) = \mathrm{Fix}(\varphi_1)$

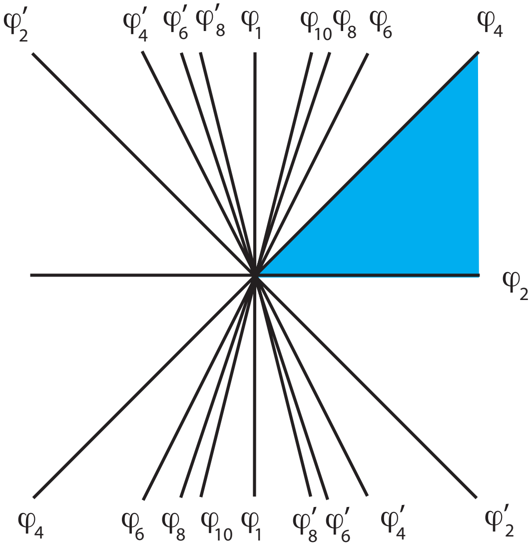
$F(x,y) = (-x, x-y)$ $\varphi_1(x,y) = (-x, x+y)$ $\varphi_2(x,y) = (x,-y)$

Dynamics

A sector between Fix(φ_2) and Fix(φ_4)

goes to

a sector between Fix(φ_2') and Fix(φ_2)



Example $\Lambda = [\varphi_1, \varphi_2]$ non-Abelian and $\mathcal{A}(\varphi_2) = \text{Fix}(\varphi_1)$

$F(x, y) = (-x, x - y)$ $\varphi_1(x, y) = (-x, x + y)$ $\varphi_2(x, y) = (x, -y)$

Dynamics

A sector between Fix(φ_2) and Fix(φ_4)

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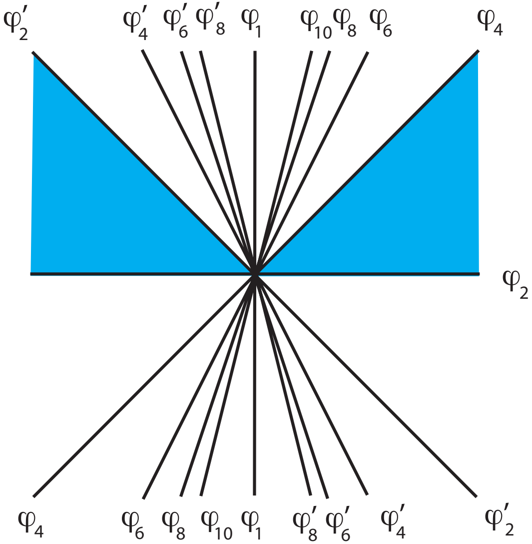
a sector between Fix(φ_2') and Fix(φ_2)



Example $\Lambda = [\varphi_1, \varphi_2]$ non-Abelian and $\mathcal{A}(\varphi_2) = \mathrm{Fix}(\varphi_1)$

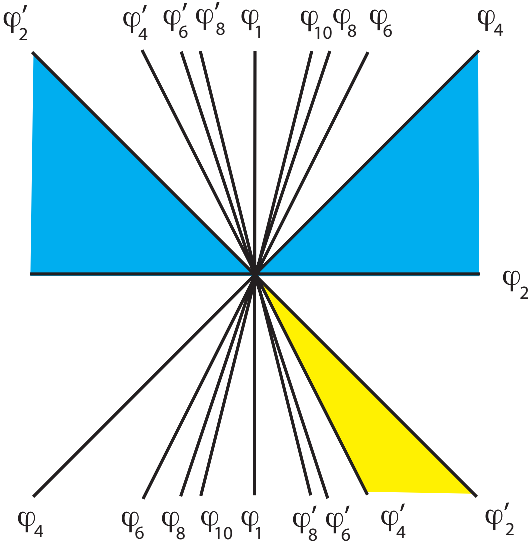
$F(x,y) = (-x, x-y)$ $\varphi_1(x,y) = (-x, x+y)$ $\varphi_2(x,y) = (x,-y)$

Dynamics

A sector between $\mathrm{Fix}(\varphi_2')$ and $\mathrm{Fix}(\varphi_4')$

goes to

a sector between $\mathrm{Fix}(\varphi_4')$ and $\mathrm{Fix}(\varphi_6')$



Example $\Lambda = [\varphi_1, \varphi_2]$ non-Abelian and $\mathcal{A}(\varphi_2) = \mathrm{Fix}(\varphi_1)$

