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MSC 1991: 54Bxx, 68U10

HODGKIN–HUXLEY SYSTEM – A system of four reaction-diffusion equations (cf. **Reaction-diffusion equation**) modelling the electrical activity of nerve cells. The equations have the form

$$\frac{\partial V}{\partial t} = \delta\frac{\partial^2 V}{\partial x^2} + I + F(V, y_1, y_2, y_3),$$
$$\frac{dy_i}{dt} = \gamma_i(V)y_i + \alpha_i(V), \qquad i = 1, 2, 3,$$

where F, γ_i and α_i are non-linear functions, fitted into experimental data and corresponding to a biochemical model, t is time and x is one-dimensional space.

When $\delta = 1$, undamped travelling-wave solutions, the action potentials (cf. **Action potential**), have been studied using the Conley index. They include single-pulse solutions, trains of finitely many impulses and periodic solutions.

The case $\delta = 0$ corresponds to a special experimental setting called a *current clamp*. The equations reduce to a four-dimensional **autonomous system** of ordinary differential equations, its homoclinic and periodic solutions, called *stationary action potentials*, arising through Hopf (or more degenerate) bifurcations (cf. also **Homoclinic point**; **Homoclinic bifurcations**; **Hopf bifurcation**).

Modifications in the equation that retain the form above, with possibly more variables, abound in the biological literature, accounting for variations in the biochemistry of cells. There is also a simplified version that has been much studied by mathematicians, the FitzHugh–Nagumo equations.

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MSC 1991: 92C20, 35K57

HOEFFDING DECOMPOSITION – Let $X_1, \ldots, X_N$ be independent identically distributed random functions with values in a **measurable space** $(E, \mathcal{E})$ (cf. **Random variable**). For $m < N$, let

$$h\colon E^m \to \mathbf{R}$$

be a measurable symmetric function in m variables and consider the U-statistics (cf. ***U*-statistic**)

$$U_N(h) = \frac{1}{\binom{N}{m}} \sum_{1 \le i_1 < \cdots < i_m \le N} h(X_{i_1}, \ldots, X_{i_m}).$$

The following theorem is called *Hoeffding's decomposition theorem*, and the representation of the U-statistic as in the theorem is called the *Hoeffding decomposition* of $U_N(h)$ (see [1]):

$$U_N(h) = \sum_{c=0}^{m} \binom{m}{c} U_N(h_c),$$

where $h_c\colon E^c \to \mathbf{R}$ is a symmetric function in c arguments and where the U-statistics $U_N(h_c)$ are degenerate, pairwise orthogonal in L_2 (uncorrelated) and satisfy

$$\mathsf{E}(U_N(h_c))^2 = \mathsf{E}(h_c(X_1, \ldots, X_c))^2.$$

The symmetric functions h_c are defined as follows:

$$h_c(x_1, \ldots, x_c) = \sum_{k=0}^{c} (-1)^{c-k} \times$$
$$\times \sum_{1 \le l_1 < \cdots < l_k \le c} E(h(x_{l_1}, \ldots, x_{l_k}, X_1, \ldots, X_{m-k})).$$

Extensions of this decomposition are known for the multi-sample case [4], under various 'uncomplete' summation procedures in the definition of a U-statistic, in some dependent situations and for non-identical distributions [3]. There are also versions of the theorem for symmetric functions that have values in a **Banach space**.

The decomposition theorem permits one to easily calculate the variance of U-statistics. Since $U_N(h_0) = \mathsf{E}h(X_1, \ldots, X_m)$ and since $U_N(h_1)$ is a sum of centred independent identically distributed random variables, the central limit theorem for non-degenerate U-statistics is an immediate consequence of the Hoeffding decomposition (cf. also **Central limit theorem**).

The terminology goes back to [2].

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MSC 1991: 60Exx, 62Bxx, 62Gxx, 62Hxx, 62Exx

HOLLEY INEQUALITY – An inequality for a finite **distributive lattice** $(\Gamma, \prec)$, saying that if μ_1 and μ_2 map Γ into $(0, \infty)$ and satisfy $\sum_\Gamma \mu_1(a) = \sum_\Gamma \mu_2(a)$ and

$$\mu_1(a)\mu_2(b) \le \mu_1(a \vee b)\mu_2(a \wedge b) \quad \text{for all } a, b \in \Gamma,$$

then

$$\sum_{a \in \Gamma} f(a)\mu_1(a) \ge \sum_{a \in \Gamma} f(a)\mu_2(a)$$