Acta Cryst A

FOUNDATIONS
ADVANCES

ISSN: 2053-2733
journals.iucr.org/a

Hexagonal projected symmetries

Juliane F. Oliveira, Sofia B. S. D. Castro and Isabel S. Labouriau

Acta Cryst. (2015). A**71**, 549–558

Acta Cryst A FOUNDATIONS ADVANCES

ISSN 2053-2733

Hexagonal projected symmetries

Juliane F. Oliveira,[a,b]* Sofia B. S. D. Castro[a,c] and Isabel S. Labouriau[a,b]

[a]Centro de Matemática da Universidade do Porto, Rua do Campo Alegre 687, 4169-007, Porto, Portugal, [b]Departamento de Matemática, Faculdade de Ciências da Universidade do Porto, Portugal, and [c]Faculdade de Economia, Universidade do Porto, Rua Dr Roberto Frias, 4200-464, Porto, Portugal. *Correspondence e-mail: juliane.oliveira@fc.up.pt

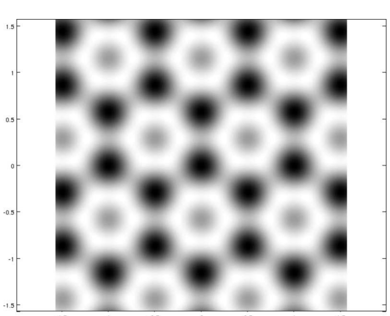
Received 3 February 2015

Accepted 4 July 2015

Edited by V. Petříček, Academy of Sciences, Czech Republic

Keywords: Symmetric patterns; projected patterns; hexagonal symmetries.

In the study of pattern formation in symmetric physical systems, a three-dimensional structure in thin domains is often modelled as a two-dimensional one. This paper is concerned with functions in $\mathbb{R}^3$ that are invariant under the action of a crystallographic group and the symmetries of their projections into a function defined on a plane. A list is obtained of the crystallographic groups for which the projected functions have a hexagonal lattice of periods. The proof is constructive and the result may be used in the study of observed patterns in thin domains, whose symmetries are not expected in two-dimensional models, like the black-eye pattern.



1. Introduction

This article is aimed at two sets of readers: crystallographers and bifurcation theorists. What is obvious to one set of readers is not necessarily so to the other. Concepts are also subject to alternative, substantially different, statements. We try to provide a bridge between the two viewpoints, conditioned by our background in bifurcations.

In the study of crystals and quasicrystals, projection is a mathematical tool for lowering dimension (Senechal, 1996; Koca *et al.*, 2014). A well developed study in crystallographic groups, their subgroups and the notion of projection used in crystallography can be found in the *International Tables for Crystallography* (ITC) Volume A (Hahn, 2005) and ITC Volume E (Kopsky & Litvin, 2002). Tables therein provide information on projections of elements of crystallographic groups.

However, we intend to use crystallographic groups for a different purpose. The symmetries of solutions of partial differential equations, under certain boundary conditions, form a crystallographic group – see, for instance, Golubitsky & Stewart (2002, ch. 5). The set of all level curves of these functions is interpreted as a pattern. In order to study three-dimensional patterns observed in a two-dimensional environment, we use the projection of symmetric functions as defined in §2. The symmetry group of the projected functions does not necessarily coincide with that of projections used in crystallography. The information contained in the ITC (Hahn, 2005; Kopsky & Litvin, 2002) has to be organized in a different way before it can be used for this purpose, and this is the object of the present article.

Regular patterns are usually seen directly in nature and experiments. Convection, reaction–diffusion systems and the Faraday waves experiment comprise three commonly studied pattern-forming systems (see, for instance, Busse, 1978; Turing, 1952; Crawford *et al.*, 1993).

Equivariant bifurcation theory has been used extensively to study pattern formation *via* symmetry-breaking steady-state bifurcation in various physical systems modelled by $E(n)$-equivariant partial differential equations. In Golubitsky & Stewart (2002, ch. 5) there is a complete description of this method used, for example, in Dionne & Golubitsky (1992), Dionne (1993), Bosch Vivancos *et al.* (1995), Callahan & Knobloch (1997) and Dionne *et al.* (1997), where the spatially periodic patterns are sometimes called planforms.

The pattern itself and its observed state can occur in different dimensions. This happens for instance when an experiment is done in a three-dimensional medium but the patterns are only observed on a surface, a two-dimensional object. This is the case for reaction–diffusion systems in the Turing instability regime (Turing, 1952), which have often been described using a two-dimensional representation (Ouyang & Swinney, 1995). The interpretation of this two-dimensional outcome is subject to discussion: the black-eye pattern observed by Ouyang & Swinney (1995) has been explained both as a mode interaction (Gunaratne *et al.*, 1994) and as a suitable projection of a three-dimensional into a two-dimensional lattice (Gomes, 1999). In her article, Gomes shows how a two-dimensional hexagonal pattern can be produced by a specific projection of a body-centred cubic (b.c.c.) lattice.

Pinho & Labouriau (2014) study projections in order to understand how these affect symmetry. Their necessary and sufficient conditions for identifying projected symmetries are used extensively in our results.

Motivated by the explanation of Gomes (1999), we look for all three-dimensional lattices that exhibit a hexagonal projected lattice. We illustrate our results using the primitive cubic lattice.

2. Projected symmetries

The study of projections is related to patterns. Patterns are level curves of functions $f:\mathbb{R}^{n+1}\to\mathbb{R}$. In our work we suppose that these functions are invariant under the action of a particular subgroup of the Euclidean group: a crystallographic group.

The Euclidean group, $E(n+1)$, is the group of all isometries on $\mathbb{R}^{n+1}$, also described by the semi-direct sum $E(n+1)\cong\mathbb{R}^{n+1}\dot{+}O(n+1)$, with elements given as an ordered pair (v,δ), in which $v\in\mathbb{R}^{n+1}$ and δ is an element of the orthogonal group $O(n+1)$ of dimension $n+1$.

Let Γ be a subgroup of $E(n+1)$. The homomorphism

$$\begin{array}{rccc}\varphi: & \Gamma & \to & O(n+1)\\ & (v,\delta) & \mapsto & \delta\end{array}$$

has as image a group $\mathbf{J}$, called the *point group* of Γ, and its kernel forms the *translation subgroup* of Γ.

We say that the translation subgroup of Γ is an $(n+1)$-*dimensional lattice*, $\mathcal{L}$, if it is generated over the integers by $n+1$ linearly independent elements $l_1,\ldots,l_{n+1}\in\mathbb{R}^{n+1}$, which we write as

$$\mathcal{L}=\langle l_1,\ldots,l_{n+1}\rangle_{\mathbb{Z}}.$$

A *crystallographic group* is a subgroup of $E(n+1)$, such that its translation subgroup is an $(n+1)$-dimensional lattice.

A description of these concepts can be found in the ITC Volume A (Hahn, 2005, ch. 8.1, pp. 720–725) and in the suggested bibliography for the chapter; see also Miller (1972).

To get symmetries of objects in $\mathbb{R}^{n+1}$, consider the group *action* of $E(n+1)$ on $\mathbb{R}^{n+1}$ given by the function:

$$\begin{array}{ccc}E(n+1)\times\mathbb{R}^{n+1} & \to & \mathbb{R}^{n+1}\\ ((v,\delta),(x,y)) & \mapsto & (v,\delta)\cdot(x,y)=v+\delta(x,y).\end{array}\qquad(1)$$

In Armstrong (1988), the reader can see that the action (1) restricted to a point group of a crystallographic group leaves its translation subgroup $\mathcal{L}$ invariant. The largest subgroup of $O(n+1)$ that leaves $\mathcal{L}$ invariant forms the *holohedry* of $\mathcal{L}$ and is denoted by $H_{\mathcal{L}}$. The holohedry is always a finite group (see Senechal, 1996, §2.4.2). Note that the term *holohedry* used here, as well as in Dionne & Golubitsky (1992) and Golubitsky & Stewart (2002), corresponds in Hahn (2005, ch. 8.2) to the definition of point symmetry of the lattice.

Crystallographic groups are related to symmetries of pattern formation by the action of the group of symmetries on a space of functions (Golubitsky & Stewart, 2002, ch. 5).

To see this, observe that equation (1) induces an action of a crystallographic group Γ on the space of functions $f:\mathbb{R}^{n+1}\to\mathbb{R}$ by

$$(\gamma\cdot f)(x,y)=f(\gamma^{-1}(x,y))\text{ for }\gamma\in\Gamma\text{ and }(x,y)\in\mathbb{R}^{n+1}.$$

Thus, we can construct a space $\mathcal{X}_\Gamma$ of Γ-invariant functions, that is

$$\mathcal{X}_\Gamma=\{f:\mathbb{R}^{n+1}\to\mathbb{R};\,\gamma\cdot f=f,\forall\gamma\in\Gamma\}.$$

In particular a Γ-invariant function is $\mathcal{L}$ invariant.

