Eliana Manuel de Matos Oliveira Pinho

Symmetries of Projected Symmetric Patterns

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Resumo

Nesta tese descrevemos como se modifica a simetria de uma função com a projecção ou a restrição — dois métodos que permitem baixar a dimensão do seu domínio.

Trabalhamos com funções definidas em domínios de dimensão $n+1$ e que são periódicas segundo $n+1$ direcções não-colineares definindo uma rede $\mathcal{L}$, isto é, funções invariantes sob a acção de um grupo cristalográfico, subgrupo do grupo euclidiano $\mathbf{E}(n+1)$.

Depois da projecção (ou da restrição) obtemos funções que são invariantes sob a acção de um subgrupo Σ de $\mathbf{E}(n)$, relacionado com o grupo cristalográfico original Γ. O nosso resultado principal descreve uma bijecção entre os dois grupos de simetria, antes e depois da redução da dimensão.

Utilizamos quer métodos algébricos quer analíticos: a simetria é enunciada algebricamente; os operadores de projecção e de restrição são definidos analiticamente. A expansão de Fourier das funções Γ-invariantes e a acção de Γ induzida no espaço dos coeficientes de Fourier são ferramentas essenciais na articulação das duas perspectivas. Os resultados intermédios relacionam os grupos de simetria com a rede dual $\mathcal{L}^*$.

Dois casos particulares são tratados em detalhe. Para funções definidas no plano, calculamos Σ explicitamente para cada um dos grupos cristalográficos planos. Para todas as dimensões, estudamos o conjunto dos períodos das funções após a projecção (ou a restrição), dado pelo subgrupo das translações de Σ. Estes resultados são usados para comparar a projecção de soluções de equações $\mathbf{E}(n+1)$-equivariantes com soluções de equações equivariantes definidas directamente em n dimensões.

Abstract

In this thesis we describe how the symmetry of a function is transformed by a projection or by a restriction — two operations that reduce the dimension of its domain.

We work with functions defined in $(n+1)$-dimensional domains and that are periodic along a lattice $\mathcal{L}$ containing $n+1$ noncolinear directions, *i.e.*, functions invariant under the action of a crystallographic group Γ, a subgroup of the Euclidean group $\mathbf{E}(n+1)$.

After projection (or restriction) we obtain functions that are invariant under the action of a subgroup Σ of $\mathbf{E}(n)$, related to the original crystallographic group Γ. Our main result describes a bijection relating the two symmetry groups, before and after the reduction in dimension.

We use both algebraic and analytic tools: the symmetry is stated algebraically; the projection and the restriction operators are defined analytically. The Fourier expansion of the Γ-invariant functions and the induced action of Γ in the space of Fourier coefficients are both essential in dealing with the two perspectives. Intermediate results relate the symmetry groups to the dual lattice $\mathcal{L}^*$.

Two particular cases are explored in detail. For functions in the plane we give explicit calculations of Σ for all the 17 plane crystallographic groups, or wallpaper groups. In all dimensions we study the set of periods of the restricted (or projected) functions, given by the subgroup of translations of Σ. This is used to compare the projection of solutions of $\mathbf{E}(n+1)$-equivariant equations to solutions of equivariant equations posed directly in n dimensions.

Résumé

Dans cette thèse on décrit les modifications de la symétrie d'une fonction avec une projection ou une restriction — deux méthodes pour réduire la dimension de leur domaine.

On travaille avec des fonctions définies dans des domaines $(n+1)$-dimensionnels et qui sont périodiques sur un réseau $\mathcal{L}$ ayant $n+1$ directions non-colineaires, *i.e.*, des fonctions invariantes sous l'action d'un groupe cristallografique Γ, sous-groupe du groupe Euclidean $\mathbf{E}(n+1)$.

Après la projection (ou la restriction) on obtient des fonctions qui sont invariantes sous l'action d'un sous-groupe Σ de $\mathbf{E}(n)$, qui est en rapport avec le groupe cristallografique originel Γ. Notre résultat principal décrit une bijéction entre les deux groupes de symétrie, avant et après la réduction de la dimension.

On utilise autant des méthodes algébriques q'analytiques: la symétrie a un énoncé algébrique; la projection et la restriction sont définies analytiquement. L'expansion de Fourier des fonctions Γ-invariantes et l'action de Γ induite sur l'espace des coefficients de Fourier sont des outils essentiels pour lier les deux perspectives. Les résultats intermédiaires mettent les groupes de symétrie en relation avec le réseau dual $\mathcal{L}^*$.

Deux cas particuliers sont traités en détail. Pour les fonctions définies dans le plan, on calcule Σ por chaq'un des 17 groupes cristalographiques planaires. Pour toutes les dimensions, on étudie les périodes des fonctions après la projection (ou la restriction), donnés par le sous-groupe des translations de Σ. On utilise ces résultats pour comparer la projection de solutions d'équations $\mathbf{E}(n+1)$-equivariantes aux solutions d'équations equivariantes définies directement en n dimensions.

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CHAPTER 1

Introduction

1.1. Context

1.1.1. symmetric functions. In the study of a mathematical model, symmetry may arise naturally from its properties. It may also appear due to the method used for finding solutions of the equations in the model. The symmetry approach of Golubitsky and Stewart [**9**] and Golubitsky et al. [**10**] is pursued in this thesis.

A particular case of symmetric solutions are patterns with spatial periodicity, or repetitive patterns, observed in many physical systems. Crystals are an example, with periodicity along three noncolinear directions. Periodic solutions are observed in convection experiments, chemical reactions and many other situations, as Faraday wave experiments, see Rucklidge et al. [**21**], or magnetic perturbation of a liquid crystal, studied by Chillingworth and Golubitsky [**4**].

Both numerical observations and the choice of mathematical tools, may also point to periodic solutions. This is the case of bifurcation problems equivariant under the action of the noncompact symmetry groups $\mathbf{E}(m)$. If they are restricted to a space of periodic functions invariant under a compact subgroup of $\mathbf{E}(m)$, the techniques of Golubitsky et al. [**10**] may be applied. In Golubitsky and Stewart [**9**, chapter 5] there is a complete description of this method, used, for example, in Dionne and Golubitsky [**6**], Dionne [**5**], Bosch Vivancos et al. [**2**], Callahan and Knobloch [**3**] and Dionne et al. [**7**], where the spatially periodic patterns are sometimes called planforms.

Wallpaper patterns are planar structures periodic along two noncolinear directions, a two-dimensional analogue to crystals. These solutions are observed in reaction-diffusion experiments on thin layers of gel. Despite the three-dimensional nature of these systems, their mathematical models usually consider planar structures due to the small thickness of the layer compared to other dimensions. Moreover, although the layers are finite, they may be seen as infinite, *i.e.*, having no boundaries. This happens, for example, in regions far from the boundaries of the gel layer, in observations not depending on its width. Thus, we may study solutions in the plane having Euclidean symmetry, a theoretical simplification.

The mathematical tools developed for the study of repetitive patterns in the plane may also be applied to bounded or partially bounded domains via Newmann boundary conditions, as described in Gomes et al. [**12**] and in Melbourne [**19**].

1.1.2. projection. The patterns observed in the reaction-diffusion experiments are usually explained by two-dimensional models. However black-eye patterns, see figure 1, are not expected in two-dimensions. Gomes [**11**] proposes a different approach, suggesting that black-eye patterns are the projection into the plane of a three-dimensional repetitive solution. In this context the thickness of the layer becomes a decisive variable, black-eye patterns occur only for specific values of the thickness.

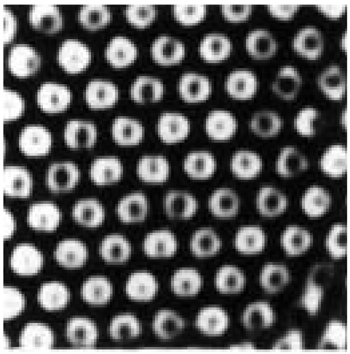

FIGURE 1. Black-eye pattern observed in reaction-diffusion experiments described in Gunaratne et al. [**13**]. (Reprinted figure with permission from [**13**]. Copyright 1994 by the American Physical Society.)

We may ask in general how a projection transforms repetitive patterns — the projection may be seen either as a physical phenomenon or as a mathematical tool. The first perspective would be used whenever we observe solutions that are the integration, along some variables, of a solution in a higher dimensional space. As a mathematical tool, projection is a way of lowering the dimension in order to obtain the desirable properties. This happens in the theoretical construction of quasicrystals, where the quasiperiodic three-dimensional structure may be obtained projecting a periodic structure in $\mathbf{R}^5$, see Senechal [**22**, section 2.6] for a description of this method and Lifshitz [**17**] for the symmetric approach to quasiperiodic crystals.

1.1.3. reducing the dimension. In this thesis we start with a symmetric function defined in $n+1$ dimensions and explore two ways of obtaining a new function on a n-dimensional subspace: either we project into the subspace (figure 2) or we take the restriction (figure 3). In both cases some structure will remain due to the initial symmetry.

Our aim is to describe the symmetries of the functions defined in a lower dimension.

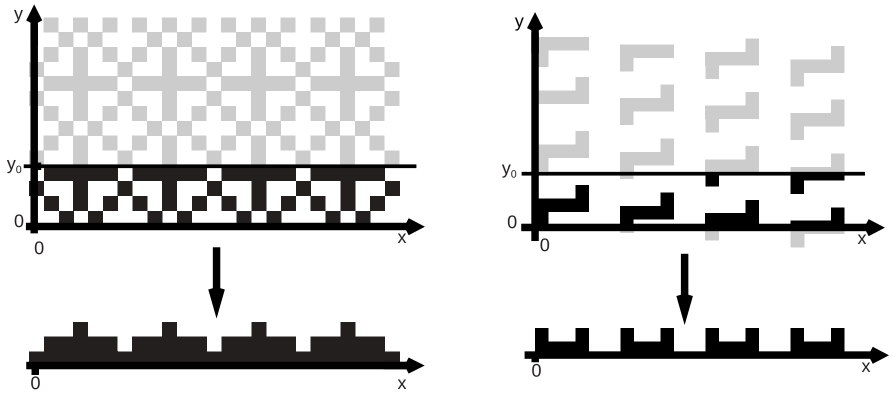

FIGURE 2. The projection of periodic patterns in $\mathbf{R}^2$ restricted to a strip of width y_0 defines functions with domain $\mathbf{R}$.

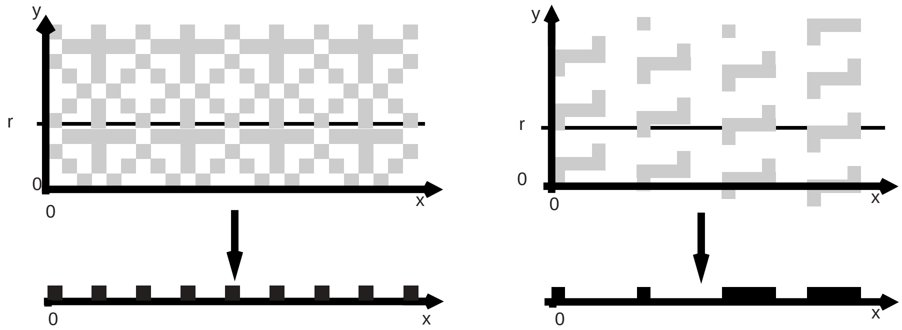

FIGURE 3. The restriction of periodic patterns in $\mathbf{R}^2$ to the line $y = r$ are functions with domain $\mathbf{R}$.

1.2. Overview

1.2.1. raw material. We study real functions with domain $\mathbf{R}^{n+1}$, periodic along $n+1$ noncolinear directions.

Having a period may be interpreted as the invariance of the function under a translation, *i.e.*, as a symmetry. Translations are isometries of the space $\mathbf{R}^{n+1}$, the set of all the isometries being the $(n+1)$-dimensional Euclidean group, $\mathbf{E}(n+1)$. Periodic functions may have more symmetry beyond periodicity. The elements of $\mathbf{E}(n+1)$ that leave these functions invariant form a group Γ with periods as a subgroup. These are called crystallographic groups since they are analogous, in the $(n+1)$-dimensional space, to the symmetry groups of crystals in $\mathbf{R}^3$.

1.2.2. spaces of functions. Our statements will concern X_Γ, a space of Γ-invariant functions where Γ is a crystallographic group. In order to make the proofs valid, we must impose some more conditions upon the space X_Γ, more precisely, conditions on the formal Fourier expansion for each of its elements. Other function spaces involved are $\Pi_{y_0}(X_\Gamma)$ and $\Phi_r(X_\Gamma)$, the spaces of, respectively, the projection and the restriction of all the functions in X_Γ. The projection of a function $f \in X_\Gamma$ is, for some $y_0 > 0$, the function defined in $\mathbf{R}^n$ by:

$$\Pi_{y_0}(f)(x) = \int_0^{y_0} f(x, y)dy,$$

where $x \in \mathbf{R}^n$ and $y \in \mathbf{R}$. This is the integration of f restricted to the domain between the hyperplanes $y = 0$ and $y = y_0$. The integration is made along the variable orthogonal to these hyperplanes and the resulting function is defined in the subspace $y = 0$.

