Two time-scales in coupled FitzHugh-Nagumo equations

XIII International Workshop on Real and Complex Singularities
celebrating the $60^{\underline{th}}$ birthday of
Maria del Carmen Romero Fuster

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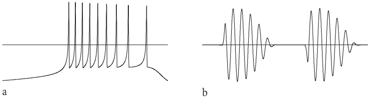
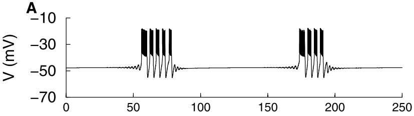
Two time-scales in coupled FitzHugh-Nagumo equations

The problem

Models for bursts: transient or periodic





The method

Ordinary differential equations with two time scales

The example

Two coupled FitzHugh-Nagumo equations

work in progress

pictures from Doss-Bachelet et al, Complexus (2003) and from Bertram et al Bull. Math. Biol. (2008)

Method — Models with two time scales

$$\begin{cases} \varepsilon \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{cases}$$

fast equation, slow equation; $\varepsilon \ll 1$

Study the two time scales separately (singular limit).

Slow time τ

Rescale time: $t = \varepsilon\tau$, $z' = \frac{dz}{d\tau}$

$$\begin{cases} x' &= f(x,y) \\ y' &= \varepsilon g(x,y) \end{cases} \qquad \varepsilon \to 0 \qquad \begin{cases} x' &= f(x,y) \\ y' &= 0 \end{cases}$$

fast equation dominates

Fast time t

$$\varepsilon \to 0 \qquad \begin{cases} 0 &= f(x,y) \\ \dot{y} &= g(x,y) \end{cases}$$

slow manifold, slow equation

Example — FitzHugh-Nagumo equation

$$\left\{ \begin{array}{rcl} \varepsilon \dot{x} & = & \varphi(x) - y \\ \dot{y} & = & x - \gamma y - \delta \end{array} \right. \qquad \begin{array}{l} \text{fast equation} \\ \text{slow equation} \end{array} \qquad \varepsilon \ll 1$$

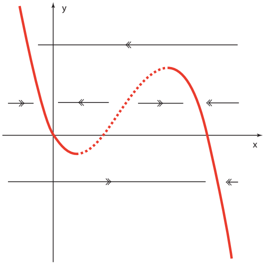
$$\varphi(x) = -x(x-a)(x-b) \qquad 0 < a < b \qquad \delta \in \mathbf{R} \qquad \gamma > 0$$

Fast dynamics $\dot{x} = \varphi(x) - y$

Equilibria lie in slow manifold.

Look at part of slow manifold that is a hyperbolic attracting equilibrium for the fast equation.

Boundary of stability for fast equation contains fold locus.



slow manifold

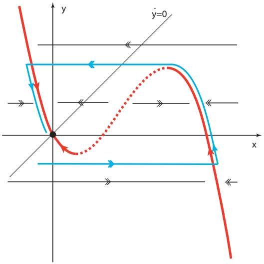
Example — FitzHugh-Nagumo equation

$$\begin{cases} \varepsilon\dot{x} = \varphi(x) - y & \text{fast equation} \\ \dot{y} = x - \gamma y - \delta & \text{slow equation} \end{cases} \qquad \varepsilon \ll 1$$

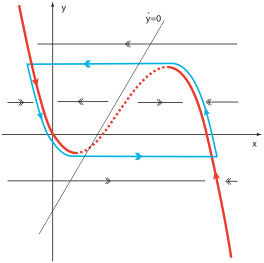
$$\varphi(x) = -x(x-a)(x-b) \qquad 0 < a < b \qquad \delta \in \mathbf{R} \qquad \gamma > 0$$

Dynamics on the slow manifold $\{y = \varphi(x)\}$

Follow slow equation, until trajectory reaches a fold.