An $\mathcal{L}$-*symmetric pattern* or $\mathcal{L}$-*crystal pattern* consists of the set of all level curves of a function $f:\mathbb{R}^{n+1}\to\mathbb{R}$ with periods in the lattice $\mathcal{L}$.

In Gomes (1999) the black-eye pattern is obtained as a projection of a function, whose level sets form a b.c.c. pattern in $\mathbb{R}^3$. In terms of symmetries, the black-eye is a hexagonal pattern, as we can see in Gomes (1999); it is the level sets of a bi-dimensional function with periods in a hexagonal plane lattice, that is, a lattice that admits as its holohedry a group isomorphic to the dihedral group of symmetries of the regular hexagon, D_6. Moreover, we expect the point group of symmetries of the black-eye to be isomorphic to D_6.

For $y_0>0$, consider the restriction of $f\in\mathcal{X}_\Gamma$ to the region between the hyperplanes $y=0$ and $y=y_0$. The projection operator Π_{y_0} integrates this restriction of f along the width y_0, yielding a new function with domain $\mathbb{R}^n$.

Definition 1. For $f\in\mathcal{X}_\Gamma$ and $y_0>0$, the *projection operator* Π_{y_0} is given by

$$\Pi_{y_0}(f)(x)=\int_0^{y_0}f(x,y)\,\mathrm{d}y.$$

The region between $y=0$ and $y=y_0$ is called the *projection band* and $\Pi_{y_0}(f):\mathbb{R}^n\to\mathbb{R}$ is the *projected function*.

The functions $\Pi_{y_0}(f)$ may be invariant under the action of some elements of the group $E(n) \cong \mathbb{R}^n \dot{+} O(n)$. The relation between the symmetries of f and those of $\Pi_{y_0}(f)$ was provided by Pinho & Labouriau (2014).

To find the group of symmetries of the projected functions $\Pi_{y_0}(\mathcal{X}_\Gamma)$, the authors consider the following data:

(*a*) For $\alpha \in O(n)$, the elements of $O(n+1)$:

$$\sigma := \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}, \alpha_+ := \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \alpha_- := \sigma\alpha_+.$$

(*b*) The subgroup $\widehat{\Gamma}$ of Γ, whose elements are of the form

$$((v, y), \alpha_\pm);\ \alpha \in O(n),\ (v, y) \in \mathbb{R}^n \times \mathbb{R},$$

the translation subgroup of $\widehat{\Gamma}$ and Γ are the same, while the point group of $\widehat{\Gamma}$ consists of those elements of Γ that fix the space $\{(v, 0) \in \mathbb{R}^n\}$.

For the three-dimensional case $\widehat{\Gamma}$ coincides with the *scanning group* defined in Kopsky & Litvin (2002, ch. 5.2).

(*c*) The projection $h : \widehat{\Gamma} \to E(n) \cong \mathbb{R}^n \dot{+} O(n)$ given by

$$h((v, y), \alpha_\pm) = (v, \alpha).$$

The group of symmetries of $\Pi_{y_0}(\mathcal{X}_\Gamma)$ is the image by the projection h of the group Γ_{y_0} defined as: (i) if $(0, y_0) \in \mathcal{L}$ then $\Gamma_{y_0} = \widehat{\Gamma}$; (ii) if $(0, y_0) \notin \mathcal{L}$ then Γ_{y_0} contains only those elements of $\widehat{\Gamma}$ that are either side preserving $((v, 0), \alpha_+)$ or side reversing $((v, y_0), \alpha_-)$.

The group $\widehat{\Gamma}$ consists of those elements of Γ that will contribute to the symmetries of the set of projected functions. Depending on whether the hypotheses above hold, the group Γ_{y_0} will be either the whole group $\widehat{\Gamma}$ or a subperiodic group of $\widehat{\Gamma}$, that is a subgroup whose lattice of translations has lower dimension than the space on which the group acts; see Hahn (2005, ch. 8.1) and Kopsky & Litvin (2002, ch. 1.2).

The group Γ_{y_0} depends on how the elements of Γ are transformed by the projection Π_{y_0} of Γ-invariant functions. The criterion that clarifies the connection between the symmetries of $\mathcal{X}_\Gamma$ and $\Pi_{y_0}(\mathcal{X}_\Gamma)$ is provided by the following result.

Theorem 1. [Theorem 1.2 in Pinho & Labouriau (2014)] All functions in $\Pi_{y_0}(\mathcal{X}_\Gamma)$ are invariant under the action of $(v, \alpha) \in E(n)$ if and only if one of the following conditions holds:

(i) $((v, 0), \alpha_+) \in \Gamma$;

(ii) $((v, y_0), \alpha_-) \in \Gamma$;

(iii) $(0, y_0) \in \mathcal{L}$ and either $((v, y_1), \alpha_+) \in \Gamma$ or $((v, y_1), \alpha_-) \in \Gamma$, for some $y_1 \in \mathbb{R}$.

3. Hexagonal projected symmetries

As we saw in the last section, there is a connection between a crystallographic group Γ in dimension $n+1$ and the group of symmetries of the set of projected functions $\Pi_{y_0}(\mathcal{X}_\Gamma)$. In this work we aim to determine which crystallographic groups in dimension 3 can yield hexagonal symmetries after projection. In other words, we want to describe how to obtain hexagonal plane patterns by projection.

Given a crystallographic group Γ, with an $(n+1)$-dimensional lattice $\mathcal{L}$, whose holohedry is $H_\mathcal{L}$, we denote by $\Pi_{y_0}(\mathcal{L})$ the translation subgroup of the crystallographic group $\Pi_{y_0}(\Gamma)$ of symmetries of $\Pi_{y_0}(\mathcal{X}_\Gamma)$, whose point group is a subset of the holohedry of $\Pi_{y_0}(\mathcal{L})$. From theorem 1 we obtain:

Corollary 1. Let $\widetilde{\Gamma}$ be a crystallographic group with lattice $\widetilde{\mathcal{L}} \subset \mathbb{R}^n$. Suppose $\widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$, and let $H_{\widetilde{\mathcal{L}}}$ and $H_\mathcal{L}$ be the holohedries of $\widetilde{\mathcal{L}}$ and $\mathcal{L} \subset \mathbb{R}^{n+1}$, respectively. If $\alpha \in H_{\widetilde{\mathcal{L}}}$ lies in the point group of $\widetilde{\Gamma}$ then either $\alpha_+ \in H_\mathcal{L}$ or $\alpha_- \in H_\mathcal{L}$.

Proof. Since $\alpha \in H_{\widetilde{\mathcal{L}}}$ implies α lies in the point group of $\widetilde{\Gamma}$, then there exists $v \in \mathbb{R}^n$ such that f is (v, α) invariant for all $f \in \Pi_{y_0}(\mathcal{X}_\Gamma)$. Hence, one of the three conditions of theorem 1 holds. Then, depending on whether (i), (ii) or (iii) is verified, either (w, α_+) or (w, α_-) is in Γ where $w \in \{(v, 0), (v, y_0), (v, y_1)\}$. By the definition of holohedry, we have either $\alpha_+ \in H_\mathcal{L}$ or $\alpha_- \in H_\mathcal{L}$. □

Remark 1. We note that there is a non-trivial relation between the lattice $\widetilde{\mathcal{L}}$ of periods of the projected functions and that of the original one. In fact, consider an $(n+1)$-dimensional lattice $\mathcal{L}$ and $\widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$. If $v \in \widetilde{\mathcal{L}}$ then (v, I_n) is a symmetry of $\Pi_{y_0}(\mathcal{X}_\Gamma)$. Applying theorem 1 with $\alpha = I_n$, one of the following holds for each $v \in \widetilde{\mathcal{L}}$:

(i) $((v, 0), \alpha_+) = ((v, 0), I_{n+1}) \in \Gamma$, or equivalently $(v, 0) \in \mathcal{L}$;

(ii) $((v, y_0), \alpha_-) = ((v, y_0), \sigma) \in \Gamma$ then $((v, y_0), \sigma)^2 \in \Gamma$, implying that $(2v, 0) \in \mathcal{L}$;

(iii) $(0, y_0) \in \mathcal{L}$ and either (v, y_1) or $(2v, 0)$ is in $\mathcal{L}$, for some $y_1 \in \mathbb{R}$.