The restriction of some $f \in X_\Gamma$ to a hyperplane $y = r$ has also domain $\mathbf{R}^n$. It is defined as

$$\Phi_r(f)(x) = f(x, r),$$

where $x \in \mathbf{R}^n$ and $r \in \mathbf{R}$, using the notation $(x, y) \in \mathbf{R}^{n+1}$.

The functions in $\Pi_{y_0}(X_\Gamma)$ and $\Phi_r(X_\Gamma)$ may have some symmetries, *i.e*, they may be invariant under the action of some nontrivial subgroup of the n-dimensional Euclidean group $\mathbf{E}(n)$.

1.2.3. aim. The symmetries of Γ-invariant functions are transformed by the projection. Some elements of Γ will yeld symmetries of the projected functions. Others may give rise to some other structure of the projected functions that is not a symmetry, *i.e.*, a structure that cannot be described as the invariance of the projected functions under the action of some element in $\mathbf{E}(n)$. An illustrative example of the second case is the quasiperiodic structure obtained by the canonical projection of a periodic one, see Senechal [**22**, section 2.6].

We study the relation between Γ and the symmetries of the projected functions. Our main result, Theorem 4.1, describes the elements of Γ that contribute to the symmetry after projection and the way these relevant elements of Γ are transformed by the projection. Formally, it states necessary and sufficient conditions upon Γ ensuring that all the projected functions are invariant under the action of some element in $\mathbf{E}(n)$.

An analogous result for the restriction is presented in Theorem 4.2.

1.2.4. symmetry components. The results obtained are best understood if we separate each isometry into two parts: a translation and an orthogonal component. This approach is based in the formulation of the Euclidean group as the semi-direct product

$$\mathbf{E}(m) \cong \mathbf{R}^m \ltimes \mathbf{O}(m).$$

Moreover, due to the definition of the projection operator, it is useful to describe the elements of $\mathbf{R}^{n+1}$ as (x, y), with $x \in \mathbf{R}^n$ and $y \in \mathbf{R}$, as in 1.2.2.

Theorem 4.1 states that the projected functions in $\Pi_{y_0}(X_\Gamma)$ inherit the symmetry from elements $\gamma \in \Gamma$ whose orthogonal component acts separately in the two subspaces $x = 0$ and $y = 0$. If γ acts in the subspace $y = 0$ by the orthogonal transformation $\alpha \in \mathbf{O}(n)$ and the translation $v \in \mathbf{R}^n$, then, under some conditions about the translation in $x = 0$ and the width y_0, this symmetry remains after projection.

1.2.5. how to. Although we state Theorem 4.1 for a space of functions, it establishes a homomorphism between two symmetry groups. Thus, it induces a projection in the space of crystallographic groups.

This relation between algebra and analysis is always present along the thesis. Our tools are either algebraic, like crystallographic groups and modules, or analytic, as Fourier expansion. The two perspectives are held together mostly by two reasons. The induced action of the symmetry group Γ in the space of the Fourier coefficients of Γ-invariant functions allows the translation of symmetry into an equation that can be traced also after projection. The formulation of the results for sufficiently large spaces of Γ-invariant functions, highlights their common characteristic, *i.e.*, the symmetry.

1.2.6. ways for generalization. Some wider approaches could be considered in future work:

- original functions whose sets of periods are not lattices, like rolls or frieze patterns;
- groups acting in the space of functions with pseudoscalar action (see section 2.2), which allows the algebraic characterization of, for example, odd functions;
- groupoid symmetry instead of group symmetry, see Stewart et al. [**23**] for an example, describing local symmetries such as the existence of a periodic pattern only in a limited region of the domain;
- projections and restrictions to spaces of higher codimension.

1.3. Description of the Chapters

This work is presented in four chapters beyond Introduction.

1.3.1. main results. The core of this work is chapter 4, *Projection and Symmetry*, where we state and prove Theorem 4.1 relating the symmetry to the projection of functions, as described above.

Along the proof of Theorem 4.1 several interesting results arise. Proposition 4.1 states necessary and sufficient conditions on a crystallographic group Γ and its dual lattice, for the projected Γ-invariant functions to be symmetric. Proposition 4.1 is the generalization of the two-dimensional result presented in Labouriau and Pinho [**15**]. After this Proposition the structure of the dual lattice and its relation to the lattice associated to Γ are studied.

Despite the name of chapter 4, Theorem 4.2 presents, for the restriction, an analogue of Theorem 4.1. Its proof uses the same arguments developed for the projection and is simpler, since it has fewer conditions. The notation and tools we have developed for the projection could all be applied to the restriction. This holds because the restriction operator was defined in a way that the projections and the restrictions of a function could easily be compared. Moreover, the formalism we developed for the projections is effective in dealing with symmetries also after the restrictions.

1.3.2. wallpaper. Before tackling functions on a space of arbitrary dimension $n+1$, we present in chapter 3 the study of real functions with domain $\mathbf{R}^2$ that are invariant under the action of a crystallographic group Γ, subgroup of $\mathbf{E}(2)$. After being projected, or restricted, these functions are defined in $\mathbf{R}$, see figures 2 and 3. The group $\Gamma \in \mathbf{E}(2)$ is called either plane crystallographic group or wallpaper group.

This case is presented in Labouriau and Pinho [**16**], with the technical sections almost coincident with those of chapter 3.

There are many reasons leading to the presentation of this particular case in a distinct chapter.

- The study of this case was our first step towards the resolution of the general case. Thus, it follows a more intuitive approach than chapter 4, where the aims are concision and simplicity in dealing with the general case. One major difference is the separate study of the periodicity and of the remaining symmetries which, for functions defined on the line, may be called parity (see section 2.4). This more naive approach may help in the first reading.
- The dimension of the objects we discuss in chapter 3, *i.e.*, functions with plane domains whose projection or restriction are functions on the line, allows illustrative examples. Moreover, the arguments involving lattices and isometries on the plane are

easier to understand. We will present geometric examples of the statements in Theorem 4.1 for the special case $n = 1$.

- For each one of the 17 wallpaper groups Γ we describe, in section 3.4, the possible symmetries of the projection and of the restriction of Γ-invariant functions to a line. This would hardly be feasible for $n > 1$, as there are 230 three-dimensional crystallographic groups.

The proofs in chapters 3 and 4 are in many ways analogous. This follows from our choice of formalism. The two-dimensional case does not require this approach, it was developed in order to make the generalization to higher dimensions easier.

The results of chapters 3 and 4 have the following major differences: first, in lower dimension there are fewer cases — compare Propositions 3.1 and 3.3 with Proposition 4.1. Second, some arguments are simpler in lower dimension and this makes for shorter proofs. Third, in chapter 4 we use a better definition for the elements $(v_\delta, \delta) \in \Gamma$ and this simplifies the proofs of some intermediate results. We point out these differences along chapters 3 and 4.

1.3.3. tools. In chapter 2 we describe the main concepts and mathematical tools used along the thesis. We also define the necessary spaces of functions and the projection and restriction operators. Each section ends with a summary.

1.3.4. conclusion. We end this thesis with chapter 5, called *Periods*, where we summarize the results concerning periodicity of the projected and of the restricted functions.

Moreover, we relate our results to problems described in section 1.1, such as equivariant perspective, models for quasicrystals and detection of projected solutions in experiments in thin layers of gel.

CHAPTER 2

Notation and Preliminaries

In this work we begin with real functions

$$f : \mathbf{R}^{n+1} \longrightarrow \mathbf{R},$$

where $n \in \mathbf{N}$, and transform them in order to obtain functions whose domain is $\mathbf{R}^n$. We will use the notation

$$(x, y) \in \mathbf{R}^{n+1}, \text{ with } x \in \mathbf{R}^n \text{ and } y \in \mathbf{R}.$$

A *symmetry* of a function f is an invertible map γ defined on the space of all functions such that

$$\gamma \cdot f = f.$$

The set of all the elements that are symmetries of a function forms a group, see Golubitsky et al. [**10**, chapter XI]. We are interested in functions whose symmetries form a special kind of group: a crystallographic group.

In this chapter we present the notation and the most relevant facts about crystallographic groups and symmetric functions. After that we develop some tools with specific interest to the process of projecting or restricting symmetric functions and studying the symmetry that remains either after the projection or after the restriction. This will be done by means of the Fourier expansion of the functions.

2.1. Groups

2.1.1. the Euclidean group. The *Euclidean group* of dimension $n+1$, $\mathbf{E}(n+1)$, is the group of all the isometries of the space $\mathbf{R}^{n+1}$ under the composition of functions. Each isometry has two components, one of which is orthogonal: for any $\gamma \in \mathbf{E}(n+1)$ we can write (see Armstrong [**1**, chapter 24] for a proof):

$$\gamma = (v, \delta), \text{ with } v \in \mathbf{R}^{n+1} \text{ and } \delta \in \mathbf{O}(n+1),$$

where $\mathbf{O}(n+1)$ is the group of orthogonal linear maps in $\mathbf{R}^{n+1}$. The elements $\gamma = (v, Id_{n+1})$, where Id_{n+1} is the identity map and $v \in \mathbf{R}^{n+1}$, are translations and $\gamma = (0, \delta)$, where $\delta \in \mathbf{O}(n+1)$ is an orthogonal matrix, correspond to rotations or reflections fixing the origin. We call $\delta \in \mathbf{O}(n+1)$ the *orthogonal component* of $(v, \delta) \in \mathbf{E}(n+1)$.

This identifies $\mathbf{E}(n+1)$ with the semi-direct product

$$\mathbf{E}(n+1) \cong \mathbf{R}^{n+1} \ltimes \mathbf{O}(n+1),$$

which is a group under the operation

$$(v_1, \delta_1) \cdot (v_2, \delta_2) = (v_1 + \delta_1 v_2, \delta_1 \delta_2)$$

for (v_1, δ_1), $(v_2, \delta_2) \in \mathbf{E}(n+1)$. On the right hand side the sum and the product are the usual sum in $\mathbf{R}^{n+1}$ and the usual product of matrices, considering the elements in $\mathbf{R}^{n+1}$ as column vectors. The inverse of $(v, \delta) \in \mathbf{E}(n+1)$ is

$$(v, \delta)^{-1} = (-\delta^{-1} v, \delta^{-1})$$

and the group *action* on $\mathbf{R}^{n+1}$ is the function:

$$\begin{array}{rcl} \mathbf{E}(n+1) \times \mathbf{R}^{n+1} & \longrightarrow & \mathbf{R}^{n+1} \\ ((v, \delta), (x, y)) & \longmapsto & (v, \delta) \cdot (x, y) = v + \delta(x, y) \end{array} .$$

Let Γ be a subgroup of $\mathbf{E}(n+1)$ (we use the notation $\Gamma \leq \mathbf{E}(n+1)$). The *orbit* of $(x, y) \in \mathbf{R}^{n+1}$ under the action of Γ is the subset of $\mathbf{R}^{n+1}$:

$$\Gamma \cdot (x, y) = \{\gamma \cdot (x, y) : \gamma \in \Gamma\}.$$

2.1.2. translation subgroups. The *translation subgroup* of Γ is the subgroup $\mathcal{L}$ of all its elements (v, Id_{n+1}) with trivial orthogonal component. We also use the symbol $\mathcal{L}$ for the orbit, on $\mathbf{R}^{n+1}$, of the origin under the action of the group $\mathcal{L}$:

$$\mathcal{L} = \{v : (v, Id_{n+1}) \in \Gamma\}.$$

The subgroup of translations is isomorphic to the group $(\mathcal{L}, +)$, where $\mathcal{L} \subset \mathbf{R}^{n+1}$.