Large transient
$\delta \approx 0$



Relaxation oscillation
$\delta \gg 0$

The problem — two coupled FitzHugh-Nagumo equations

$$
\begin{cases}
\varepsilon \dot{x}_1 &= \varphi(x_1) - y_1 - k_1(x_1 - x_2) \\
\dot{y}_1 &= x_1 - \gamma y_1 - \delta \\
\varepsilon \dot{x}_2 &= \varphi(x_2) - y_2 - k_2(x_2 - x_1) \\
\dot{y}_2 &= x_2 - \gamma y_2 - \delta
\end{cases}
$$

$$
\begin{aligned}
0 < k_1 &\le k_2 \\
0 < \varepsilon &\ll 1 \\
\varphi(x) &= -x(x-a)(x-b) \\
0 < a &< b \\
\gamma &> 0 \\
\delta &\in \mathbf{R}
\end{aligned}
$$

Two dimensional fast equation

Two dimensional slow manifold

Two coupled FitzHugh-Nagumo equations

$$\begin{cases} \varepsilon\dot{x}_1 = \varphi(x_1) - y_1 - k_1(x_1 - x_2) & \dot{y}_1 = x_1 - \gamma y_1 - \delta \\ \varepsilon\dot{x}_2 = \varphi(x_2) - y_2 - k_2(x_2 - x_1) & \dot{y}_2 = x_2 - \gamma y_2 - \delta \end{cases}$$

Synchrony $x_1 = x_2$, $y_1 = y_2$

Theorem

For the coupled FitzHugh-Nagumo equations,
the diagonal $\{x_1 = x_2 \;\; y_1 = y_2\}$ *is always flow-invariant.*
For large $k_1 + k_2 \gg 0$ *the diagonal is attracting.*
On the diagonal the two equations are synchronised and behave like a single FitzHugh-Nagumo equation.

Want to study the dynamics for small $k_1 + k_2 > 0$

Method — Models with two time scales

$$\begin{cases} \varepsilon\dot{x} = f(x,y) & \text{fast equation} & \varepsilon \ll 1 \\ \dot{y} = g(x,y) & \text{slow equation} & \end{cases}$$

Geometric part : Hybrid system in the singular limit $\varepsilon \to 0$

- fast equation $\dot{x} = f(x,y)$
- slow manifold $f(x,y) = 0$ (equilibria of fast equation)
- constrained slow equation $\dot{y} = g(x,y)$ on attracting part of slow manifold.
- when slow flow runs into folds — jump out of slow manifold into other attracting part.

Two dimensional fast equation

- equilibria of fast equation may change stability outside the fold line.

Two dimensional slow manifold

- slow equation may have equilibria on the fold line (generic in one parameter families).

Coupled FitzHugh-Nagumo

Fast equation $(\dot{x}_1, \dot{x}_2) = f(x_1, x_2, y_1, y_2)$
Slow manifold $\{(x_1, x_2, y_1, y_2) : \ f(x_1, x_2, y_1, y_2) = 0\}$.
Equilibria of the fast equation.

Lemma

The slow manifold of the coupled FitzHugh-Nagumo equations is a graph $(y_1, y_2) = F(x_1, x_2)$.

Trivial proof:

Fast equation:

$$\begin{cases} \dot{x}_1 & = & \varphi(x_1) - y_1 - k_1(x_1 - x_2) \\ \dot{x}_2 & = & \varphi(x_2) - y_2 - k_2(x_2 - x_1) \end{cases}$$

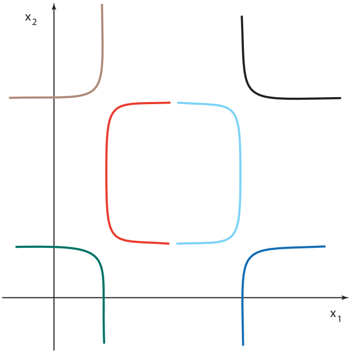
$$0 < k_1 \leq k_2 \qquad 0 < a < b \qquad \varphi(x) = -x(x-a)(x-b)$$

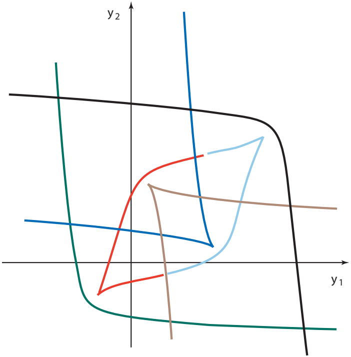
Generic singularities of the projection into the (y_1, y_2)-plane are folds, with four cusp points.