While condition (i) implies that $\mathcal{L} \cap \{(x, 0) \in \mathbb{R}^{n+1}\} \subseteq \widetilde{\mathcal{L}}$, the other conditions show that this inclusion is often strict. Furthermore, conditions (ii) and (iii) show that we may have no element of the form (v, y_1) in $\mathcal{L}$ and yet $v \in \widetilde{\mathcal{L}}$. This is due to a possible non-zero translation vector associated with $\sigma \in J$.

As a converse to corollary 1 we have:

Corollary 2. Let Γ be a crystallographic group with an $(n+1)$-dimensional lattice $\mathcal{L}$ and let $\widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$, with $H_{\widetilde{\mathcal{L}}}$ and $H_\mathcal{L}$ the holohedries of $\widetilde{\mathcal{L}} \subset \mathbb{R}^n$ and of $\mathcal{L} \subset \mathbb{R}^{n+1}$, respectively. Suppose either α_+ or α_- is in $H_\mathcal{L}$. If one of the following conditions holds

(i) $\sigma \notin H_\mathcal{L}$;

(ii) either $(0, \alpha_+)$ or $(0, \alpha_-)$ is in Γ; then, for any $y_0 \in \mathbb{R}$, $\alpha \in H_{\widetilde{\mathcal{L}}}$.

Proof. Consider $v \in \widetilde{\mathcal{L}}$ and suppose condition (i) holds. By theorem 1, either $(v, 0) \in \mathcal{L}$ or $(0, y_0)$ and $(v, y_1) \in \mathcal{L}$.

If $(v, 0) \in \mathcal{L}$ then $(\alpha v, 0) \in \mathcal{L}$. Otherwise, $(0, y_0)$ and $(\alpha v, y_2) \in \mathcal{L}$ for $y_2 \in \{-y_1,\ y_1\}$. Applying theorem 1 we have $\alpha v \in \widetilde{\mathcal{L}}$ in both cases. Therefore, α is a symmetry of $\widetilde{\mathcal{L}}$.

Suppose now that condition (ii) holds. If $(0, \alpha_+) \in \Gamma$ then $(0, \alpha)$ belongs to $\widetilde{\Gamma}$, for all $y_0 \in \mathbb{R}$, by condition (i) of theorem 1. The other possibility is that $(0, \alpha_-) \in \Gamma$. If for $v \in \widetilde{\mathcal{L}}$ either condition (i) or condition (iii) of theorem 1 holds, the proof follows as in the case of condition (i). Suppose then that $((v, y_0), \sigma) \in \Gamma$, then $((v, y_0), \sigma) \cdot (0, \alpha_-) = ((v, y_0), \alpha_+) \in \Gamma$. Therefore, $(v, \alpha) \in \widetilde{\Gamma}$, by theorem 1, completing the proof. □

The analysis in Kopsky & Litvin (2002, ch. 5.1) aims to find *sectional layer groups* and *penetration rod groups*, subgroups of the crystallographic group that leave a *crystallographic plane*, defined by three lattice points, and a *crystallographic straight line* invariant, respectively, by a method of scanning a given crystallographic group.

When a pattern is projected, it is not apparent that the plane of projection is crystallographic, as it may not contain three lattice points.

We say that a lattice $\mathcal{L}_1$ is *rationally compatible* with a lattice $\mathcal{L}_2$ if there exists $r \in \mathbb{Z} \setminus \{0\}$ such that $r\mathcal{L}_1 \subset \mathcal{L}_2$. A vector $v \in \mathbb{R}^k$ is *rational with respect to a lattice* $\mathcal{L} \subset \mathbb{R}^k$ if $\langle v, \ell \rangle \in \mathbb{Q}$ for all $\ell \in \mathcal{L}$, where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^k$. Given a lattice $\widetilde{\mathcal{L}} \subset \mathbb{R}^k$, we define its *suspension* $\widetilde{\mathcal{L}}_s \subset \mathbb{R}^{k+1}$ as $\widetilde{\mathcal{L}}_s = \{(v, 0);\, v \in \widetilde{\mathcal{L}}\}$.

The next proposition provides conditions for a suspension of the projected lattice to be rationally compatible with the original lattice.

Proposition 1. Consider a crystallographic group Γ with a lattice $\mathcal{L} \subset \mathbb{R}^{n+1}$ and let $\widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L}) \subset \mathbb{R}^n$ be the translation subgroup of $\Pi_{y_0}(\Gamma)$ and denote its suspension by $\widetilde{\mathcal{L}}_s \subset \mathbb{R}^{n+1}$.

If $(0, y_0) \notin \mathcal{L}$, then $\widetilde{\mathcal{L}}_s$ is always rationally compatible with $\mathcal{L}$.

If $(0, y_0) \in \mathcal{L}$, then $\widetilde{\mathcal{L}}_s$ is rationally compatible with $\mathcal{L}$ if and only if the normal vector $(0, y_0)$ to the projection hyperplane is rational with respect to $\mathcal{L}$.

Note that, if $(0, y_0) \in \mathcal{L}$, we are projecting the values of functions on a band of the width of one (or more) cells along a crystallographic direction. Otherwise the projected group is smaller. So, we must use different results in the ITC according to the case.

Proof. If $(0, y_0) \notin \mathcal{L}$, then only conditions (i) or (ii) of remark 1 are applicable. Therefore, if $v \in \widetilde{\mathcal{L}}$ then $(2v, 0) \in \mathcal{L}$. Hence, $\widetilde{\mathcal{L}}$ is rationally compatible with $\mathcal{L}$.

If $(0, y_0) \in \mathcal{L}$, then, using remark 1, it follows that $v \in \widetilde{\mathcal{L}}$ for all v such that $(v, y_1) \in \mathcal{L}$ for some $y_1 \in \mathbb{R}$.

Suppose first that $\widetilde{\mathcal{L}}_s$ is rationally compatible with $\mathcal{L}$ and let $(v, y_1) \in \mathcal{L}$. Then $r(v, 0) \in \mathcal{L}$ and hence $(0, ry_1) \in \mathcal{L}$ for some $r \in \mathbb{Z}$. It follows that $y_1 = (p/q)y_0$ for some p, q non-zero integers. Therefore $(0, y_0)$ is rational with respect to $\mathcal{L}$.

Now suppose $(0, y_0)$ is rational with respect to $\mathcal{L}$. We claim that for each $\widetilde{l}_j$ one of the following conditions holds:

(i) $((\widetilde{l}_j, 0), I_{n+1}) \in \Gamma$;

(ii) $((\widetilde{l}_j, y_1), \sigma) \in \Gamma$, for some $y_1 \in \mathbb{R}$;

(iii) $(0, y_0)$ and $(\widetilde{l}_j, (p/q)y_0) \in \mathcal{L}$, for some p, q non-zero integers.

Condition (iii) is a stronger version of condition (iii) in remark 1. The other conditions follow from remark 1. Any generator $\widetilde{l}_j$ of $\widetilde{\mathcal{L}}$ such that $(\widetilde{l}_j, y) \in \mathcal{L}$ must satisfy either (i) or (iii) above, because $(0, y_0)$ is rational with respect to $\mathcal{L}$. Any other generator of $\widetilde{\mathcal{L}}$ must satisfy (ii), proving our claim.

Conversely, if one of the conditions (i) or (ii) is true, then $(2v, 0) \in \mathcal{L}$, using remark 1. If condition (iii) holds for some $j \in \{1, \ldots, n\}$, then $(0, y_0), (\widetilde{l}_j, (p_j/q_j)y_0) \in \mathcal{L}$, where p_j, q_j are non-zero integers. Since $\mathcal{L}$ is a lattice $(q_j\widetilde{l}_j, 0) \in \mathcal{L}$. Therefore $\widetilde{\mathcal{L}}_s$ is rationally compatible with $\mathcal{L}$. □

As an illustration, take $\Gamma = \mathcal{L} = \langle (0, 1), (2^{1/2}, 1/2) \rangle_{\mathbb{Z}}$, for which $\widetilde{\mathcal{L}}_s = \langle (2^{1/2}, 0) \rangle_{\mathbb{Z}}$ is always rationally compatible with $\mathcal{L}$, independently of y_0. Another example is given by $\Gamma = \mathcal{L} = \langle (0, 1), (1, 2^{1/2}) \rangle_{\mathbb{Z}}$. For $y_0 = 1$ we have that $\widetilde{\mathcal{L}}_s = \langle (1, 0) \rangle_{\mathbb{Z}}$ is not rationally compatible with $\mathcal{L}$, whereas for $y_0 \notin \mathbb{Z}$, we get $\widetilde{\mathcal{L}}_s = \{(0, 0)\}$.