$\mathcal{L}$ is a normal subgroup of Γ and is the kernel of the homomorphism

$$\begin{array}{rcl} \Gamma & \longrightarrow & \mathbf{O}(n+1) \\ (v, \delta) & \longmapsto & \delta \end{array} ,$$

whose image is

$$\mathbf{J} = \{\delta : (v, \delta) \in \Gamma \text{ for some } v \in \mathbf{R}^{n+1}\}.$$

Thus, $\mathbf{J}$ is isomorphic to the quotient $\Gamma/\mathcal{L}$,

$$\Gamma/\mathcal{L} \cong \mathbf{J}. \tag{1}$$

Moreover, the action of $\mathbf{J}$ on $\mathbf{R}^{n+1}$ leaves $\mathcal{L}$ invariant as shown in Armstrong [**1**, chapter 25]. Thus

$$\mathbf{J}\mathcal{L} = \{\delta l : \delta \in \mathbf{J}, l \in \mathcal{L}\} = \mathcal{L}.$$

2.1.3. lattices. Using the definition of Senechal [**22**, page 37], a subset $\mathcal{L} \subset \mathbf{R}^{n+1}$ is a *lattice* if it is generated over the integers by $n+1$ linearly independent elements $l_1, \ldots, l_{n+1} \in \mathbf{R}^{n+1}$, which we write:

$$\mathcal{L} = \{l_1, \ldots, l_{n+1}\}_{\mathbf{Z}} = \left\{ \sum_{i=1}^{n+1} m_i l_i, m_i \in \mathbf{Z} \right\}.$$

A lattice is a $\mathbf{Z}$-module and we will often refer properties of a lattice $\mathcal{L}$ that come from its module structure, like the invariance of $\mathcal{L}$ under the rotation of π around the origin:

$$-\mathcal{L} = \{-l : l \in \mathcal{L}\} = \mathcal{L}.$$

The vectors $l_1, \ldots, l_{n+1}$ that generate $\mathcal{L}$, define a $(n+1)$-dimensional parallelepiped called the *fundamental cell* of the lattice. Although the fundamental cell depends on the choice of generators, its volume ρ is an invariant of the lattice (see Senechal [**22**, page 38]), where the volume is the determinant of the matrix whose lines are the coordinates of the generators l_i (matrix M in section 2.3, below), *i.e.*, the wedge product

$$l_1 \wedge \cdots \wedge l_{n+1} = \rho.$$

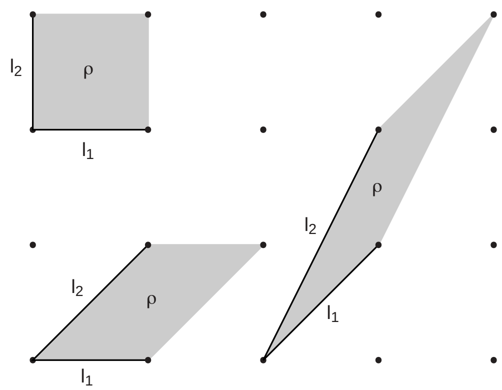

FIGURE 1. The area of the fundamental cell, $\rho = l_1 \wedge l_2$, does not depend on the choice of the generators l_1 and l_2 such that $\mathcal{L} = \{l_1, l_2\}_{\mathbf{Z}}$.

For any $l \in \mathcal{L}$ there is some $m \in \mathbf{Z}$ such that l/m is the smallest element of $\mathcal{L}$ colinear with l. Then l/m is a generator of $\mathcal{L}$, *i.e.*, there are elements $g_1, \ldots, g_n \in \mathcal{L}$ such that $\mathcal{L} = \{l/m, g_1, \ldots, g_n\}_{\mathbf{Z}}$ and $l/m \wedge g_1 \wedge \cdots \wedge g_n = \rho$.

2.1.4. crystallographic groups. A group $\Gamma \leq \mathbf{E}(n+1)$ is a *crystallographic group* if its translation subgroup $\mathcal{L}$ is a lattice.

This concept is a generalization of the three-dimensional crystallographic group as defined by Miller [**20**, page 55]. In Miller it is necessary to refer the number of generators of the lattice $\mathcal{L}$ because lattices are defined there as more general structures than in Senechal [**22**]. On the other hand, Senechal presents a definition of crystallographic group more general than the one we use here.

When $n = 1$ and $\Gamma \leq \mathbf{E}(2)$ the objects above are also called *plane lattices* and either *plane crystallographic groups* or *wallpaper groups* and there is some specific terminology described in chapter 3.

Let Γ be a crystallographic group. $\mathbf{J}$ is called the *point group* of $\mathcal{L}$ and is a subgroup of the *holohedry* of $\mathcal{L}$: the largest subgroup of $\mathbf{O}(n+1)$ that leaves $\mathcal{L}$ invariant. The holohedry is always a finite group, see Senechal [**22**, subsection 2.4.2].

If $\delta \in \mathbf{J}$ then by definition there is some $v \in \mathbf{R}^{n+1}$ such that $(v, \delta) \in \Gamma$ and $(l, Id_{n+1}) \cdot (v, \delta) = (l + v, \delta)$ also belongs to Γ for all $l \in \mathcal{L}$. Thus, we can relate each $\delta \in \mathbf{J}$ to a coset $\mathcal{L} \cdot (v, \delta)$. We use the symbol (v_δ, δ) for any element of that coset, *i.e.*, v_δ is the non-orthogonal component of $(v, \delta) \in \Gamma$ defined up to elements of $\mathcal{L}$. Moreover, by (1), there is no other $u \in \mathbf{R}^{n+1}$ such that $(u, \delta) \in \Gamma$ and $(u, \delta) \notin \mathcal{L} \cdot (v, \delta)$. To understand the action of Γ we only need to look at a finite number of its elements, by the isomorphism

$$\Gamma/\mathcal{L} \cong \{(v_\delta, \delta) : \delta \in \mathbf{J}\} \cong \mathbf{J}.$$

Thus, we can study a subgroup of the compact group

$$(\mathbf{R}^{n+1}/\mathcal{L}) \ltimes \mathbf{H},$$

where $\mathbf{H}$ is the holohedry of $\mathcal{L}$.

In chapter 3, v_δ has a different meaning. It is the translation associated to δ that belongs to the fundamental cell of the lattice.

For a complete description of plane crystallographic groups, with a notation similar to the one defined above, see chapter 25 and 26 of Armstrong [**1**]. Miller [**20**, chapter 2] studies crystallographic groups in $\mathbf{R}^3$ with a somewhat different approach and Senechal [**22**, chapter 2] presents some results on lattices with the restriction, as from page 41, of being integral lattices.

crystallographic groups

Until the end of this work, Γ is a crystallographic group of dimension $n+1$ with lattice $\mathcal{L}$ and point group $\mathbf{J}$, *i.e.*:

i. $\Gamma \leq \mathbf{E}(n+1) \cong \mathbf{R}^{n+1} \ltimes \mathbf{O}(n+1)$,
ii. the translation subgroup of Γ is isomorphic to $(\mathcal{L}, +)$,
iii. $\mathcal{L} = \{l_1, l_2, \ldots, l_{n+1}\}_{\mathbf{Z}}$ such that $\mathbf{R}^{n+1} = \{l_1, l_2, \ldots, l_{n+1}\}_{\mathbf{R}}$,
iv. $\mathbf{J} = \{\delta : (v, \delta) \in \Gamma \text{ for some } v \in \mathbf{R}^{n+1}\}$.

examples — Let $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathbf{O}(2)$ and let ${}^s\Gamma$ be the sample - crystallographic group generated by the following three elements of $\mathbf{E}(2)$:

- the translation $\gamma_1 = ((0,1), Id_2)$,
- the reflection in the vertical axis $\gamma_2 = ((0,0), -\sigma)$,
- the reflection in the horizontal axis followed by a translation, or *glide reflection*, $\gamma_3 = \left(\left(\frac{1}{2}, 0\right), \sigma\right)$.

See figure 2 for a description of their action on $\mathbf{R}^2$.

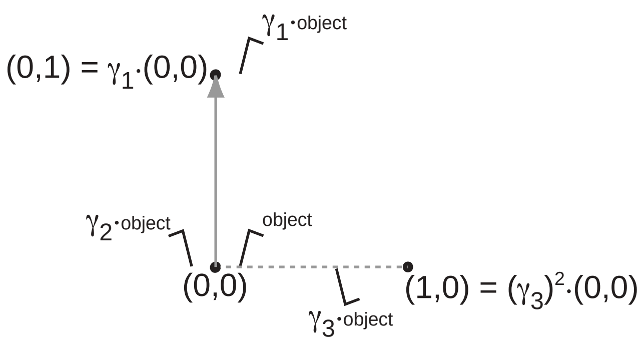

FIGURE 2. Group ${}^s\Gamma$ acting on $\mathbf{R}^2$.

${}^s\Gamma$ is a wallpaper group usually called *p2mg*, see Armstrong [**1**, chapter 26].
From γ_3 we obtain

$$\gamma_3^2 = \left(\left(\frac{1}{2}, 0\right), \sigma\right) \cdot \left(\left(\frac{1}{2}, 0\right), \sigma\right) = ((1,0), Id_2) \in^s \Gamma$$

and thus the orbit of $(0,0) \in \mathbf{R}^2$ by its subgroup of translations is the square lattice

$${}^s\mathcal{L} = \{(1,0), (0,1)\}_{\mathbf{Z}}.$$

A fundamental cell of ${}^s\mathcal{L}$ is represented in figure 3 with all the symmetries of ${}^s\Gamma$.

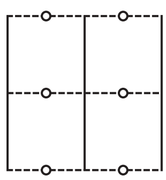

FIGURE 3. Fundamental cell of ${}^s\mathcal{L}$. Rotations of π around the circles transform ${}^s\mathcal{L}$ in itself. The dashed lines are invariant under glide reflections and the continuous lines are invariant under reflections.

The holohedry of ${}^s\mathcal{L}$ is

$$\mathbf{D}_4 = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sigma \right\rangle,$$

where $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the rotation of $\frac{\pi}{2}$ in the clockwise direction around the origin.

The point group of ${}^s\Gamma$ is the subgroup of $\mathbf{D}_4$:

$${}^s\mathbf{J} = \mathbf{Z} \times \mathbf{Z} =< \sigma, -\sigma >,$$

whose elements are associated to the translations
$v_{Id_2} = (0,0)$, $v_\sigma = \left(\frac{1}{2}, 0\right)$, $v_{-\sigma} = (0,0)$ and $v_{-Id_2} = \left(\frac{1}{2}, 0\right)$.

2.2. Invariant Functions

2.2.1. $\mathbf{E}(n+1)$ acting on functions. The action of Γ in $\mathbf{R}^{n+1}$ induces the *scalar action* on the functions whose domain is $\mathbf{R}^{n+1}$:

$$(\gamma \cdot f)(x,y) = f(\gamma^{-1} \cdot (x,y))$$

for $\gamma \in \Gamma$ and $(x,y) \in \mathbf{R}^{n+1}$. This is a particular case of the physical action, as defined in Melbourne [**18**, section 2.1]. The orbit of f under the action of Γ is the subset of the space of functions:

$$\Gamma \cdot f = \{\gamma \cdot f : \gamma \in \Gamma\}.$$

When γ is a symmetry of a function f, *i.e.*,

$$(\gamma \cdot f)(x,y) = f(x,y) \quad \forall (x,y) \in \mathbf{R}^{n+1},$$

we say that f is a *γ-invariant* function or that γ *fixes* f. If f is γ-invariant for all the elements γ of a group Γ then f is a *Γ-invariant* function and Γ is the *isotropy subgroup* of f.

Let X_Γ be a space of Γ-invariant functions $f : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}$. Subsequently we will impose some more restrictions on the elements of X_Γ, the space of the functions we will study.

2.2.2. $\mathcal{L}$-periodic functions. The functions that are invariant under the action of a crystallographic group Γ are, in particular, $\mathcal{L}$-invariant functions. This invariance is also named *$\mathcal{L}$-periodicity.* This periodicity is the basic requirement of our work. The objects we study are $\mathcal{L}$-periodic functions, where $\mathcal{L}$ is a lattice and their symmetry group may be any crystallographic group having translations subgroup $\mathcal{L}$.

These functions are the generalization, for any dimension, of functions on the plane whose level curves form a periodic tiling. We will use the expressions *periodic pattern* or *repetitive pattern* when refering to the functions in X_Γ whose level sets have that property of space rhythm, visually very easy to detect.

invariant functions

X_Γ is the space of Γ-invariant functions f, for the scalar action:

i. $f : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}$,
ii. $x \in \mathbf{R}^n$ and $y \in \mathbf{R}$,
iii. $(\gamma \cdot f)(x,y) = f(\gamma^{-1} \cdot (x,y))$ for all $\gamma \in \Gamma$ and for all $(x,y) \in \mathbf{R}^{n+1}$,
iv. $(\gamma \cdot f)(x,y) = f(x,y)$ for all $\gamma \in \Gamma$ and for all $(x,y) \in \mathbf{R}^{n+1}$.

---

examples — Let ${}^s\Gamma$ be the sample - crystallographic group defined in page 27. The function

$$\begin{array}{rcl} {}^s f : \mathbf{R}^2 & \longrightarrow & \mathbf{R} \\ (x,y) & \longmapsto & \cos(\pi(y + \cos(2\pi x)))\cos(\pi y) \end{array}$$

is ${}^s\Gamma$-invariant. In fact, ${}^s f$ remains the same under the action of each one of the generators of group ${}^s\Gamma$:

- $\gamma_1 \cdot {}^s f(x,y) = \cos(\pi(y + \cos(2\pi x) - \pi)\cos(\pi y - \pi)$,
- $\gamma_2 \cdot {}^s f(x,y) = \cos(\pi(y + \cos(-2\pi x))\cos(\pi y)$,
- $\gamma_3 \cdot {}^s f(x,y) = \cos(\pi(-y + \cos(2\pi x - \pi))\cos(-\pi y)$.

In particular, ${}^s f$ is a ${}^s\mathcal{L}$-periodic function.

The level sets of ${}^s f$ define a pattern with symmetry ${}^s\Gamma$. Figure 4 shows this pattern and figure 5 shows the fundamental cell of ${}^s\mathcal{L}$ with this pattern and its symmetries.