Coupled FitzHugh-Nagumo — the slow manifold

Projection
into the (x_1, x_2)-plane
regular

Projection
into the (y_1, y_2)-plane
with folds and cusps

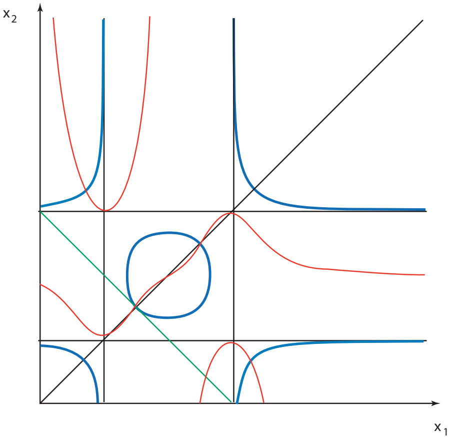




Fold points for coupled FitzHugh-Nagumo, with $a = 1$, $b = 3$, $k_1/2 = k_2 = 1/2$

Coupled FitzHugh-Nagumo — the slow manifold

Projection into the (x_1, x_2)-plane



$a = 1,\ b = 3,\ k_1 = k_2 = 1$

Fold points (blue)

Cusp points where red line meets blue.

On green line: symmetric saddles.

Coupled FitzHugh-Nagumo

Stability of equilibria of the fast equation
Attracting component of the slow manifold

Proposition

The linearisation of the fast equation at an equilibrium point in the coupled FitzHugh-Nagumo equations does not have purely imaginary eigenvalues.

Consequence

The boundary of stability for equilibria of the fast equation is contained in the closure of the fold locus of the slow manifold.

Coupled FitzHugh-Nagumo

Stability of equilibria of the fast equation

Attracting component of the slow manifold

Proposition

For k_1, k_2 sufficiently small, there are two intervals $[m_1, M_1]$, $[m_2, M_2]$ such that all the asymptotically stable equilibria of the coupled FitzHugh-Nagumo equations satisfy $x_1 \notin [m_1, M_1]$ and $x_2 \notin [m_2, M_2]$.
Moreover, the lines $x_1 = m_1$, $x_1 = M_1$, $x_2 = m_2$, $x_2 = M_2$ are asymptotes to the fold locus on the (x_1, x_2)-plane.

Proposition

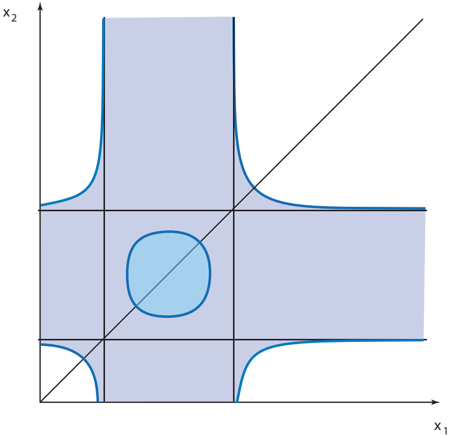
For k_1, k_2 sufficiently small, the attracting component of the slow manifold is contained in the set $\{(x_1, x_2) : \det DF(x_1, x_2) \geq 0\}$, where $\{(x_1, x_2) : \det DF(x_1, x_2) = 0\}$ is the fold locus on the (x_1, x_2)-plane .