For three-dimensional lattices, and if the generators of $\widetilde{\mathcal{L}}$ are related by an orthogonal transformation, then we can remove the condition on $(0, y_0)$ from the statement of proposition 1, at the price of having some more complicated conditions. This provides an alternative means of obtaining rational compatibility.

Our starting point is a specific two-dimensional lattice $\tilde{\mathcal{L}}$ and we want to characterize the three-dimensional lattices $\mathcal{L}$ that project onto this. The first step is to establish that $\mathcal{L}$ must have a non-trivial intersection with the plane $X0Y = \{(x, y, 0);\, x, y \in \mathbb{R}\}$.

Theorem 2. Let Γ be a crystallographic group with a lattice $\mathcal{L} \subset \mathbb{R}^3$ such that its projection $\widetilde{\Gamma} = \Pi_{y_0}(\Gamma)$ has a plane lattice $\widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$ generated by two linearly independent vectors $\widetilde{l}_1$ and $\widetilde{l}_2 = \rho\widetilde{l}_1$ for ρ in the point group, $\widetilde{J}$, of $\widetilde{\Gamma}$.

Then the suspension $\widetilde{\mathcal{L}}_s \subset \mathbb{R}^3$ is rationally compatible with $\mathcal{L}$ if for each $v \in \{\widetilde{l}_1, \widetilde{l}_2\}$ one of the following conditions holds:

(i) $((v, 0), I_3) \in \Gamma$;

(ii) $((v, y_1), \sigma) \in \Gamma$, for some $y_1 \in \mathbb{R}$;

(iii) $(v, y_1) \in \mathcal{L}$, for some $y_1 \in \mathbb{R}$.

That the conditions are also necessary is apparent from remark 1. Note that the statement of theorem 2 excludes oblique and primitive rectangular lattices.

Proof. Since $\widetilde{l}_2 = \rho\widetilde{l}_1$, it is sufficient to show that one of $r(\widetilde{l}_1, 0)$ or $r(\widetilde{l}_2, 0)$ is in $\mathcal{L}$. To see this, suppose, without loss of generality, that $r(\widetilde{l}_1, 0) \in \mathcal{L}$. Then since $\rho \in \widetilde{J}$, by corollary 1, either $\rho_+ \in H_{\mathcal{L}}$ or $\rho_- \in H_{\mathcal{L}}$. As $\rho_+(r\widetilde{l}_1, 0) = \rho_-(r\widetilde{l}_1, 0) = (r\widetilde{l}_2, 0)$, it implies that $r(\widetilde{l}_2, 0) \in \mathcal{L}$ and therefore $\mathcal{L}$ has a sublattice $\mathcal{L}_r$.

If for some $v \in \{\widetilde{l}_1, \widetilde{l}_2\}$ one of the conditions (i) or (ii) is true then, by remark 1, $(rv, 0) \in \mathcal{L}$, for $r = 1$ or $r = 2$. Hence, all that remains to prove is the case when $\widetilde{l}_1$ and $\widetilde{l}_2$ only satisfy condition (iii).

By hypothesis,

$$(\widetilde{l}_1, y_1) \text{ and } (\widetilde{l}_2, y_2) \text{ are in } \mathcal{L}, \quad \text{for some } y_1, \;\; y_2 \in \mathbb{R}. \qquad (2)$$

This implies that

$$(\widetilde{l}_1 + \widetilde{l}_2, y_1 + y_2) \in \mathcal{L}. \qquad (3)$$

Using (2) and corollary 1 either

$$\rho_+(\widetilde{l}_1, y_1) = (\widetilde{l}_2, y_1) \in \mathcal{L}$$

or

$$\rho_-(\widetilde{l}_1, y_1) = (\widetilde{l}_2, -y_1) \in \mathcal{L}.$$

If $(\widetilde{l}_2, y_1) \in \mathcal{L}$ then

$$(\widetilde{l}_2, y_1) + (\widetilde{l}_2, y_2) = (2\widetilde{l}_2, y_1 + y_2) \in \mathcal{L}.$$

Thus, using (3)

$$(\widetilde{l}_1 + \widetilde{l}_2, y_1 + y_2) - (2\widetilde{l}_2, y_1 + y_2) = (\widetilde{l}_1 - \widetilde{l}_2, 0) \in \mathcal{L}.$$

Since $\{\widetilde{l}_1, \widetilde{l}_2\}$ is a basis to $\widetilde{\mathcal{L}}$ and $\rho \in \widetilde{J}$ then

$$\rho(\widetilde{l}_1 - \widetilde{l}_2) = m\widetilde{l}_1 + n\widetilde{l}_2, m, n \in \mathbb{Z}$$

where m, n are not both equal to zero. Suppose that $n \neq 0$, then

$$n(\widetilde{l}_1 - \widetilde{l}_2, 0), (m\widetilde{l}_1 + n\widetilde{l}_2, 0) \in \mathcal{L} \quad (4)$$

implying that the sum of these last two vectors $((n+m)\widetilde{l}_1, 0) \in \mathcal{L}$. Therefore, if $n \neq -m$, $\mathcal{L}_r$ is a sublattice of $\mathcal{L}$, where $r = m + n \in \mathbb{Z}$. If $n = -m$, we subtract the two expressions in (4) to get $(2n\widetilde{l}_1, 0) \in \mathcal{L}$.

If $(\widetilde{l}_2, -y_1) \in \mathcal{L}$ then

$$-(\widetilde{l}_2, -y_1) + (\widetilde{l}_2, y_2) = (0, y_1 + y_2) \in \mathcal{L}.$$

Thus, using (3),

$$(\widetilde{l}_1 + \widetilde{l}_2, y_1 + y_2) - (0, y_1 + y_2) = (\widetilde{l}_1 + \widetilde{l}_2, 0) \in \mathcal{L}.$$

An analogous argument applied to $\rho(\widetilde{l}_1 + \widetilde{l}_2, 0)$ finishes the proof. □

Let $\mathcal{L} \subset \mathbb{R}^3$ be a lattice and $P \subset \mathbb{R}^3$ be a plane such that $P \cap \mathcal{L} \neq \emptyset$. Given $v \in P \cap \mathcal{L}$ there is a rotation $\gamma \in O(3)$ such that $\gamma(P - v)$ is the plane $X0Y = \{(x, y, 0); x, y \in \mathbb{R}\}$. Then we define the y_0-*projection* of $\mathcal{L}$ into P as the lattice $\gamma^{-1}(\widetilde{\mathcal{L}}) \subset E(2)$ where $\widetilde{\mathcal{L}}$ is the symmetry group of $\Pi_{y_0}(\mathcal{X}_{\gamma(\mathcal{L}-v)})$.

We say that the y_0-*projection* of $\mathcal{L}$ into the plane P *is a hexagonal plane lattice* if and only if the lattice $\widetilde{\mathcal{L}}$ admits as its holohedry a group isomorphic to D_6.

Our main result is the following theorem. Note that the hypothesis of having a threefold rotation is not restrictive when one is looking for projections yielding a pattern with D_6 symmetry.

Theorem 3. Let $\mathcal{L} \subset \mathbb{R}^3$ be a lattice of a crystallographic group Γ. Suppose for some y_* the group $\Pi_{y_*}(\Gamma)$ contains a threefold rotation. Then, for any y_0, the y_0-projection of $\mathcal{L}$ into the plane P is a hexagonal plane lattice if and only if: (i) $P \cap \mathcal{L}$ contains at least two elements; (ii) there exists $\beta \in H_{\mathcal{L}}$ such that: β is a threefold rotation; P is β invariant.

Proof. Suppose first that $(0, 0, 0) \in P \cap \mathcal{L}$. To show that conditions (i) and (ii) are necessary let us consider, without loss of generality, that $P = X0Y$. Therefore, the conditions hold by theorem 2.

To prove that conditions (i) and (ii) are sufficient, consider the threefold rotation $\beta \in H_{\mathcal{L}}$. By Miller (1972), theorem 2.1 and the proof of the crystallographic restriction theorem, in the same reference, there exists only one subspace of dimension 2 invariant by β. Such a plane is the plane perpendicular to its rotation axis. So, let P be this plane.

Since $P \cap \mathcal{L} \neq \{(0, 0, 0)\}$, let v be a non-zero element of minimum length in $P \cap \mathcal{L}$ and consider the lattice $\mathcal{L}' = \langle v, \beta v \rangle_{\mathbb{Z}}$. As β has order three the sublattice $\mathcal{L}'$ is a hexagonal plane lattice.