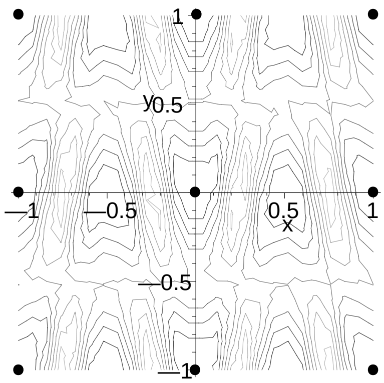

FIGURE 4. The level sets of ${}^s f$ with black dots indicating the elements of ${}^s\mathcal{L}$.

2.3. Fourier Expansion

2.3.1. dual lattices. We study Γ-invariant functions that have formal Fourier expansion in terms of the *waves*

$$\omega_k(x,y) = \mathrm{e}^{2\pi i <k,(x,y)>},$$

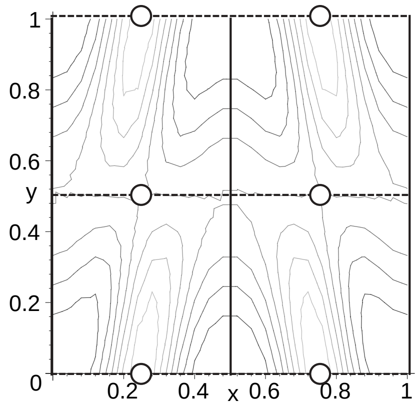

FIGURE 5. Fundamental cell of ${}^s\mathcal{L}$ with level sets of ${}^sf$ and its symmetries. The circles are centers of rotations of π. The dashed lines are invariant under glide reflections and the continuous lines are invariant under reflections.

where $k \in \mathbf{R}^{n+1}$ and $< \cdot, \cdot >$ is the usual inner product in $\mathbf{R}^{n+1}$. As the functions are, in particular, $\mathcal{L}$-periodic, the waves in the Fourier expansion must also be $\mathcal{L}$-periodic functions. Since

$$\omega_k\left((x,y)+l\right) = \omega_k(x,y)\omega_k(l),$$

periodicity implies $\omega_k(l) = 1$ for all $l \in \mathcal{L}$ and this happens if and only if

$$< k, l > \in \mathbf{Z} \quad \forall l \in \mathcal{L}.$$

Let $\mathcal{L} = \{l_1, \ldots, l_{n+1}\}_{\mathbf{Z}}$. The set of all the elements $k \in \mathbf{R}^{n+1}$ such that ω_k is a $\mathcal{L}$-periodic function is the *dual lattice* of $\mathcal{L}$:

$$\mathcal{L}^* = \{k \in \mathbf{R}^{n+1} : < k, l_i > \in \mathbf{Z},\ i = 1, \ldots, n+1\}.$$

It may be written as

$$\mathcal{L}^* = \{l_1^*, \ldots, l_{n+1}^*\}_{\mathbf{Z}},$$

where $l_i^* \in \mathbf{R}^{n+1}$ and $< l_i^*, l_j >= \delta_{ij}$ for all $i, j \in \{1, \ldots, n+1\}$.

Notice that if $\delta\mathcal{L} = \mathcal{L}$ then $\delta^{-1}\mathcal{L} = \mathcal{L}$. Let k be any element of $\mathcal{L}^*$, then, by orthogonality of δ,

$$< \delta k, l > = < k, \delta^{-1} l > \in \mathbf{Z} \quad \forall l \in \mathcal{L},$$

i.e., $\delta k \in \mathcal{L}^*$. It follows that $\mathcal{L}^*$ has the same holohedry as $\mathcal{L}$. Therefore, $-\mathcal{L}^* = \mathcal{L}^*$ and $\mathbf{J}\mathcal{L}^* = \mathcal{L}^*$. Moreover, if we define a matrix with the generators of $\mathcal{L}$, which we call *matrix of the basis* of $\mathcal{L}$,

$$M = \begin{pmatrix} l_1 \\ \vdots \\ l_{n+1} \end{pmatrix}$$

then the rows of M^*,

$$M^* = \left(M^{-1}\right)^T = \begin{pmatrix} l_1^* \\ \vdots \\ l_{n+1}^* \end{pmatrix},$$

are generators of $\mathcal{L}^*$.

2.3.2. formal Fourier expansion. The *formal Fourier expansion* of a function $f \in X_\Gamma$ is

$$f(x,y) = \sum_{k \in \mathcal{L}^*} \omega_k(x,y) C(k)$$

where $C : \mathcal{L}^* \longrightarrow \mathbf{C}$ are the *Fourier coefficients.*

We assume that in X_Γ this expansion is unique.

For a real function f, the coefficients have the restriction

$$\overline{C(k)} = C(-k).$$

2.3.3. Γ action on the Fourier coefficients. From the action of Γ on X_Γ we get:

$$\begin{aligned}(v_\delta, \delta) \cdot f(x,y) &= f((v_\delta, \delta)^{-1} \cdot (x,y)) \\ &= \textstyle\sum_{k \in \mathcal{L}^*} \omega_k(\delta^{-1}(x,y)) \omega_k(-\delta^{-1} v_\delta) C(k) \\ &= \textstyle\sum_{k \in \mathcal{L}^*} \omega_{\delta k}(x,y) \omega_{\delta k}(-v_\delta) C(k), \text{ by orthogonality,} \\ &= \textstyle\sum_{k \in \delta\mathcal{L}^*} \omega_k(x,y) \omega_k(-v_\delta) C(\delta^{-1} k) \\ &= \textstyle\sum_{k \in \mathcal{L}^*} \omega_k(x,y) \omega_k(-v_\delta) C(\delta^{-1} k), \text{ because } \delta\mathcal{L}^* = \mathcal{L}^*.\end{aligned}$$

By the unicity of the Fourier expansion, this induces an action of Γ on the space of Fourier coefficients:

$$(v_\delta, \delta) \cdot C(k) = \omega_k(-v_\delta) C(\delta^{-1} k).$$

Analogously, the (v_δ, δ)-invariance of f implies

$$C(k) = \omega_k(-v_\delta) C(\delta^{-1} k)$$

for all its Fourier coefficients.

The induced action of Γ in the space of Fourier coefficients connects the algebraic and analytical perspectives. It translates into analytic language those properties of a function arising from symmetry.

2.3.4. a remark on symmetry groups. From the above calculations it follows that the symmetry groups fixing $\mathcal{L}$-periodic functions are crystallographic groups with lattice $\mathcal{L}$. Moreover, if f is a $\mathcal{L}$-periodic function and δ is in the holohedry of $\mathcal{L}$ then $(v, \delta) \cdot f$ is also $\mathcal{L}$-periodic for all $v \in \mathbf{R}^{n+1}$.

Thus, we must distinguish several symmetry groups related to the same lattice $\mathcal{L}$:

- the crystallographic group that leaves the lattice $\mathcal{L}$ invariant — figure 6;
- $\mathcal{L}$-periodic functions are fixed by crystallographic groups Γ with lattice $\mathcal{L}$ — figure 7;
- the group $\mathbf{R}^{n+1} \ltimes \mathbf{H}$, where $\mathbf{H}$ is the holohedry of $\mathcal{L}$, leaves invariant the space of all $\mathcal{L}$-periodic functions — figure 8.

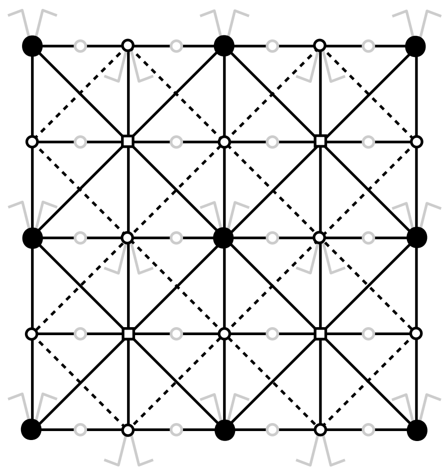

FIGURE 6. The lattice $\mathcal{L}$ (black dots) is invariant under the reflections in the black continuous lines and under the glide reflections in the black dashed lines. Open circles and squares are centers of rotations of, respectively, order two and four that leave the lattice invariant. There are also centers of order four in the black dots. The symmetry group of this lattice is called *p4mm*.

2.3.5. the functions I_k. The simplest Γ-invariant functions in X_Γ are the real and imaginary components of I_k, for $k \in \mathcal{L}^*$, given by:

$$I_k(x,y) = \sum_{\delta \in \mathbf{J}} \omega_{\delta k}(x,y)\omega_{\delta k}(-v_\delta)$$

and we will assume that they lie in X_Γ. Each function I_k, for $k \in \mathcal{L}^*$, is the sum of all the elements in the orbit of ω_k under the action of Γ.

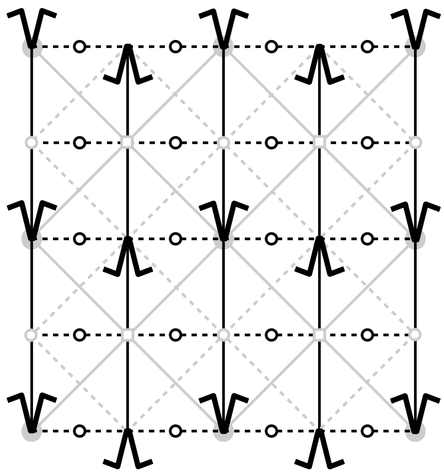

FIGURE 7. The black pattern is invariant under the reflections in the black continuous lines and under the glide reflections in the black dashed lines. Open circles are centers of rotations of order two that leave the pattern invariant. The symmetry group of this pattern is called *p2mg*.

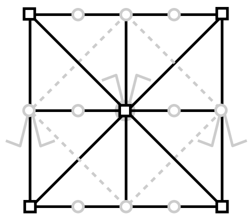

FIGURE 8. $\mathbf{H}$, the holohedry of lattice $\mathcal{L}$, has reflections and rotations of order two and four. For this lattice $\mathbf{H} = \mathbf{D}_4$, so the space of all the $\mathcal{L}$-periodic functions is invariant under the action of group $\mathbf{R}^{n+1} \ltimes \mathbf{H}$.

Fourier expansion

If $f \in X_\Gamma$ then f has a formal Fourier expansion:

i. $f(x,y) = \sum_{k \in \mathcal{L}^*} \omega_k(x,y) C(k)$,

ii. $\mathcal{L}^* = \{l_1^*, \ldots, l_{n+1}^*\}_{\mathbf{Z}}$, with $<l_i^*, l_j> \in \mathbf{Z}$ for all $i, j \in \{1, \ldots, n+1\}$,

iii. $C : \mathcal{L}^* \longrightarrow \mathbf{C}$,

iv. $\omega_k(x,y) = \mathrm{e}^{2\pi i <k,(x,y)>}$.

Moreover, $Re(I_k), Im(I_k) \in X_\Gamma$, for $I_k(x,y) = \sum_{\delta \in \mathbf{J}} \omega_{\delta k}(x,y) \omega_{\delta k}(-v_\delta)$ and for all $k \in \mathcal{L}^*$.

examples — Let ${}^s\Gamma$ be the sample - crystallographic group defined in page 27. The translation subgroup of ${}^s\Gamma$ defines the lattice ${}^s\mathcal{L}$:

$$ {}^s\mathcal{L} = \{(1,0),(0,1)\}_{\mathbf{Z}}. $$

The dual lattice of ${}^s\mathcal{L}$ is:

$$ {}^s\mathcal{L}^* = {}^s\mathcal{L} = \{(1,0),(0,1)\}_{\mathbf{Z}}. $$

For an example where the dual lattice $\mathcal{L}^*$ is not equal to $\mathcal{L}$, consider the hexagonal lattice

$$ \mathcal{L} = \left\{(1,0), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\right\}_{\mathbf{Z}}, $$

whose dual lattice is

$$ \mathcal{L}^* = \left\{\left(1, \frac{-\sqrt{3}}{3}\right), \left(0, \frac{2\sqrt{3}}{3}\right)\right\}_{\mathbf{Z}}. $$

The dual lattice is also hexagonal with hexagons that are dual to the hexagons of $\mathcal{L}$, see figure 9.

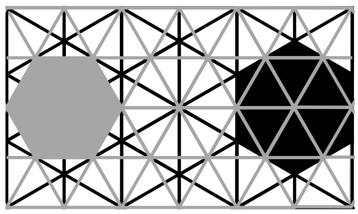

FIGURE 9. The hexagonal lattice $\mathcal{L}$ is defined by the vertices of the gray triangles and its dual lattice is $\mathcal{L}^*$, the set of vertices of the black triangles.

The areas of the fundamental cells of these two lattices are, respectively, $\rho = \frac{\sqrt{3}}{2}$ and $\frac{1}{\rho} = \frac{2\sqrt{3}}{3}$, see figure 10.

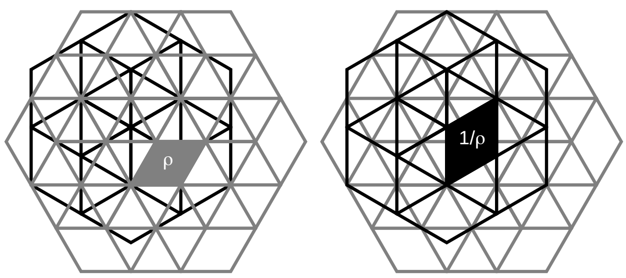

FIGURE 10. The fundamental cells of $\mathcal{L}$ and $\mathcal{L}^*$ have areas, respectively, ρ and $1/\rho$.