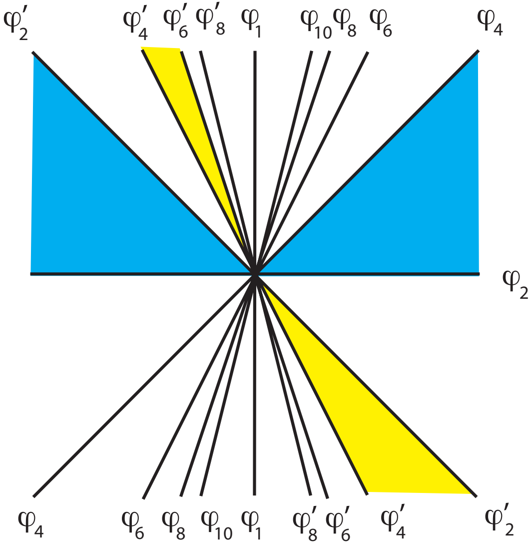
$F(x,y) = (-x, x-y)$ $\varphi_1(x,y) = (-x, x+y)$ $\varphi_2(x,y) = (x, -y)$

Dynamics

A sector between
$\mathrm{Fix}(\varphi_2')$ and $\mathrm{Fix}(\varphi_4')$

goes to

a sector between
$\mathrm{Fix}(\varphi_4')$ and $\mathrm{Fix}(\varphi_6')$



Example $\Lambda = [\psi_1, \psi_2]$ non-Abelian and $\mathcal{A}(\psi_2) \neq \mathrm{Fix}(\psi_1)$

$F = \psi_1 \circ \psi_2$ on $(\mathbb{R}^2, 0)$

Mancini *et al* $\quad \Rightarrow \quad t = \mathrm{tr}(\psi_1 \circ \psi_2)$ invariant $\quad (\psi_1, \psi_2) \sim (\varphi_1, \varphi_2)$

$$\varphi_1(x, y) = (-x, y + (2+t)x) \quad \text{and} \quad \varphi_2(x, y) = (x + y, -y)$$

$$F = \begin{pmatrix} -1 & -1 \\ 2+t & 1+t \end{pmatrix}$$

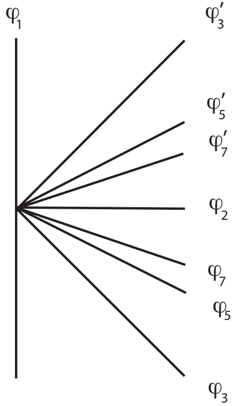
eigenvalues of $F \quad \lambda_+ = (t + \sqrt{t^2 - 4})/2 \quad \lambda_- = (t - \sqrt{t^2 - 4})/2$

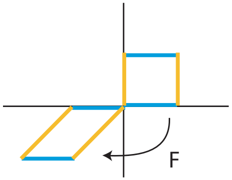
$$\lambda_+ \lambda_- = 1$$

Example $\Lambda = [\varphi_1, \varphi_2]$ non-Abelian and $\mathcal{A}(\varphi_2) \neq \mathrm{Fix}(\varphi_1)$

$t = \mathrm{tr}(F) = -2$

$$F = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$$





Example $\Lambda = [\varphi_1, \varphi_2]$ non-Abelian and $\mathcal{A}(\varphi_2) \neq \mathrm{Fix}(\varphi_1)$

$|t| = |\,\mathrm{tr}(F)| < 2$ $\lambda_\pm = \mathrm{e}^{\pm i\theta}$ eigenvectors $R + iI$

coordinates in the basis $\beta = \{R, I\}$ of $\mathbb{R}^2$

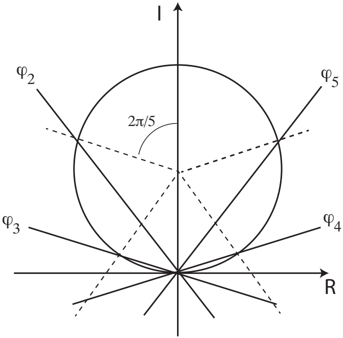
$$F|_\beta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\alpha_k = (k-1)\theta$$

$$\varphi_k|_\beta = \begin{pmatrix} -\cos\alpha_k & \sin\alpha_k \\ \sin\alpha_k & \cos\alpha_k \end{pmatrix}$$

$$\varphi'_k|_\beta = \begin{pmatrix} -\cos\alpha_k & -\sin\alpha_k \\ -\sin\alpha_k & \cos\alpha_k \end{pmatrix}$$

$$\mathrm{Fix}(\varphi_k) = \langle\, (\sin\alpha_k, 1 + \cos\alpha_k)|_\beta \rangle$$
$$\mathrm{Fix}(\varphi'_k) = \langle\, (-\sin\alpha_k, 1 + \cos\alpha_k)|_\beta \rangle$$