To finish the proof, consider $y_0 \in \mathbb{R}$; we prove that the y_0-projection of $\mathcal{L}$ into the plane P is a hexagonal plane lattice. Let $(0, \alpha) \in \Pi_{y_*}(\Gamma)$, where α is a threefold rotation. Then, by theorem 1, one of the conditions holds:

(i) $(0, \alpha_+) \in \Gamma$;

(ii) $((0, y_*), \alpha_-) \in \Gamma$;

(iii) $(0, y_*) \in \mathcal{L}$ and either $((0, y_1), \alpha_+)$ or $((0, y_1), \alpha_-)$ is in Γ.

Since the order of α is finite, we have that either $(0, \alpha_+) \in \Gamma$ or $(0, \alpha_-) \in \Gamma$. Then, the result follows by condition (ii) of corollary 2.

If $(0, 0, 0) \notin P \cap \mathcal{L}$, note that the proof can be reduced to the previous case by a translation. □

Remark 2. Theorem 3 shows that a possible way to obtain patterns with hexagonal symmetry, by y_0-projection, is to project the functions $f \in \mathcal{X}_{\mathcal{L}}$ in a plane invariant by the action of some element $\beta \in H_{\mathcal{L}}$ with order three. After finding one of those planes, in order to obtain projections as in definition 1, we only need to change coordinates. The reader can see an example with the b.c.c. lattice in Gomes (1999).

We are grateful to an anonymous referee who pointed out that for certain specific widths of the projection this can be obtained by other means. However, in these cases the symmetry group of the projected functions, for most y_0, has a very small point group and this is not interesting for the study of bifurcating patterns. More specifically, we are interested in relating hexagonal patterns of different complexity in solutions of the same differential equation with symmetry. As the projection width y_0 varies, one may obtain hexagonal patterns with different symmetry groups, corresponding to different patterns, as illustrated in the figures at the end of this article. Bifurcation occurs *via* symmetry breaking and, hence, more symmetry (a bigger point group) makes the bifurcation problem more interesting.

As a consequence of theorem 3, we are able to list all the Bravais lattices that may be projected to produce a two-dimensional hexagonal pattern.

Theorem 4. The Bravais lattices that project to a hexagonal plane lattice, under the conditions of theorem 3, are: (i) primitive cubic lattice; (ii) body-centred cubic lattice; (iii) face-centred cubic lattice; (iv) hexagonal lattice; and (v) rhombohedral lattice. Moreover, up to a change of coordinates, for the first three lattices the plane of projection must be parallel to one of the planes in Table 1. For the hexagonal and rhombohedral lattices the plane of projection must be parallel to the plane $X0Y$.

Proof. It is apparent from theorem 3 that we can exclude the following Bravais lattices: triclinic, monoclinic, orthorhombic and tetragonal, since the holohedries of these lattices do not have elements of order three.

To see if the other Bravais lattices have hexagonal projected symmetries, we need to examine the rotations of order three and six in their holohedries and see if the plane perpendicular to their rotation axes intersects the lattice.

The group of rotational symmetries of the cubic lattice (as well as the body-centred cubic lattice and the face-centred cubic lattice) is isomorphic to S_4, the group of permutations of four elements. So, in the holohedry of the cubic lattice we only have rotations of order one, two or three. Consider a system of generators for a representative for the cubic lattice $\mathcal{L}$, in the standard basis of $\mathbb{R}^3$, given by

$$(1, 0, 0), (0, 1, 0), (0, 0, 1).$$

Then, the matrix representation of the rotations of order three in $H_{\mathcal{L}}$ is

$$\gamma_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \; \gamma_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix},$$
$$\gamma_3 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \; \gamma_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Two-dimensional spaces perpendicular to the rotation axis of each one of these rotations are given in Table 1.

This means that, for the first three lattices in the list, the projection of functions $f \in \mathcal{X}_{\mathcal{L}}$ into a plane have hexagonal symmetries only if the plane is parallel to one of the plane subspaces given in Table 1.

Consider now a three-dimensional hexagonal lattice. Its group of rotational symmetries has order 12 and it has a subgroup of order six consisting of the rotational symmetries of the rhombohedral lattice.

Let the representatives for the hexagonal and rhombohedral lattices be generated by

$$(1, 0, 0), \left(\frac{1}{2}, \frac{3^{1/2}}{2}, 0\right), (0, 0, c) \; c \neq 0, \; \pm 1$$

and

$$(1, 0, 1), \left(-\frac{1}{2}, \frac{3^{1/2}}{2}, 1\right), (-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 1),$$

respectively. Then, the 12 rotations in the holohedry of the hexagonal lattice are generated by

$$\rho_z = \begin{pmatrix} \frac{1}{2} & -\frac{3^{1/2}}{2} & 0 \\ \frac{3^{1/2}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \gamma_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The generators of the group of rotational symmetries of the rhombohedral lattice are then ρ_z^2 and γ_x.

Table 1
Two-dimensional spaces perpendicular to the rotation axis of each one of the rotations γ_i.

Here we denote by $\langle v \rangle$, $v \in \mathbb{R}^3$ the subspace generated by v.

Rotation	Rotation axis	Perpendicular plane
γ_1	$\langle(1, 1, -1)\rangle$	$P_1 = \{(x, y, z); z = x + y\}$
γ_2	$\langle(1, -1, -1)\rangle$	$P_2 = \{(x, y, z); z = x - y\}$
γ_3	$\langle(1, -1, 1)\rangle$	$P_3 = \{(x, y, z); z = -x + y\}$
γ_4	$\langle(1, 1, 1)\rangle$	$P_4 = \{(x, y, z); z = -(x + y)\}$

We conclude that the only rotations of order six in the holohedry of the hexagonal lattice are ρ_z and ρ_z^5, and of order three ρ_z^2 and ρ_z^4.

Therefore, the y_0-projection of the hexagonal and rhombohedral lattices is a hexagonal plane sublattice if and only if the y_0-projection is made into a plane parallel to the plane $X0Y$. □

4. Hexagonal projected symmetries of the primitive cubic lattice

We conclude the article with an example to illustrate the hexagonal symmetries obtained by z_0-projection of functions with periods in the primitive cubic lattice, for all $z_0 \in \mathbb{R}$.

Consider a three-dimensional crystallographic group, $\Gamma = \mathcal{L} \dot{+} H_{\mathcal{L}}$, where $\mathcal{L}$ is the primitive cubic lattice generated by the vectors (1, 0, 0), (0, 1, 0) and (0, 0, 1) over $\mathbb{Z}$, and $H_{\mathcal{L}}$ its holohedry.

Without loss of generality, consider the projection of Γ on P_1 (see Table 1).

From theorem 3, the cubic lattice has a hexagonal plane sublattice that intersects P_1. This sublattice is generated by

$$(0, 1, 1), (1, 0, 1). \tag{5}$$

To make our calculations easier and to set up the hexagonal symmetries in the standard way, consider the new basis $\{(0, 1, 1), (1, 0, 1), (0, 0, 1)\}$ for the lattice $\mathcal{L}$. Now multiply $\mathcal{L}$ by the scalar $1/2^{1/2}$ in order to normalize the vectors of (5). With these changes the crystallographic group Γ has the new translational subgroup generated by the vectors

$$v_1 = \left(0, \frac{1}{2^{1/2}}, \frac{1}{2^{1/2}}\right), v_2 = \left(\frac{1}{2^{1/2}}, 0, \frac{1}{2^{1/2}}\right), v_3 = \left(0, 0, \frac{1}{2^{1/2}}\right).$$

Projection of Γ on P_1, as in definition 1, can be done after a change of coordinates that transforms P_1 into $X0Y$. Consider that change given by the orthonormal matrix:

$$A = \begin{pmatrix} 0 & \frac{1}{2^{1/2}} & \frac{1}{2^{1/2}} \\ \frac{2}{6^{1/2}} & \frac{-1}{6^{1/2}} & \frac{1}{6^{1/2}} \\ \frac{1}{3^{1/2}} & \frac{1}{3^{1/2}} & \frac{-1}{3^{1/2}} \end{pmatrix}.$$

Then, in the new system of coordinates $X = Ax$, we obtain the base for the primitive cubic lattice given by

$$l_1 = (1, 0, 0), l_2 = \left(\frac{1}{2}, \frac{3^{1/2}}{2}, 0\right), l_3 = \left(\frac{1}{2}, \frac{3^{1/2}}{6}, \frac{-6^{1/2}}{6}\right). \tag{6}$$

Table 2
Projection of $\Gamma = \mathcal{L} \dot{+} H_{\mathcal{L}}$ for each $z_0 \in \mathbb{R}$.