Coupled FitzHugh-Nagumo

Attracting component of the slow manifold

Stability of equilibria of fast equation

Projection into the (x_1, x_2)-plane



blue - unstable nodes

gray - saddles

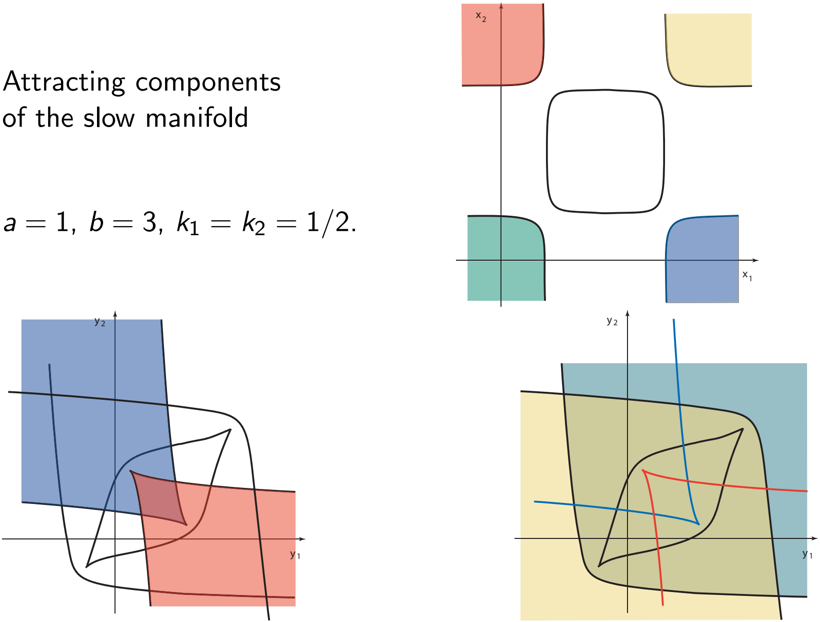
white - stable nodes

$a = 1,\ b = 3,\ k_1 = k_2 = 1$

Coupled FitzHugh-Nagumo — the slow manifold

Attracting components
of the slow manifold

$a = 1,\ b = 3,\ k_1 = k_2 = 1/2.$



Method — Models with two time scales

$$\begin{cases} \varepsilon\dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{cases}$$

fast equation
slow equation

$\varepsilon \ll 1$

Study the two time scales separately (singular limit).

Slow time τ

Rescale time: $t = \varepsilon\tau$, $z' = \frac{dz}{d\tau}$

$$\begin{cases} x' &= f(x,y) \\ y' &= \varepsilon g(x,y) \end{cases} \qquad \varepsilon \to 0 \qquad \begin{cases} x' &= f(x,y) \\ y' &= 0 \end{cases}$$

fast equation dominates

Fast time t

$$\varepsilon \to 0 \qquad \begin{cases} 0 &= f(x,y) \\ \dot{y} &= g(x,y) \end{cases}$$

slow manifold
slow equation

Coupled FitzHugh-Nagumo equations

Slow equation constrained to the slow manifold

$$\begin{cases} \dot{y}_1 & = & x_1 - \gamma y_1 - \delta \\ \dot{y}_2 & = & x_2 - \gamma y_2 - \delta \end{cases} \qquad (y_1, y_2) = F(x_1, x_2)$$

with

$$F(x_1, x_2) = (\varphi(x_1) - k_1(x_1 - x_2), \varphi(x_2) - k_2(x_2 - x_1))$$

$$0 < k_1 \leq k_2 \qquad 0 < a < b \qquad \varphi(x) = -x(x-a)(x-b)$$

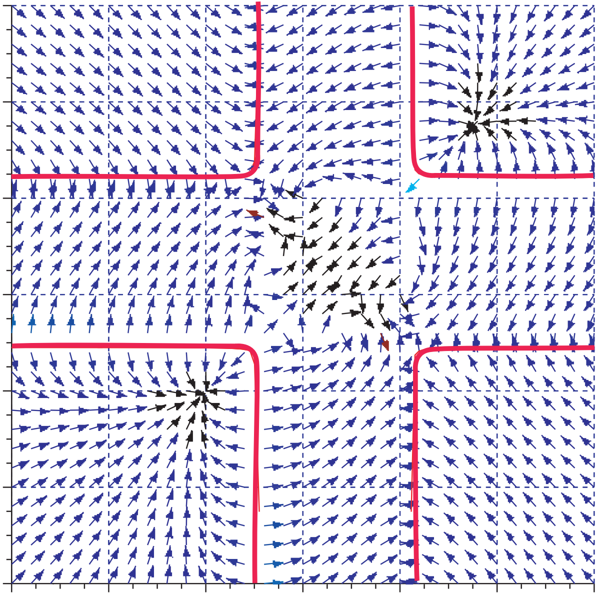
Projection into the (x_1, x_2)-plane

$DF(x_1, x_2)\ (\dot{x}_1, \dot{x}_2) = (\dot{y}_1, \dot{y}_2)$ i.e.

$$DF(x_1, x_2) \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_1 - \gamma y_1 - \delta \\ x_2 - \gamma y_2 - \delta \end{pmatrix} = \begin{pmatrix} x_1 - \delta \\ x_2 - \delta \end{pmatrix} - \gamma F(x_1, x_2)$$

Reverses flow orientation in some non attracting parts of the slow manifold.