For the sample - crystallographic group defined in page 27, ${}^{s}\Gamma$, consider the functions

$$I_k(x,y) = \sum_{\delta\in\{Id_2,\sigma,-\sigma,-Id_2\}} \omega_{\delta k}(x,y)\omega_{\delta k}(-v_\delta)$$

for two different elements of $\mathcal{L}^*$, $k=(0,1)$ and $k=(1,2)$:

$$\begin{aligned} I_{(0,1)}(x,y) &= \mathrm{e}^{2\pi i y} + \mathrm{e}^{-2\pi i y} + \mathrm{e}^{2\pi i y} + \mathrm{e}^{-2\pi i y} \\ &= 2\left(\mathrm{e}^{2\pi i y} + \mathrm{e}^{-2\pi i y}\right) \end{aligned}$$

and

$$\begin{aligned} I_{(1,2)}(x,y) = \;& \mathrm{e}^{2\pi i(x+2y)} + \mathrm{e}^{2\pi i(x-2y-1/2)} + \\ & + \mathrm{e}^{-2\pi i(x-2y)} + \mathrm{e}^{-2\pi i(x+2y-1/2)}. \end{aligned}$$

Two of the simplest ${}^{s}\Gamma$-invariant functions are, therefore,

$$Re\left(I_{(0,1)}(x,y)\right) = 4\cos(2\pi y)$$

and

$$Im\left(I_{(1,2)}(x,y)\right) = -4\cos(2\pi x)\sin(4\pi y),$$

whose level sets are presented in figure 11.

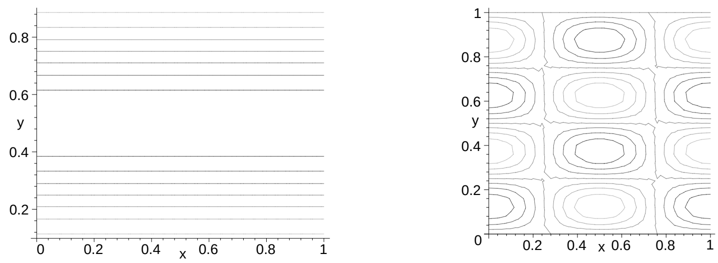

FIGURE 11. The fundamental cell of ${}^{s}\mathcal{L}$ with the level sets of two simple ${}^{s}\Gamma$-invariant functions, $4\cos(2\pi y)$ and $-4\cos(2\pi x)\sin(4\pi y)$.

2.4. Projection

The projection operator acting on X_Γ is defined with respect to some *hyperplanes* - these are affine subspaces of $\mathbf{R}^{n+1}$ with codimension one.

2.4.1. the projection operator. For $y_0 > 0$, consider the restriction of f to the region between the hyperplanes $y=0$ and $y=y_0$. The *projection operator* Π_{y_0} integrates this restriction of f along the width y_0:

$$\Pi_{y_0}(f)(x) = \int_0^{y_0} f(x,y)dy.$$

The region between $y=0$ and $y=y_0$ is called the *projected band* or the *projection band*, and y_0 is called the *width of projection* or the *width of the projected band.* Moreover, we will call $\Pi_{y_0}(f) : \mathbf{R}^n \longrightarrow \mathbf{R}$ the *projected function* or the *function f after projection.*

If $f \in X_\Gamma$ then

$$\Pi_{y_0}(f)(x) = \int_0^{y_0} \sum_{k \in \mathcal{L}^*} \omega_k(x,y) C(k) dy$$

and, when the integral and the summation commute,

$$\begin{aligned} \Pi_{y_0}(f)(x) &= \textstyle\sum_{k \in \mathcal{L}^*} \int_0^{y_0} \omega_k(x,y) C(k) dy \\ &= \textstyle\sum_{k \in \mathcal{L}^*} \omega_{k_1}(x) C(k_1,k_2) \int_0^{y_0} \omega_{k_2}(y) dy, \end{aligned}$$

where

$$k = (k_1, k_2), \text{ with } k_1 \in \mathbf{R}^n \text{ and } k_2 \in \mathbf{R}.$$

Grouping the terms with common n first components in $\mathcal{L}^*$, we obtain

$$\Pi_{y_0}(f)(x) = \sum_{k_1 \in \mathcal{L}_1^*} \omega_{k_1}(x) \sum_{k_2 : (k_1,k_2) \in \mathcal{L}^*} C(k_1,k_2) \int_0^{y_0} \omega_{k_2}(y) dy,$$

where

$$\mathcal{L}_1^* = \{k_1 : (k_1,k_2) \in \mathcal{L}^*\}.$$

2.4.2. functions after projection. For

$$D(k_1) = \sum_{k_2 : (k_1,k_2) \in \mathcal{L}^*} C(k_1,k_2) \int_0^{y_0} \omega_{k_2}(y) dy,$$

the projection of $f \in X_\Gamma$ has the expression:

$$\Pi_{y_0}(f)(x) = \sum_{k_1 \in \mathcal{L}_1^*} \omega_{k_1}(x) D(k_1).$$

The coefficients $D(k_1)$ could be written $D_{y_0}(k_1)$ since they depend on y_0. We avoid that notation in order to simplify the formalism.

Notice that if $\mathcal{L}_1^*$ is a lattice in $\mathbf{R}^n$ then the expression above is the Fourier expansion of a $\mathcal{L}_1$-periodic function, where $\mathcal{L}_1 \subset \mathbf{R}^n$ is the lattice with dual $\mathcal{L}_1^*$. This and other considerations about periodicity are presented in chapter 5.

2.4.3. symmetry of projected functions. The functions $\Pi_{y_0}(f)$ may be invariant under the action of some elements of the group $\mathbf{E}(n) \cong \mathbf{R}^n \ltimes \mathbf{O}(n)$. Using a notation similar to the $(n+1)$-dimensional case, $(v_\alpha, \alpha) \in \mathbf{E}(n)$ is a symmetry of $\Pi_{y_0}(f)$ if

$$(v_\alpha, \alpha) \cdot \Pi_{y_0}(f)(x) = \Pi_{y_0}(f)(x) \quad \forall x \in \mathbf{R}^n.$$

For $f \in X_\Gamma$ this is equivalent to

$$\sum_{k_1 \in \mathcal{L}_1^*} \omega_{k_1}(x) D(k_1) = \sum_{k_1 \in \mathcal{L}_1^*} \omega_{k_1}(\alpha^{-1}x)\omega_{k_1}(-\alpha^{-1}v_\alpha) D(k_1).$$

This equation imposes restrictions on the coefficients $D(k_1)$, as formulated on Lemma 4.3 of chapter 4.

When $n = 1$, the function $\Pi_{y_0}(f)$ has domain $\mathbf{R}$ and may be invariant under the action of elements in $\mathbf{E}(1) \cong \mathbf{R} \ltimes \mathbf{O}(1)$. Moreover, $\mathbf{O}(1) \cong \{1, -1\}$. In chapter 3 we study the symmetries of $\Pi_{y_0}(f)$, for $n = 1$: section 3.1 for the symmetry with orthogonal component $\alpha = 1$, called *periodicity*, and section 3.2 when the orthogonal component is $\alpha = -1$, which we call *parity*.

projection

If $f \in X_\Gamma$ and $y_0 > 0$ then:

i. $\Pi_{y_0}(f)(x) = \int_0^{y_0} f(x,y)dy = \sum_{k_1 \in \mathcal{L}_1^*} \omega_{k_1}(x) D(k_1)$,

ii. $\mathcal{L}_1^* = \{k_1 : (k_1, k_2) \in \mathcal{L}^*\}$, $k_1 \in \mathbf{R}^n$ and $k_2 \in \mathbf{R}$,

iii. $D(k_1) = \sum_{k_2:(k_1,k_2) \in \mathcal{L}^*} C(k_1, k_2) \int_0^{y_0} \omega_{k_2}(y)dy$.

---

examples — Let ${}^s\Gamma$ be the sample - crystallographic group defined in page 27 and ${}^sf$ the ${}^s\Gamma$-invariant function ${}^sf(x,y) = \cos(\pi(y + \cos(2\pi x)))\cos(\pi y)$.

The projection

$$\begin{aligned}\Pi_{y_0}({}^sf)(x) = & \tfrac{1}{4\pi}\cos(\pi\cos(2\pi x))\,(\sin(2\pi y_0) + 2\pi y_0) + \\ & + \tfrac{1}{4\pi}\sin(\pi\cos(2\pi x))\,(\cos(2\pi y_0) - 1)\end{aligned}$$

is invariant under the action of $(1,1)$ and $(0,-1) \in \mathbf{R} \ltimes \mathbf{O}(1)$ for all $y_0 > 0$ since $\cos(2\pi x - 2\pi) = \cos(-2\pi x) = \cos(2\pi x)$. In figure 12 we see the projection $\Pi_{y_0}({}^sf)$ for $y_0 = 0.3$.

For $y_0 = 1$, the projection $\Pi_1({}^sf)(x) = 1/2\ \cos(\pi\cos(2\pi x))$ is invariant under the action of $(1/2, 1) \in \mathbf{R} \ltimes \mathbf{O}(1)$:

$$(1/2, 1) \cdot \Pi_1({}^sf)(x) = \cos(\pi\cos(2\pi x - \pi)) = \Pi_1(f)(x).$$

See figure 13.

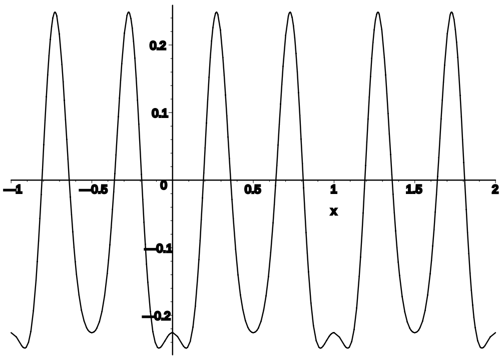

FIGURE 12. The graphic of a ${}^s\Gamma$-invariant function after projection. The observed symmetries do not depend on the projection width y_0.

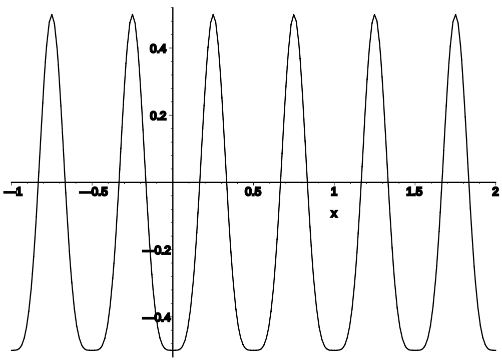

FIGURE 13. The graphic of a ${}^s\Gamma$-invariant function after projection. The period of the projected function, for the widths $y_0 \in \mathbf{N}$, is half the period of the projected functions for the remaining widths.

2.5. Restriction

Let Φ_r be the operator that restricts the functions to the hyperplane $y = r$:

$$\Phi_r(f)(x) = f(x, r).$$

If $f \in X_\Gamma$ then

$$\begin{aligned} \Phi_r(f)(x) &= \sum_{k\in\mathcal{L}^*} \omega_k(x, r)C(k) \\ &= \sum_{k_1\in\mathcal{L}_1^*} \omega_{k_1}(x) \sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} C(k_1, k_2)\omega_{k_2}(r). \end{aligned}$$

The functions $\Phi_r(f)$ may be invariant under the action of some elements of the group $\mathbf{E}(n) \cong \mathbf{R}^n \ltimes \mathbf{O}(n)$, as described above for the projected functions.

restriction

If $f \in X_\Gamma$ and $r \in \mathbf{R}$ then:

i. $\Phi_r(f)(x) = f(x,r) = \sum_{k_1 \in \mathcal{L}_1^*} \omega_{k_1}(x) D(k_1)$,
ii. $\mathcal{L}_1^* = \{k_1 : (k_1,k_2) \in \mathcal{L}^*\}$, $k_1 \in \mathbf{R}^n$ and $k_2 \in \mathbf{R}$,
iii. $D(k_1) = \sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} C(k_1,k_2)\omega_{k_2}(r)$.

examples — Let ${}^s\Gamma$ be the sample - crystallographic group defined in page 27 and ${}^sf$ the ${}^s\Gamma$-invariant function ${}^sf(x,y) = \cos(\pi(y+\cos(2\pi x)))\cos(\pi y)$.

The restriction $\Phi_r({}^sf)(x) = \cos(\pi(r+\cos(2\pi x)))\cos(\pi r)$ is invariant under the action of $(1,1)$ and $(0,-1) \in \mathbf{R} \ltimes \mathbf{O}(1)$ for all $r \in \mathbf{R}$:

- $(1,1) \cdot \Phi_r({}^sf)(x) = \cos(\pi(r+\cos(2\pi x - 2\pi)))\cos(\pi r)$;
- $(0,-1) \cdot \Phi_r({}^sf)(x) = \cos(\pi(r+\cos(-2\pi x)))\cos(\pi r)$.