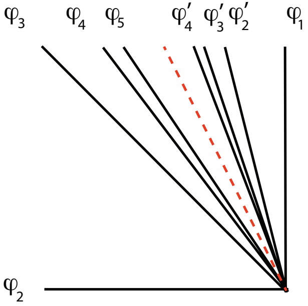


$\theta = 2\pi/5$

Example $\Lambda = [\varphi_1, \varphi_2]$ non-Abelian and $\mathcal{A}(\varphi_2) \neq \mathrm{Fix}(\varphi_1)$

$t = \mathrm{tr}(F) = 2$

$$F = \begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix}$$



$F = I_2 + N \qquad N^2 = 0$

$\mathrm{Fix}(F) = \langle(1, -2)\rangle$

eigenspace of $\lambda = 1$

$\mathrm{Fix}(\varphi_k) = \langle(k-1, 4-2k)\rangle$

$\mathrm{Fix}(\varphi'_k) = \langle(k-1, -2k)\rangle$

$\mathrm{Fix}(\varphi_k)$ and $\mathrm{Fix}(\varphi'_k)$

$\to \mathrm{Fix}(F)$ as $k \to \infty$

half-lines in $\mathrm{Fix}(F)$ are accumulation points of the boundary of

$$\mathcal{C} = \mathbb{R}^2 \setminus \bigcup_{k=1}^{\infty} \left(\mathrm{Fix}(\varphi_k) \cup \mathrm{Fix}(\varphi'_k)\right)$$

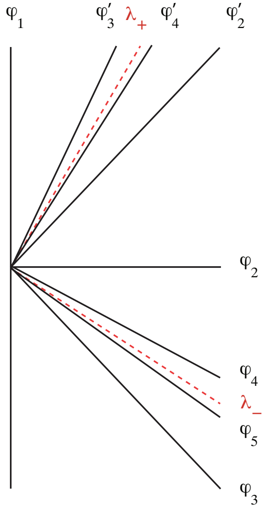
Example $\Lambda = [\varphi_1, \varphi_2]$ non-Abelian and $\mathcal{A}(\varphi_2) \neq \mathrm{Fix}(\varphi_1)$

$|t| = |\,\mathrm{tr}(F)| > 2$ $\qquad$ F has real eigenvalues $0 < |\lambda_c| < 1 < |\lambda_e|$

F hyperbolic — no fixed points

coordinates in the basis β of eigenvectors

$\mathrm{Fix}(\varphi_k) = \langle (1, -\lambda_+^{k-1})|_\beta \rangle$
$\mathrm{Fix}(\varphi'_k) = \langle (1, -\lambda_-^{k-1})|_\beta \rangle$



$t = \mathrm{tr}(F) < -2$

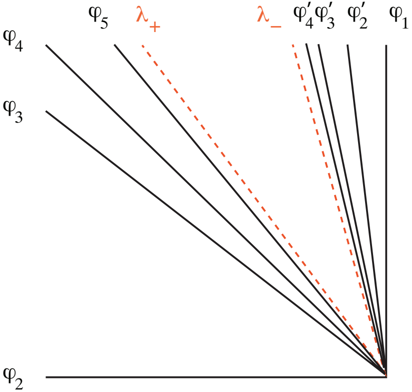
Example $\Lambda = [\varphi_1, \varphi_2]$ non-Abelian and $\mathcal{A}(\varphi_2) \neq \mathrm{Fix}(\varphi_1)$

$|t| = |\,\mathrm{tr}(F)| > 2$ F has real eigenvalues $0 < |\lambda_-| < 1 < |\lambda_+|$

F hyperbolic — no fixed points

coordinates in the basis β of eigenvectors

$\mathrm{Fix}(\varphi_k) = \langle (1, -\lambda_+^{k-1})|_\beta \rangle$
$\mathrm{Fix}(\varphi'_k) = \langle (1, -\lambda_-^{k-1})|_\beta \rangle$



$t = \mathrm{tr}(F) > 2$

The end.