Coupled FitzHugh-Nagumo — the slow equation



slow equation,
constrained to
slow manifold,
projected into
(x_1, x_2)-plane

red line:
fold points

$a = 1, \quad b = 3,$
$k_1 = k_2 = 1/2,$
$\gamma = 17/16,$
$\delta = 0$

Coupled FitzHugh-Nagumo — the slow equation, $\delta = 0$

Proposition

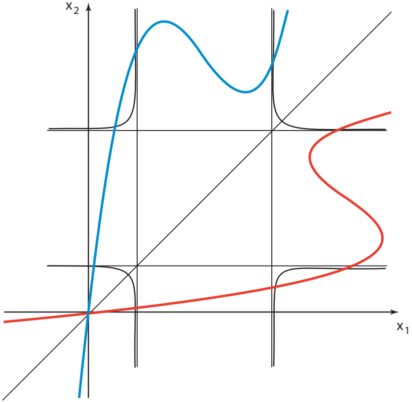
For $0 < a < b$ and $k_1, k_2 > 0$, if $\delta = 0$ and $0 < \gamma < 4/(b-a)^2$ then the only equilibrium of the slow equation is the origin.
For $\gamma = 4/(b-a)^2$ a saddle-node appears at

$$(x_1, x_2) = \frac{1}{2}(a+b, a+b).$$

Symmetric coupling

For $0 < a < b$ and $\delta = 0$, if $k_1 = k_2 = (b-a)^2/4$, then for $\gamma = 4/(b-a)^2$ the slow equation has a saddle-node at a fold point.
For $\gamma > 4/(b-a)^2$ the saddle-node gives rise to a sink.
For $k_1 = k_2 > (b-a)^2/4$, the sink moves into the fold line.

Coupled FitzHugh-Nagumo — the slow equation



slow equation
constrained to slow manifold
projected on
(x_1, x_2)-plane

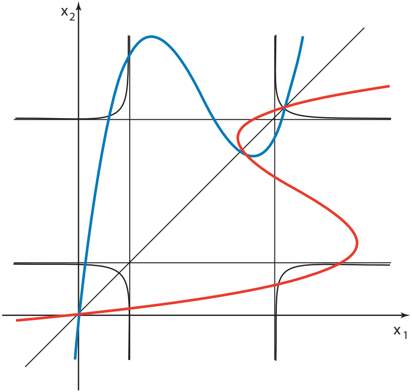
Black curves: fold points

$\dot{y}_1 = 0$ blue curve
$\dot{y}_2 = 0$ red curve
intersections at equilibria

$a = 1,\ b = 3,\ k_1 = k_2 = 1/2,\ \delta = 0$ $\gamma = 6/7$

saddle-node at $\gamma = 1$.

Coupled FitzHugh-Nagumo — the slow equation



slow equation
constrained to slow manifold
projected on
(x_1, x_2)-plane

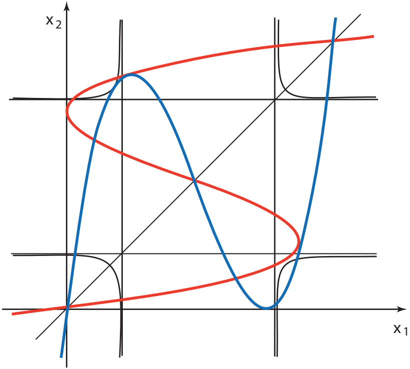
Black curves: fold points

$\dot{y}_1 = 0$ blue curve
$\dot{y}_2 = 0$ red curve
intersect at equilibria

$a = 1,\ b = 3,\ k_1 = k_2 = 1/2,\ \delta = 0$ $\gamma = 17/16$

saddle-node at $\gamma = 1$.