Figure 14 presents $\Phi_r({}^sf)$ for $r = 0.3$.

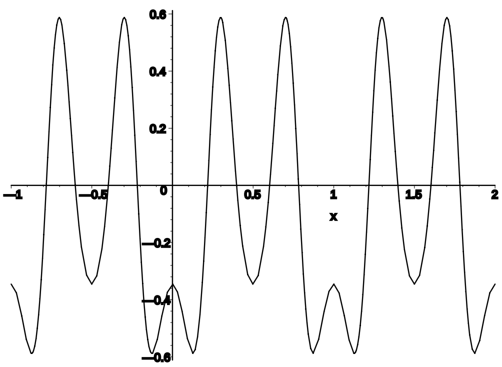

FIGURE 14. The graphic of a ${}^s\Gamma$-invariant function restricted to the line $y = r$, for $r = 0.3$. The restricted function exhibits some symmetry that does not depend on r.

For $r = 1$, the restriction $\Phi_1({}^sf)(x) = \cos(\pi\cos(2\pi x))$, in figure 15, is invariant under the action of $(1/2,1) \in \mathbf{R} \ltimes \mathbf{O}(1)$:

- $(1/2,1) \cdot \Phi_1({}^sf)(x) = \cos(\pi\cos(2\pi x - \pi))$.

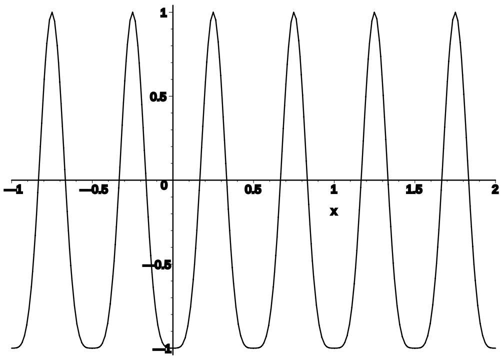

FIGURE 15. The graphic of a ${}^s\Gamma$-invariant function restricted to the line $y = 1$. The period of the restricted function, for $y = r$, $r \in \mathbf{N}$, is half the period of the functions restricted to $y = r$ with $r \notin \mathbf{N}$.

2.6. The Space X_Γ

We may now summarize the characterization of the space of functions X_Γ.

the space X_Γ

We assume X_Γ is a vector space of functions such that:

1. Γ is a $(n+1)$-dimensional crystallographic group with lattice $\mathcal{L}$, dual lattice $\mathcal{L}^*$ and point group $\mathbf{J}$,
2. if $f \in X_\Gamma$ then:
   (i) $f : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}$,
   (ii) f is a Γ-invariant function,
   (iii) f has a unique formal Fourier expansion in waves $\omega_k(x, y)$, $k \in \mathcal{L}^*$,
   (iv) the integral and the summation commute in the projection of f,
3. $Re(I_k), Im(I_k) \in X_\Gamma$, where $I_k(x, y) = \sum_{\delta \in \mathbf{J}} \omega_{\delta k}(x, y)\omega_{\delta k}(-v_\delta)$, for all $k \in \mathcal{L}^*$.

CHAPTER 3

Symmetries of Projected Wallpaper Patterns

In this chapter we describe the symmetries of the projection of functions defined in $\mathbf{R}^2$ onto a line and compare them to those of the restriction of the functions to a line. We work with functions $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$ invariant under the action of a wallpaper group Γ, *i.e.*, a plane crystallographic group.

The results are presented in a form suitable for generalization to higher dimension, we hope they will make it easier to understand the case of dimension $n+1$, presented in chapter 4. The results in this chapter appear in Labouriau and Pinho [**16**] and some of the examples were presented as a poster in "VIIIème Rencontre Internationale de São Carlos sur les Singularités Réelles et Complexes", CIRM, Marseille, France, 2004.

The main result on periodicity (section 3.1) is the following: start with all the functions f with symmetry Γ and a strip of width y_0. Define the projection of f to be its integral over the width of the strip, a new function defined on the line. Suppose all the projected functions have a common period $P > 0$. If $(P, 0) \in \mathbf{R}^2$ is not already a period of all the original functions, then either the projection width is a period in Γ or Γ contains a glide reflection related to the projection width.

In section 3.2 we show that if the projected functions are invariant for the reflection in a point and if Γ does not already contain a reflection then it contains a rotation of order two related to y_0.

In each of these sections we relate, by means of Propositions 3.1 and 3.3, the symmetry of the functions in X_Γ to some conditions over Γ and its dual lattice $\mathcal{L}^*$. After that we translate these conditions to restrictions over Γ and $\mathcal{L}$, obtaining Theorems 3.1 and 3.2. These theorems do not substantially differ from Theorem 4.1, for the $(n+1)$-dimensional case. However, the generalization of Propositions 3.1 and 3.3, stated in Proposition 4.1, gives rise to a new condition and qualitativelly different proofs, with the development of new tools and strategies. Along this chapter we indicate how these proofs and those of the next chapter differ.

A study for the restriction of $f(x, y)$ to the line $y = r$ is carried out in section 3.3 and the symmetries of projections and of restrictions are compared in section 3.4 for each of the 17 wallpaper groups. In section 3.5 we discuss one-dimensional models that are direct solutions of an equivariant equation comparing them to the projections or restrictions of two-dimensional solutions of equivariant problems. We show that the symmetry of the projected problem is in many ways richer than that of simple one-dimensional problems.

3.1. Projection and Symmetry - Periodicity on $\Pi_{y_0}(X_\Gamma)$

Let Γ be a wallpaper group and X_Γ be a space of Γ-invariant functions, $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$, with the properties described in section 2.6.

In this chapter we will use the symbol σ for the reflection

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Given $\delta \in \mathbf{J}$, there are several choices of v such that $(v, \delta) \in \Gamma$. Along this chapter, let v_δ satisfy $(v_\delta, \delta) \in \Gamma$ with v_δ in the fundamental cell of the lattice. If v_δ lies in one of the edges of the parallelogram, we choose v_δ in the edge that contains the origin and if v_δ is one of the vertices we choose the origin.

A different definition for (v_δ, δ) appears in chapters 2 and 4, where the non-orthogonal component v_δ is defined up to elements of the lattice $\mathcal{L}$. This will make the proofs simpler.

3.1.1. periodicity of $\Pi_{y_0}(X_\Gamma)$ related to $\mathcal{L}^*$.

Proposition 3.1. *All functions in $\Pi_{y_0}(X_\Gamma)$ have a common period $P \in \mathbf{R} - \{0\}$ if and only if, for each $k = (k_1, k_2) \in \mathcal{L}^*$, one of the following conditions holds:*

A. $k_1 P \in \mathbf{Z}$,
B. $k_2 y_0 \in \mathbf{Z} - \{0\}$,
C. $\sigma \in \mathbf{J}$ *and* $< k, \sigma v_\sigma + (0, y_0) > + \frac{1}{2} \in \mathbf{Z}$.

Before proving Proposition 3.1, note that a function of one variable with Fourier expansion $\sum_{k_1 \in \mathcal{L}_1^*} D(k_1)\omega_{k_1}(x)$ has period P if and only if, for each $k_1 \in \mathcal{L}_1^*$, one of the two following conditions holds:

A'. ω_{k_1} has period P, or
B'. $D(k_1) = 0$.

For each $k_1 \in \mathcal{L}_1^*$, conditions A and A' are equivalent. The Fourier coefficients for the projection of $f \in X_\Gamma$ are given by

$$D(k_1) = \sum_{k_2 : (k_1, k_2) \in \mathcal{L}^*} C(k_1, k_2) \int_0^{y_0} \omega_{k_2}(y) dy.$$

and condition A is equivalent to $\int_0^{y_0} \omega_{k_2}(y) dy = 0$.

Conditions A, B and C hold, respectively, in the following subsets of the dual lattice $\mathcal{L}^*$:

$$\mathcal{M}_P^* = \{k \in \mathcal{L}^* : < k, (P, 0) > \in \mathbf{Z}\},$$
$$\mathcal{N}_{y_0}^* = \{k \in \mathcal{L}^* : < k, (0, y_0) > \in \mathbf{Z} - \{0\}\} \quad \text{and}$$
$$\mathcal{N}_\sigma^* = \left\{k \in \mathcal{L}^* : < k, \sigma v_\sigma + (0, y_0) > + \frac{1}{2} \in \mathbf{Z}\right\} \quad \text{if } \sigma \in \mathbf{J},$$
$$\mathcal{N}_\sigma^* = \emptyset \quad \text{if } \sigma \notin \mathbf{J}.$$

We will refer to condition A either as $k_1 P \in \mathbf{Z}$ or as $k \in \mathcal{M}_P^*$. The same will happen with conditions B and C.

The set $\mathcal{M}_P^*$ is a submodule of $\mathcal{L}^*$. Let $\mathcal{M}_{y_0}^*$ be the module

$$\mathcal{M}_{y_0}^* = \{k \in \mathcal{L}^* : < k, (0, y_0) >= 0\}.$$

We have $\mathcal{M}_{y_0}^* \cap \mathcal{N}_{y_0}^* = \emptyset$ and the smallest module containing $\mathcal{N}_{y_0}^*$ is $\overline{\mathcal{N}_{y_0}^*} = \mathcal{M}_{y_0}^* \cup \mathcal{N}_{y_0}^*$. Similarly, if $\mathcal{M}_\sigma^*$ is the module

$$\mathcal{M}_\sigma^* = \{k \in \mathcal{L}^* : < k, \sigma v_\sigma + (0, y_0) > \in \mathbf{Z}\},$$

then the disjoint union $\mathcal{M}_\sigma^* \cup \mathcal{N}_\sigma^* = \overline{\mathcal{N}_\sigma^*}$ is the smallest module containing $\mathcal{N}_\sigma^*$.

The set $\mathcal{N}_\sigma^*$ has the property:

$$(2) \qquad \begin{array}{c} v_1, v_2 \in \mathcal{N}_\sigma^* \\ n_1, n_2 \in \mathbf{Z} \end{array} \Rightarrow n_1 v_1 + n_2 v_2 \in \left\{ \begin{array}{ll} \mathcal{M}_\sigma^* & \text{if } \ n_1 + n_2 \text{ is even} \\ \mathcal{N}_\sigma^* & \text{if } \ n_1 + n_2 \text{ is odd} \end{array} \right. .$$

In figures 1, 2 and 3 we show some examples of lattices defined by these subsets of $\mathcal{L}^*$.

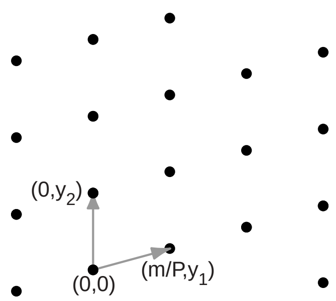

FIGURE 1. Example of the submodule $\mathcal{M}_P^*$ with two noncolinear generators. If, for $m \in \mathbf{Z}$, $y_1 \in \mathbf{R}$, the elements $(m/P, y_1)$ belong to $\mathcal{L}^*$ then they lie in $\mathcal{M}_P^*$.

With this notation we can restate Proposition 3.1 as:

PROPOSITION 3.2. *All functions in* $\Pi_{y_0}(X_\Gamma)$ *have a common period* $P \in \mathbf{R} - \{0\}$ *if and only if* $\mathcal{L}^* = \mathcal{M}_P^* \cup \mathcal{N}_{y_0}^* \cup \mathcal{N}_\sigma^*$.

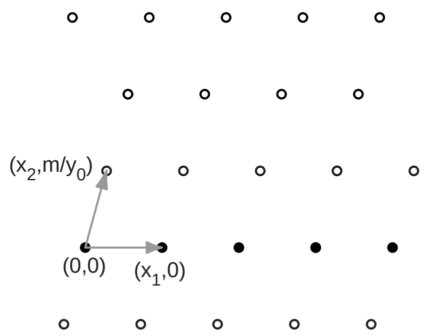

FIGURE 2. In $\mathcal{L}^*$, the elements $(x_1, 0)$ belong to $\mathcal{M}^*_{y_0}$ for all $x_1 \in \mathbf{R}$ (black dots) and the elements $(x_2, m/y_0)$ belong to $\mathcal{N}^*_{y_0}$ for all $m \in \mathbf{Z} - \{0\}$ and $y_1 \in \mathbf{R}$ (open circles). Here we present the module $\overline{\mathcal{N}^*_{y_0}} = \mathcal{M}^*_{y_0} \cup \mathcal{N}^*_{y_0}$, in an example with $\mathcal{M}^*_{y_0} \neq \{(0,0)\}$ and $\mathcal{N}^*_{y_0} \neq \emptyset$.