$z_0 \in \mathbb{R}$	Γ_{z_0}	$\Sigma_{z_0} = \mathcal{L}_{z_0} \dot{+} J_{z_0}$
$z_0 = \frac{3n}{6^{1/2}}, n \in \mathbb{Z}\backslash\{0\}$	$\Gamma_{z_0} = \widehat{\Gamma}$	$\mathcal{L}_{z_0} = \left\langle \left(\frac{1}{2}, \frac{3^{1/2}}{6}\right), \left(\frac{1}{2}, \frac{-3^{1/2}}{6}\right) \right\rangle_{\mathbb{Z}}$
Then $\left(0, 0, \frac{3n}{6^{1/2}}\right) \in \mathcal{L}$		$J_{z_0} = D_6 = \langle \gamma', \kappa' \rangle$
$z_0 = \frac{3n-1}{6^{1/2}}, n \in \mathbb{Z}\backslash\{0\}$	Γ_{z_0} contains $\left(\left(\frac{1}{2}, \frac{3^{1/2}}{6}, \frac{3n-1}{6^{1/2}}\right), \gamma\right)$	$\mathcal{L}_{z_0} = \left\langle (1, 0), \left(\frac{1}{2}, \frac{3^{1/2}}{2}\right) \right\rangle_{\mathbb{Z}}$
Then $\left(\frac{1}{2}, \frac{3^{1/2}}{6}, \frac{3n-1}{6^{1/2}}\right) \in \mathcal{L}$		$J_{z_0} = \left\langle \left(\left(\frac{1}{2}, \frac{3^{1/2}}{6}\right), \gamma'\right) \kappa' \right\rangle$
$z_0 = \frac{3n+1}{6^{1/2}}, n \in \mathbb{Z}\backslash\{0\}$	Γ_{z_0} contains $\left(\left(\frac{1}{2}, \frac{-3^{1/2}}{6}, \frac{3n+1}{6^{1/2}}\right), \gamma\right)$	$\mathcal{L}_{z_0} = \left\langle (1, 0), \left(\frac{1}{2}, \frac{3^{1/2}}{2}\right) \right\rangle_{\mathbb{Z}}$
Then $\left(\frac{1}{2}, \frac{-3^{1/2}}{6}, \frac{3n+1}{6^{1/2}}\right) \in \mathcal{L}$		$J_{z_0} = \left\langle \left(\left(\frac{1}{2}, \frac{-3^{1/2}}{6}\right), \gamma'\right), \kappa' \right\rangle$
For z_0 different to the cases before	$\Gamma_{z_0} = H$	$\mathcal{L}_{z_0} = \left\langle (1, 0), \left(\frac{1}{2}, \frac{3^{1/2}}{2}\right) \right\rangle_{\mathbb{Z}}$
		$J_{z_0} = \langle -\gamma', \kappa' \rangle$

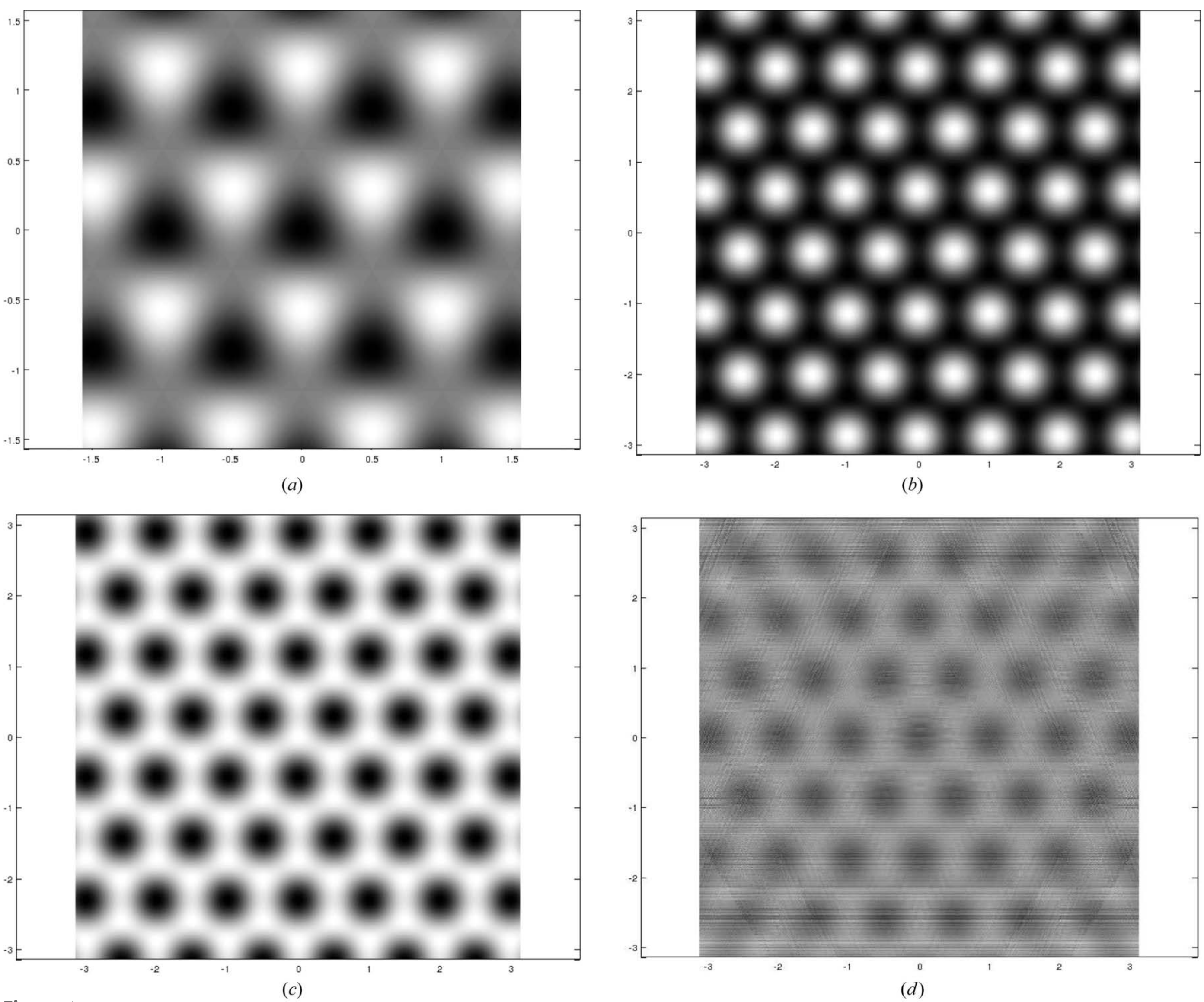


Figure 1
Projection of pattern u in a six-dimensional representation with primitive cubic lattice periodicity. Contour plots of the integral of u over different depths z_0. (*a*) $z_0 = 1/[2(6^{1/2})]$, (*b*) $z_0 = 1/6^{1/2}$, (*c*) $z_0 = 2/6^{1/2}$, (*d*) $z_0 = 3/6^{1/2}$. The same pictures occur for the projection of a strip of half this height of a pattern in an eight-dimensional representation with face-centred cubic periodicity.

Observe that we changed the position of $\mathcal{L}$ as prescribed by theorem 3.

We proceed to describe the symmetries of the space $\Pi_{z_0}(\mathcal{X}_\Gamma)$, for each $z_0 \in \mathbb{R}$. For this, we need to obtain the subgroups $\widehat{\Gamma}$ and Γ_{z_0} of Γ. Denote by $\Sigma_{z_0} = \mathcal{L}_{z_0} \dot{+} J_{z_0}$ the subgroup of $E(2)$ of all symmetries of $\Pi_{z_0}(\mathcal{X}_\Gamma)$.

It is straightforward to see that the elements of Γ with orthogonal part $\alpha_\pm$ are in the group

$$\widehat{\Gamma} = \{((v, z), \rho); (v, z) \in \mathcal{L}, \rho \in \widehat{J}\}$$

where $\widehat{J}$ is the group generated by

$$\gamma = \begin{pmatrix} \frac{1}{2} & -\frac{3^{1/2}}{2} & 0 \\ \frac{3^{1/2}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } \kappa = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the group Γ_{z_0} has a subgroup $H = \overline{\mathcal{L}} \dot{+} \overline{J}$, for all $z_0 \in \mathbb{R}$, where $\overline{\mathcal{L}}$ is the translation subgroup $\overline{\mathcal{L}} = \langle l_1, l_2 \rangle_\mathbb{Z}$ and $\overline{J}$ is the subgroup generated by $((0,0,0), \kappa)$ and $((0,0,0), -\gamma)$. Using statement (i) of theorem 1, for all $z_0 \in \mathbb{R}$ all the functions $f \in \Pi_{z_0}(\mathcal{X}_\Gamma)$ are $(1, 0)$, and $(\frac{1}{2}, \frac{3^{1/2}}{2})$ periodic and invariant for the action of

$$\kappa' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } -\gamma' = \begin{pmatrix} -\frac{1}{2} & \frac{3^{1/2}}{2} \\ -\frac{3^{1/2}}{2} & -\frac{1}{2} \end{pmatrix}.$$

In Table 2 we list the group Γ_{z_0}, for each $z_0 \in \mathbb{R}$, and describe the respective projected symmetries.