Coupled FitzHugh-Nagumo — the slow equation



slow equation
constrained to slow manifold
projected on
(x_1, x_2)-plane

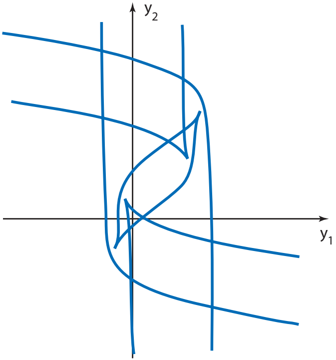
Black curves: fold points

$\dot{y}_1 = 0$ blue curve
$\dot{y}_2 = 0$ red curve
intersections at equilibria

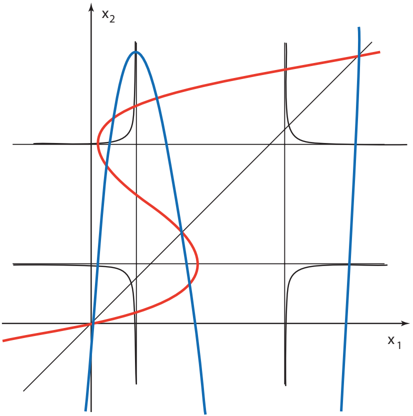
$a = 1,\ b = 3,\ k_1 = k_2 = 1/2,\ \delta = 0$ $\gamma = 2$

saddle-node at $\gamma = 1$.

Coupled FitzHugh-Nagumo — asymmetric coupling



Fold points
in the (y_1, y_2) plane



Equilibria for the slow equation
$\dot{y}_1 = 0$ blue $\qquad \dot{y}_2 = 0$ red
Black curves — folds

$a = 1$, $b = 3$, $k_1 = 1/4$, $k_2 = 1$, $\gamma = 40$, $\delta = 0$.
$\delta = 0$ two cubics through (0,0)

Coupled FitzHugh-Nagumo — the slow equation

Proposition

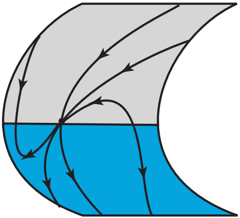
The equilibria of the slow equation that lie on the attracting component of the slow manifold are always asymptotically stable.

For a set of parameters with codimension 1, equilibria of the slow equation occur on the fold line.

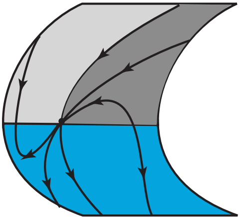
Corollary

When an equilibrium of the slow equation occurs at the boundary of the attracting component, it is a folded node.

Folded node

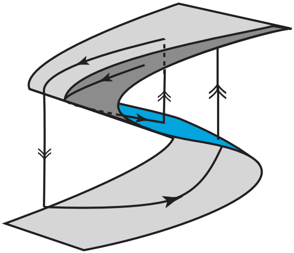


Folded node



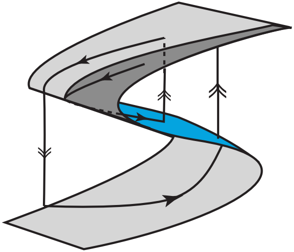
Trapping region.

Folded node



Trajectories that reach the trapping region get funneled into unstable part of slow manifold and jump back.

Folded node

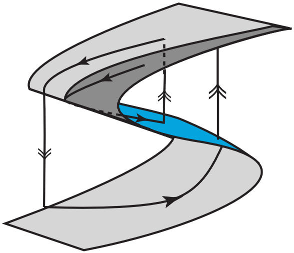


Trajectories that reach the trapping region get funneled into unstable part of slow manifold and jump back.

Canard

A solution that moves in the attracting part of the slow manifold,
passes close to the fold line,
and then follows the repelling part of the slow manifold for some time.

Folded node



Trajectories that reach the trapping region get funneled into unstable part of slow manifold and jump back.

Canard

A solution that moves in the attracting part of the slow manifold,
passes close to the fold line,
and then follows the repelling part of the slow manifold for some time.

Bursts!

THE END

Thank you for your attention.