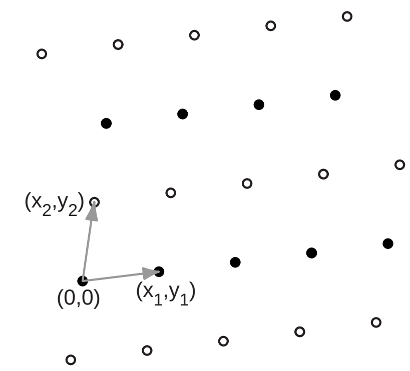

FIGURE 3. An example of the module $\overline{\mathcal{N}^*_\sigma} = \mathcal{M}^*_\sigma \cup \mathcal{N}^*_\sigma$ where $\mathcal{M}^*_\sigma \neq \{(0,0)\}$ (black dots) and $\mathcal{N}^*_\sigma \neq \emptyset$ (open circles).

3.1.2. proof.

PROOF OF PROPOSITION 3.1 — SUFFICIENCY. We study the expression for $D(k_1)$ in the case $k_1 P \notin \mathbf{Z}$ and $k_2 y_0 \notin \mathbf{Z} - \{0\}$, when C holds. Therefore, $\sigma \in \mathbf{J}$ and:

$$\omega_{\sigma k}(-v_\sigma) = \mathrm{e}^{2\pi i\left(<k,(0,y_0)>+\frac{1}{2}\right)} = -\omega_{k_2}(y_0).$$

From the invariance of Fourier coefficients $C(\sigma k') = C(k')\omega_{\sigma k'}(-v_\sigma)$ for all $k' \in \mathcal{L}^*$. Therefore $2D(k_1)$ equals

$$\sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} \left(C(k_1,k_2) \int_0^{y_0} \omega_{k_2}(y)dy + C(k_1,-k_2) \int_0^{y_0} \omega_{-k_2}(y)dy \right)$$

$$= \sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} C(k_1,k_2) \left(\int_0^{y_0} \omega_{k_2}(y)dy - \omega_{k_2}(y_0) \int_0^{y_0} \omega_{-k_2}(y)dy \right)$$

and because of the identity

$$\int_0^{y_0} \omega_{k_2}(y)dy = \omega_{k_2}(y_0) \int_0^{y_0} \omega_{-k_2}(y)dy \tag{3}$$

the expression multiplying each coefficient $C(k_1, k_2)$ is always zero. □

PROOF — NECESSITY. The set $\Pi_{y_0}(X_\Gamma)$ inherits periodicity from $\mathcal{L}$ through the invariant functions $I_k(x, y)$. Suppose $D(k_1) = 0$ for the real and imaginary parts of $I_k(x, y)$, for all $k \in \mathcal{L}^*$. The projection of I_k has the Fourier expansion:

$$\Pi_{y_0}(I_k)(x) = \sum_{\delta \in \mathbf{J}} \omega_{\delta k|_1}(x) D'(\delta, k)$$

where $\delta k|_j$ denotes the $j^{\underline{\text{th}}}$ coordinate of δk and

$$D'(\delta, k) = \omega_{\delta k}(-v_\delta) \int_0^{y_0} \omega_{\delta k|_2}(y)dy.$$

The coefficient $D(k_1)$ in $\Pi_{y_0}(I_k)$ is

$$S(k) = \sum_{\delta \in \mathrm{J}^+(k)} D'(\delta, k) \qquad \text{where} \qquad \mathrm{J}^+(k) = \{\delta \in \mathbf{J} \, : \, \delta k|_1 = k_1\}.$$

The remainder of the proof is divided in three lemmas. We describe the possibilities for $\mathrm{J}^+(k)$ in Lemma 3.1. In Lemma 3.2 we show that $S(k) = 0$ implies $k \in \mathcal{N}^*_{y_0} \cup \mathcal{N}^*_\sigma \cup \mathcal{P}^*$, for a certain subset $\mathcal{P}^*$ of $\mathcal{L}^*$ and, since this holds for all $k \in \mathcal{L}^*$, therefore $\mathcal{L}^* = \mathcal{M}^*_P \cup \mathcal{N}^*_{y_0} \cup \mathcal{N}^*_\sigma \cup \mathcal{P}^*$. In Lemma 3.3 we show that this implies $\mathcal{L}^* = \mathcal{M}^*_P \cup \mathcal{N}^*_{y_0} \cup \mathcal{N}^*_\sigma$, and thus the result will follow. □

3.1.3. the set $\mathbf{J}^+$.

LEMMA 3.1. *Let $\mathbf{J}^+ = \mathbf{J} \cap \{I, \sigma\}$ and $k = (k_1, k_2) \in \mathcal{L}^*$. Then*

1. $\mathrm{J}^+(k) = \{\delta \in \mathbf{J} \, : \, \delta k = k$ *or* $\delta k = \sigma k\}$.
2. $\mathbf{J}^+ \subset \mathrm{J}^+(k)$ *and* $\mathrm{J}^+(0,0) = \mathbf{J}$.
3. *If $k \neq (0,0)$, and if $\delta \in \mathrm{J}^+(k) - \mathbf{J}^+$ then $\delta k = (k_1, -|\delta| k_2)$ where $|.|$ is the determinant.*
4. *If $\mathbf{J}^+ = \{I\}$ and $k \neq (0,0)$ then $\mathrm{J}^+(k)$ contains at most one element $\delta \neq I$.*
5. *If $\mathbf{J}^+ = \{I, \sigma\}$ and $k \neq (0,0)$, then either $\mathrm{J}^+(k) - \mathbf{J}^+$ contains exactly two elements, either δ and $\sigma\delta$, or $\mathrm{J}^+(k) = \mathbf{J}^+$.*

PROOF. Assertions 1 and 2 are immediate from the orthogonality of $\mathbf{J}$. To prove 3, let $\delta \in \mathrm{J}^+(k) - \mathbf{J}^+$ with $k \neq (0,0)$. If $\delta k = k$ then $|\delta| = -1$, since an element of $\mathbf{O}(2)$ with determinant 1, other than the identity, does not fix any point besides the origin. Similarly if $\delta k = \sigma k$ then $|\sigma\delta| = -1$ and $|\delta| = 1$.

For assertions 4 and 5, suppose $\xi \in \mathrm{J}^+(k) - \mathbf{J}^+$, $\xi \neq \delta$, then either $\xi k = \delta k$ or $\xi k = \sigma\delta k$. In the first case either both $\xi k = k$ and $\delta k = k$ or both $\xi k = \sigma k$ and $\delta k = \sigma k$. Therefore, by 3, $|\xi| = |\delta|$. Since $k = \xi^{-1}\delta k$ then by the proof of 3 it follows that $\xi^{-1}\delta = I$, contradicting our hypothesis. Thus $\xi k = \sigma\delta k$ and, by 3, $|\xi| = -|\delta|$. This implies $\xi^{-1}\sigma\delta = I$ and both $\xi = \sigma\delta$ and σ are in $\mathbf{J}$. □

In figure 4 we present two examples where $\mathbf{J}^+ \neq \mathrm{J}^+(k)$.

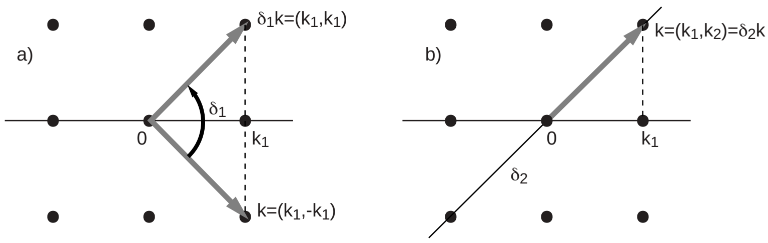

FIGURE 4. a) In a square lattice, if $\mathbf{J}$ contains the rotation δ_1 by $\pi/2$, then $\delta_1 \in \mathrm{J}^+(k)$ for the elements $k = (k_1, -k_1) \in \mathcal{L}^*$. b) For all the lattices, the reflection δ_2, in the line containing k, belongs to $\mathrm{J}^+(k)$. However, neither δ_1 nor δ_2 lies in $\mathbf{J}^+$.

3.1.4. the sets $\mathcal{O}^*$ and $\mathcal{P}^*$. Let $\mathcal{O}^* = \{k \in \mathcal{L}^* : \mathrm{J}^+(k) = \mathbf{J}^+\}$. If $\mathbf{J} \neq \mathbf{J}^+$, then for $\delta \in \mathbf{J} - \mathbf{J}^+$, let

$$\mathcal{P}^*_\delta = \{k \in \mathcal{L}^* \ : \ \delta k = (k_1, -|\delta| k_2)\}.$$

In cases 4 and 5 of Lemma 3.1, δ may be either a rotation such that $\delta k = \sigma k$, or a reflection fixing k. Therefore $\mathcal{P}^*_\delta$ is the intersection of $\mathcal{L}^*$ with a line through the origin, to wit, the line fixed by either $\sigma\delta$ or δ, and $\mathcal{P}^*_\delta$ may contain only the origin. If $\sigma \in \mathbf{J}$ and if $\delta \in \mathbf{J} - \mathbf{J}^+$, then $\sigma\delta \in \mathbf{J} - \mathbf{J}^+$ and $\mathcal{P}^*_\delta = \mathcal{P}^*_{\sigma\delta}$. Let $\mathcal{P}^* = \bigcup_{\delta \in \mathbf{J}-\mathbf{J}^+} \mathcal{P}^*_\delta$, with the convention that $\mathcal{P}^* = \{(0,0)\}$ if $\mathbf{J} - \mathbf{J}^+ = \emptyset$. We have

$$\mathcal{L}^* = \mathcal{O}^* \cup \mathcal{P}^* = \mathcal{O}^* \cup \bigcup_{\delta \in \mathbf{J}-\mathbf{J}^+} \mathcal{P}^*_\delta$$

where any two sets in the right hand side either are disjoint or coincide.

In the $(n+1)$-dimensional case, in chapter 4, there is no analogue for item 3 of Lemma 3.1. Thus $\mathcal{P}^*$ is defined as the complement of $\mathcal{O}^*$ in $\mathcal{L}^*$ and its geometrical description is given by Lemma 4.5 in page 87.

LEMMA 3.2. *If* $S(k) = 0$ *for some* $k \in \mathcal{L}^*$*, then* $k \in \left(\mathcal{N}^*_{y_0} \cup \mathcal{N}^*_\sigma \cup \mathcal{P}^*\right)$.

PROOF. We will show that if $S(k) = \sum_{\delta \in \mathrm{J}^+(k)} D'(\delta, k) = 0$, with $k \in \mathcal{O}^*$, then $k \in \left(\mathcal{N}^*_{y_0} \cup \mathcal{N}^*_\sigma\right)$.

If $\mathrm{J}^+(k) = \mathbf{J}^+ = \{I\}$ then $S(k) = D'(I,k) = \int_0^{y_0} \omega_{k_2}(y)dy = 0$ and therefore $k_2 y_0 \in \mathbf{Z} - \{0\}$ or, equivalently, $k \in \mathcal{N}^*_{y_0}$.

If $\mathrm{J}^+(k) = \mathbf{J}^+ = \{I, \sigma\}$ then $S(k) = D'(I,k) + D'(\sigma,k)$ and thus $S(k) = \int_0^{y_0} \omega_{k_2}(y)dy + \omega_{\sigma k}(-v_\sigma)\int_0^{y_0} \omega_{-k_2}(y)dy = 0$. If $\int_0^{y_0} \omega_{k_2}(y)dy = \int_0^{y_0} \omega_{-k_2}(y)dy = 0$ then $k_2 y_0 \in \mathbf{Z} - \{0\}$. Otherwise, using (3) we get $\omega_{\sigma k}(-v_\sigma) = -\omega_{k_2}(y_0)$, which is equivalent to $< k, \sigma v_\sigma + (0, y_0) > + \frac{1}{2} \in \mathbf{Z}$, *i.e.*, $k \in \mathcal{N}^*_\sigma$. □

LEMMA 3.3. *If* $\mathcal{L}^* = \mathcal{M}^* \cup \mathcal{N}^*_{y_0} \cup \mathcal{N}^*_\sigma \cup \mathcal{P}^*$*, where* $\mathcal{M}^* \subset \mathcal{L}^*$ *is a submodule, then* $\mathcal{L}^* = \mathcal{M}^* \cup \mathcal{N}^*_{y_0} \cup \mathcal{N}^*_\sigma$.

PROOF. For $k \neq (0,0)$, $k \in \mathcal{L}^*$, let $\mathcal{L}^* = \{g, h\}_{\mathbf{Z}}$ for some $g = \frac{1}{m}k$, $m \in \mathbf{N}$, and let $\mathcal{Q}^*_k = \{k + nh : n \in \mathbf{Z}\}$. We will show that $\mathcal{Q}^*_k \subset \left(\mathcal{M}^* \cup \overline{\mathcal{N}^*_{y_0}} \cup \mathcal{N}^*_\sigma\right)$.

First we show that if $\mathcal{R}^*$ is any one of the sets $\mathcal{M}^*$, $\overline{\mathcal{N}^*_{y_0}}$ or $\mathcal{N}^*_\sigma$, then $\mathcal{Q}^*_k \cap \mathcal{R}^*$ is either empty or it is a single point or it contains infinitely many equally spaced points.