We assume that all the functions $f : \mathbb{R}^3 \to \mathbb{R}$ in $\mathcal{X}_\mathcal{L}$ admit a unique formal Fourier expansion in terms of the waves

$$w_k(x, y, z) = \exp(2\pi i \langle k, (x, y, z) \rangle)$$

where k is a *wavevector* in the dual lattice, $\mathcal{L}^* = \{k \in \mathbb{R}^3; \langle k, l_i \rangle \in \mathbb{Z}, i = 1, 2, 3\}$, of $\mathcal{L}$ given in (6), with *wavenumber* $|k|$, where $(x, y, z) \in \mathbb{R}^3$ and $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^3$. Thus,

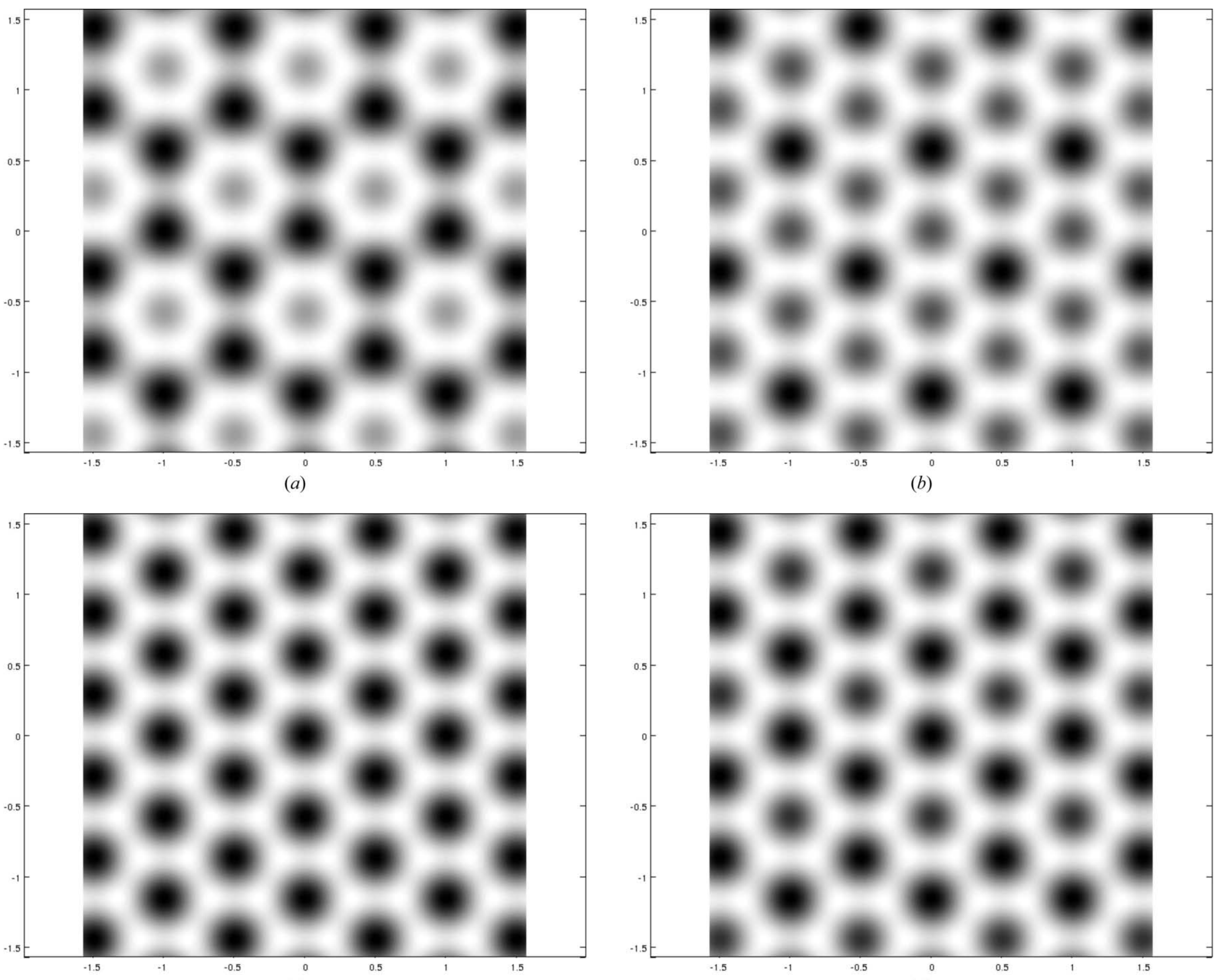


Figure 2
Projection of pattern u in a 12-dimensional representation with body-centred cubic lattice periodicity. Contour plots of the integral of u over different depths z_0. (*a*) $z_0 = 1/[2(6^{1/2})]$,. (*b*) $z_0 = 1/6^{1/2}$, (*c*) $z_0 = 3/[2(6^{1/2})]$, (*d*) $z_0 = 2/6^{1/2}$.

$$f(x, y, z) = \sum_{k \in \mathcal{L}^*} z_k w_k(x, y, z)$$

where z_k is the Fourier coefficient, for each $k \in \mathcal{L}^*$, and with the restriction $z_{-k} = \overline{z_k}$.

Therefore, we can write

$$\mathcal{X}_{\mathcal{L}} = \bigoplus_{k \in \mathcal{L}'} V_k$$

for

$$\mathcal{L}' = \{k = (k_1, k_2) \in \mathcal{L}^*;\; k_1 > 0 \text{ or } k_1 = 0 \text{ and } k_2 > 0\}$$

and

$$V_k = \{\mathrm{Re}(a w_k(x, y, z));\, a \in \mathbb{C}\} \cong \mathbb{C}.$$

Note that $\mathcal{X}_\Gamma$ is a subspace of $\mathcal{X}_{\mathcal{L}}$.

We say that the space

$$\bigoplus_{|k|=a} V_k = V_{k_1} \oplus V_{k_2} \oplus \cdots \oplus V_{k_s}$$

is a $2s$-dimensional representation of the action of Γ on the space $\mathcal{X}_{\mathcal{L}}$.

A straightforward calculation shows that the function

$$u(x, y, z) = \sum_{|k|=2^{1/2}} \exp(2\pi i k \cdot (x, y, z)) \tag{7}$$

is Γ invariant.

The contour plots of the projections of u are shown in Fig. 1, with the symmetries given in Table 2. In Dionne (1993) it is shown that the function u belongs to a six-dimensional representation.

The body-centred cubic lattice shows a different configuration, illustrated in Fig. 2.

As an illustration, consider a system of generators:

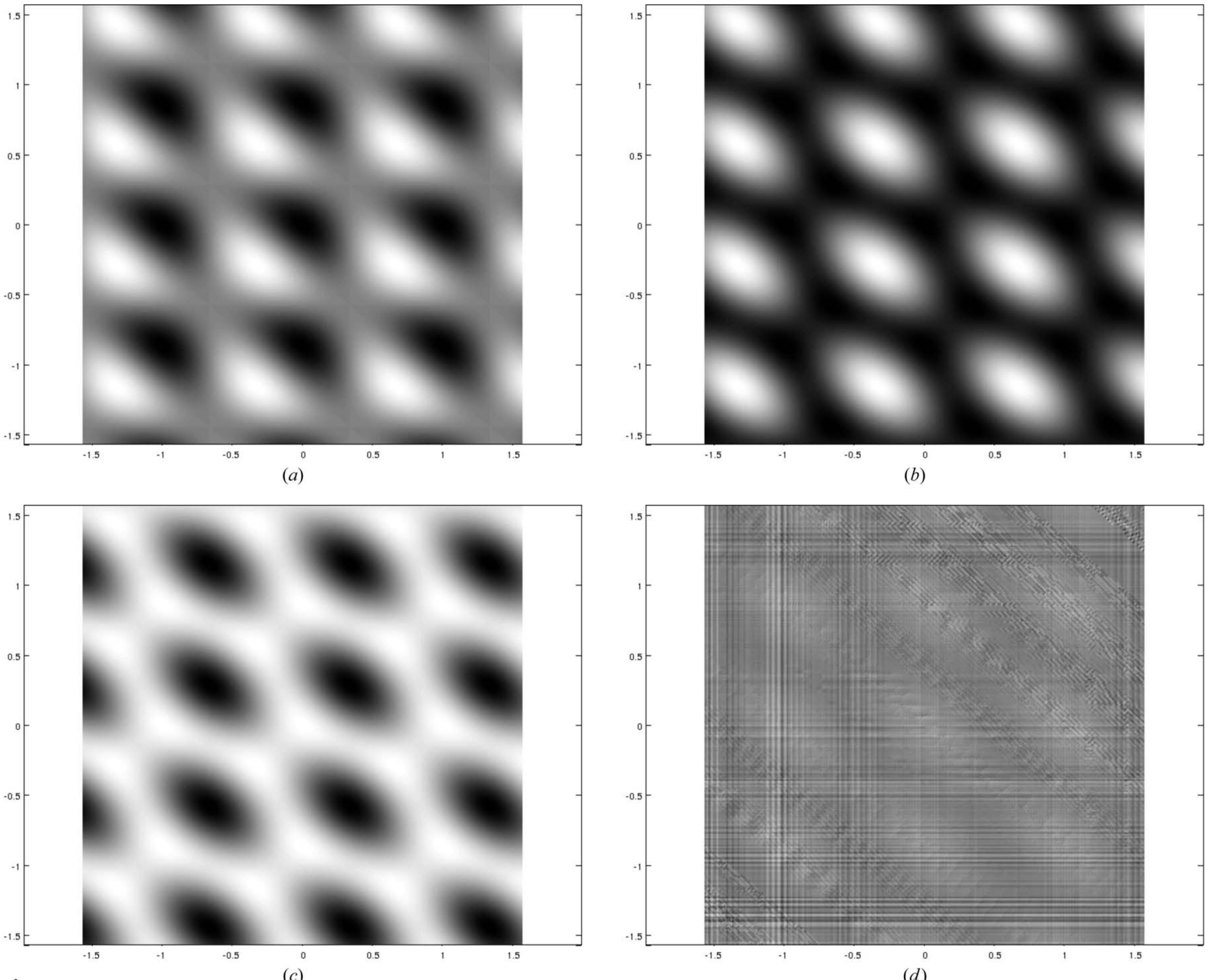


Figure 3
Projection of pattern u in a six-dimensional representation with rhombohedral lattice periodicity. Contour plots of the integral of u over different depths z_0 with parameter $a = 2$. (*a*) $z_0 = 2/6$, (*b*) $z_0 = 2/3$, (*c*) $z_0 = 4/3$, (*d*) $z_0 = 2$.

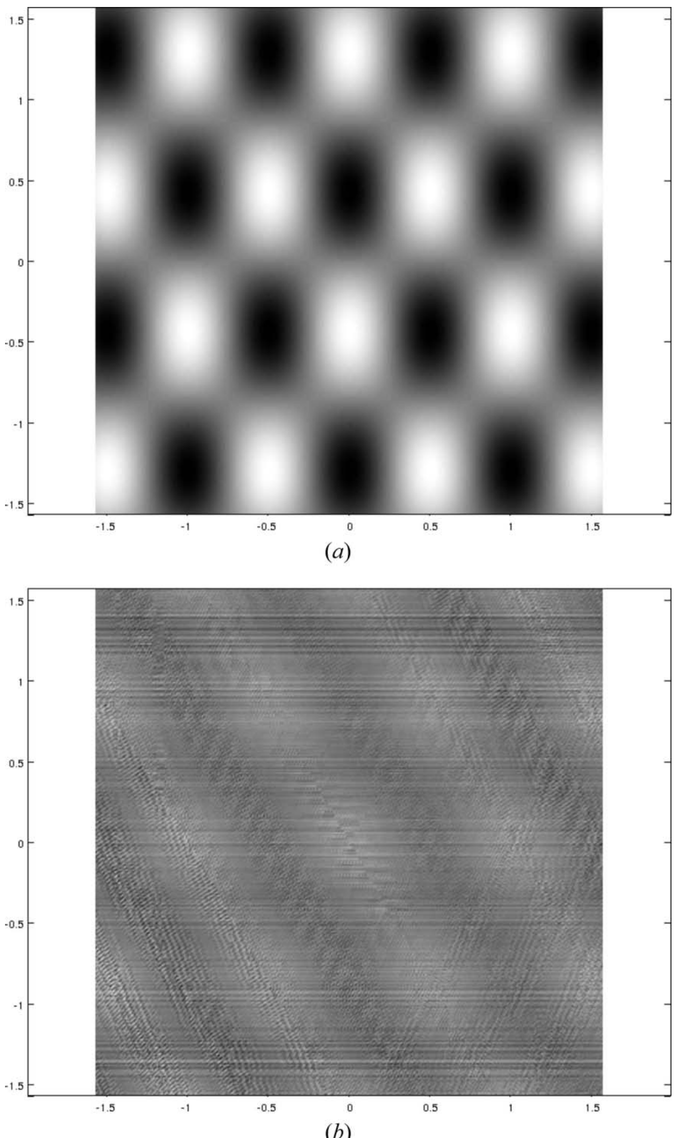


Figure 4
Projection of pattern u in a 12-dimensional representation with hexagonal lattice periodicity. Contour plots of the integral of u over different depths z_0 with parameter $c = 2$. (a) $z_0 = 1$, (b) $z_0 = 2$.

$$l_1 = (1, 0, 0), l_2 = \left(\frac{1}{2}, \frac{3^{1/2}}{2}, 0\right), l_3 = (0, 0, c) \ c \neq 0, \pm 1$$

and

$$r_1 = (1, 0, 0), r_2 = \left(\frac{1}{2}, \frac{3^{1/2}}{2}, 0\right), r_3 = \left(\frac{-1}{2}, \frac{3^{1/2}}{6}, \frac{a}{3}\right), \ a \neq 0$$

for the hexagonal and rhombohedral lattices, respectively. A construction similar to that used for the primitive cubic lattice may be applied to these two cases, but here the parameters a and c will change the pattern of the projected functions. Examples are shown in Figs. 3 and 4.

Acknowledgements

CMUP (UID/MAT/00144/2013) is supported by the Portuguese government through the Fundação para a Ciência e a Tecnologia (FCT) with national (MEC) and European structural funds through the programs FEDER, under the partnership agreement PT2020. JFO was supported by a grant from the Conselho Nacional de Desenvolvimento Cientifico e Tecnologico (CNPq) of Brazil.

References

Armstrong, M. A. (1988). *Groups and Symmetry*. New York: Springer Verlag.
Bosch Vivancos, I., Chossat, P. & Melbourne, I. (1995). *Arch. Rat. Mech. Anal.* **131**, 199–224.
Busse, F. H. (1978). *Rep. Prog. Phys.* **41**, 1929.
Callahan, T. K. & Knobloch, E. (1997). *Nonlinearity*, **10**, 1179–1216.
Crawford, J. D., Gollub, J. P. & Lane, D. (1993). *Nonlinearity*, **6**, 119.
Dionne, B. (1993). *Z. Angew. Math. Phys.* **44**, 673–694.
Dionne, B. & Golubitsky, M. (1992). *Z. Angew. Math. Phys.* **43**, 36–62.
Dionne, B., Silber, M. & Skeldon, A. C. (1997). *Nonlinearity*, **10**, 321–353.
Golubitsky, M. & Stewart, I. (2002). *The Symmetry Perspective*. Berlin: Birkhauser Verlag.
Gomes, M. G. M. (1999). *Phys. Rev. E*, **60**, 3741–3747.
Gunaratne, H. G., Ouyang, Q. & Swinney, H. L. (1994). *Phys. Rev. E*, **50**, 2802–2806.
Hahn, T. (2005). *International Tables for Crystallography*, Vol. A, 5th ed. Dordrecht: Kluwer.
Koca, N. O., Koca, M. & Koc, R. (2014). *Acta Cryst.* A**70**, 605–615.
Kopsky, V. & Litvin, D. B. (2002). *International Tables for Crystallography*, Vol. E, 1st ed. Chichester: Wiley.
Miller, W. (1972). *Symmetry Groups and their Applications*. New York: Academic Press.
Ouyang, Q. & Swinney, H. L. (1995). *Chemical Waves and Patterns. Onset and Beyond Turing Pattern Formation*, edited by R. Kapral & K. Showalter, pp. 269–295. Dordrecht: Kluwer.
Pinho, E. M. & Labouriau, I. S. (2014). *J. Pure Appl. Algebra*, **218**, 37–51.
Senechal, M. (1996). *Quasicrystals and Geometry*. Cambridge University Press.
Turing, A. M. (1952). *Philos. Trans. R. Soc. London. Ser. B*, **237**, 37–72.