Suppose $\mathcal{R}^*$ is a module. If there are two distinct points $k + n_1 h$ and $k + n_2 h$ in $\mathcal{Q}^*_k \cap \mathcal{R}^*$, then $(n_2 - n_1)h \in \mathcal{R}^*$. Thus the claim follows for $\mathcal{R}^* = \mathcal{M}^*$ and for $\mathcal{R}^* = \overline{\mathcal{N}^*_{y_0}}$. If $\mathcal{Q}^*_k \cap \mathcal{R}^*$ is an infinite set, then it has a characteristic period $\tau \in (\mathcal{Q}^*_k \cap \mathcal{R}^*)$ given by the smallest difference between two elements of $\mathcal{Q}^*_k \cap \mathcal{R}^*$. We denote it by τ_1 for $\mathcal{Q}^*_k \cap \mathcal{M}^*$ and τ_2 for $\mathcal{Q}^*_k \cap \overline{\mathcal{N}^*_{y_0}}$.

By (2), the smallest difference between any two elements of $\mathcal{Q}^*_k \cap \mathcal{N}^*_\sigma$ defines a period $\tau_3 \in \mathcal{M}^*_\sigma$. Thus, whenever $\mathcal{Q}^*_k \cap \mathcal{N}^*_\sigma$ has more than one element, if $k + n_1 h \in \mathcal{N}^*_\sigma$ then $\{k + n_1 h + n\tau_3 : n \in \mathbf{Z}\} = \mathcal{Q}^*_k \cap \mathcal{N}^*_\sigma$, proving the claim for $\mathcal{R}^* = \mathcal{N}^*_\sigma$.

Finally, note that $\mathcal{Q}^*_k$ is contained in a line in $\mathbf{R}^2$ that does not go through the origin. On the other hand, each $\mathcal{P}^*_\delta$ is contained in a line through the origin, and therefore $\mathcal{P}^*$ is contained in a finite union of one-dimensional vector subspaces of $\mathbf{R}^2$. It follows that $\mathcal{Q}^*_k \cap \mathcal{P}^*$ is finite and at least one of the periods τ_1, τ_2 or τ_3 must exist. The least common multiple of the existing periods is a period, τ, of $\mathcal{Q}^*_k \cap (\mathcal{M}^* \cup \overline{\mathcal{N}^*_{y_0}} \cup \mathcal{N}^*_\sigma)$, see figure 5. Therefore $\mathcal{Q}^*_k - (\mathcal{M}^* \cup \overline{\mathcal{N}^*_{y_0}} \cup \mathcal{N}^*_\sigma)$ is either the empty set or an infinite set with period τ and, since it is a subset of the finite set $\mathcal{Q}^*_k \cap \mathcal{P}^*$, then $\mathcal{Q}^*_k - (\mathcal{M}^* \cup \overline{\mathcal{N}^*_{y_0}} \cup \mathcal{N}^*_\sigma) = \emptyset$, implying $\mathcal{Q}^*_k \subset (\mathcal{M}^* \cup \overline{\mathcal{N}^*_{y_0}} \cup \mathcal{N}^*_\sigma)$.

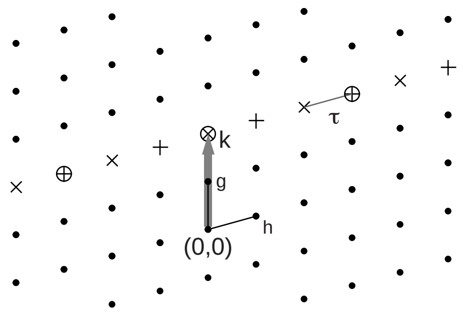

FIGURE 5. The lattice $\mathcal{L}^*$, where $\mathcal{Q}_k^*$ is the union of elements that are not black dots. The elements indicated by $+$ and $\times$ belong, respectively, to $\mathcal{M}^*$ and $\overline{\mathcal{N}_{y_0}^*}$, the open circle indicates elements that also lie in $\mathcal{N}_\sigma^*$. Thus, $\tau_1 = \tau_2 = 2\tau$ and $\tau_3 = 3\tau$.

We have established that $\mathcal{L}^* = \mathcal{M}^* \cup \overline{\mathcal{N}_{y_0}^*} \cup \mathcal{N}_\sigma^*$. Moreover, $\mathcal{M}_{y_0}^* \cap \mathcal{P}^* = \{(0,0)\}$, because if $k \in \mathcal{M}_{y_0}^*$ then $k = (k_1, 0)$ for some $k_1 \in \mathbf{R}$. If $k_1 \neq 0$ and $k \in \mathcal{P}_\delta^*$, for some $\delta \in \mathbf{J} - \mathbf{J}^+$, then $\delta k = (k_1, 0)$ and, by orthogonality, $\delta \in \mathbf{J}^+$, a contradiction. It follows that $\mathcal{M}_{y_0}^* \subset (\mathcal{M}^* \cup \mathcal{N}_\sigma^*)$, completing the proof. □

Proving the analogue of $\mathcal{Q}_k^* \cap \mathcal{P}^*$ being finite for the general case, in Lemma 4.6 of chapter 4, will require new strategies, due to the dimensions involved.

3.1.5. periodicity of $\Pi_{y_0}(X_\Gamma)$ related to $\mathcal{L}$.

THEOREM 3.1. *All functions in $\Pi_{y_0}(X_\Gamma)$ have a common period $P \neq 0$ if and only if one of the following conditions holds:*

I. $(P, 0) \in \mathcal{L}$,
II. $((P, y_0), \sigma) \in \Gamma$,
III. $(0, y_0)$ *and* $(P, y_1) \in \mathcal{L}$ *for some* $y_1 \in \mathbf{R}$,
IV. $(0, y_0) \in \mathcal{L}$ *and* $((P, y_1), \sigma) \in \Gamma$ *for some* $y_1 \in \mathbf{R}$.

We present in figures 6, 7, 8 and 9 some examples illustrating the conditions in Theorem 3.1, in the form of patterns. These may be interpreted as the level sets of functions $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$, taking only the values 0 and 1, with $f(x, y) = 0$ on the white regions. After projection we obtain a function whose value for each $x \in \mathbf{R}$ is the width of the black region above it. Figures 8 and 9 appear after the definition of glide reflection in page 50.

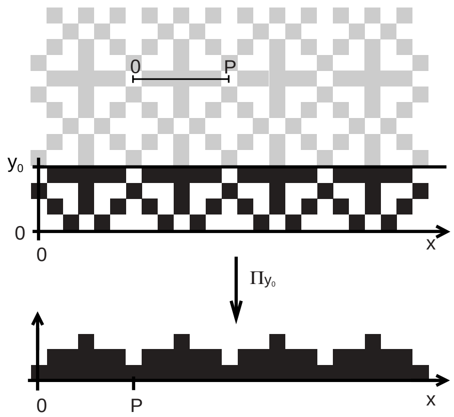

FIGURE 6. By condition I of Theorem 3.1, if a period $(P, 0) \in \mathcal{L}$ then, for all y_0, we have the period P after projection.

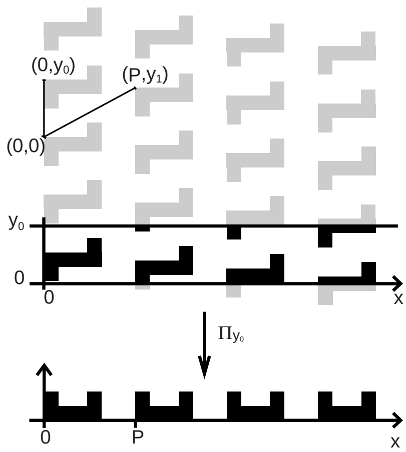

FIGURE 7. If $(0, y_0) \in \mathcal{L}$ then (P, y_1) acts as a translation by P after a projection of width y_0 (condition III of Theorem 3.1).

3.1.6. structure of the proof. A direct calculation of the integrals shows that the conditions are sufficient, see section 4.2 (ahead) for details in the general case. The proof that the conditions are necessary ocupies the rest of this section and is divided in lemmas as follows: first we use Proposition 3.2 to get $\mathcal{L}^* = \mathcal{M}_P^* \cup \mathcal{N}_{y_0}^* \cup \mathcal{N}_\sigma^*$.

The simpler case $\sigma \notin \mathbf{J}$, where $\mathcal{N}_\sigma^* = \emptyset$ is treated in Lemma 3.4 below, where we show that either I or III holds.

Suppose now that $\sigma \in \mathbf{J}$. This imposes restrictions on the generators of $\mathcal{L}$, that we describe in Lemma 3.5. In particular, we obtain the classical result that the oblique lattice cannot have a reflection in its holohedry.

There are two possibilities for $\mathcal{L}$. The first is that $\mathcal{L}$ is either the square or the rectangular lattice in *standard position*, with generators on the coordinate axes. The second possibility is that $\mathcal{L}$ is either the square or the centred rectangular or the hexagonal lattice, in what we call *diamond position*. See page 52, after Lemma 3.5, for a formal description of these lattices, illustrated in figure 10.

If $\delta \in \mathbf{J}$ with $\det \delta = -1$ and $\delta v_\delta = -v_\delta$ then (v_δ, δ) is a *reflection*, otherwise (v_δ, δ) is a *glide* (short for *glide reflection*).

In Lemma 3.6 we relate the coordinates of the translation component of (v_σ, σ) to the generators of $\mathcal{L}$. For the square or rectangular lattices in standard position, (v_σ, σ) is either a reflection or a glide reflection. For the lattices in diamond position, (v_σ, σ) must be a reflection, see the examples in figures 8 and 9.

In section 3.4 we show that in the second case, the presence of a reflection $\sigma \in \mathbf{J}$ on a horizontal line implies that Γ contains glide reflections along the same direction.

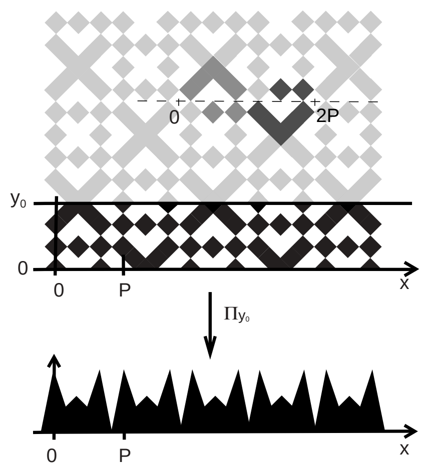

FIGURE 8. The glide reflection in the dashed line imposes a period P for the projected pattern, by condition II of Theorem 3.1. Notice that, by the translation $(0, -2y_0)$, there is a glide reflection on the line along the middle of the projection band. Thus the glide reflection is a symmetry of the band.

We use the information that $\mathcal{L}^* = \mathcal{M}_P^* \cup \mathcal{N}_{y_0}^* \cup \mathcal{N}_\sigma^*$ from Proposition 3.2 to relate in Lemma 3.7 the period P to the coordinates of one of the generators of $\mathcal{L}$.

If $\sigma \in \mathbf{J}$ then it will follow from Lemma 3.5 that $\mathcal{M}_{y_0}^* \neq \{(0,0)\}$ and so $\mathcal{M}_{y_0}^*$ contains some generator l_1^* of $\mathcal{L}^*$. In Lemma 3.8 we use this to obtain restrictions on the other generator l_2^* of $\mathcal{L}^*$ and we then

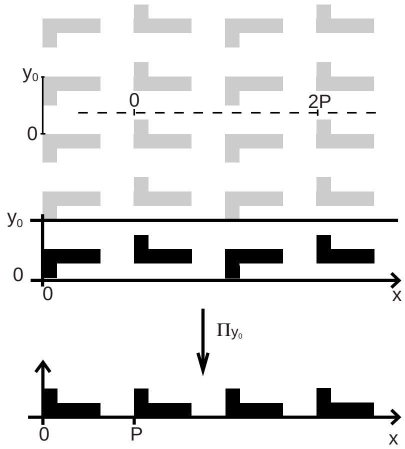

FIGURE 9. If $(0, y_0) \in \mathcal{L}$ then, by condition IV of Theorem 3.1 the glide reflection in the dashed line acts as a translation by P after the projection. The glide reflection is not a symmetry of the band.

show that conditions I, II,III, IV hold — for the square and rectangular lattices in standard position in Lemma 3.9, and for the lattices in diamond position in Lemma 3.10 — completing the proof.

3.1.7. the simple case.

LEMMA 3.4. *If $\mathcal{L}^* = \mathcal{M}_P^* \cup \mathcal{N}_{y_0}^*$, then one of the following conditions holds:*
I. $(P, 0) \in \mathcal{L}$,
III. $(0, y_0) \in \mathcal{L}$ *and* $(P, y_1) \in \mathcal{L}$ *for some* $y_1 \in \mathbf{R}$.

PROOF. Since $\mathcal{L}^*$ is the union of the two modules $\mathcal{L}^* = \mathcal{M}_P^* \cup \overline{\mathcal{N}_{y_0}^*}$, then either $\mathcal{L}^* = \mathcal{M}_P^*$, implying condition I, or $\mathcal{L}^* = \overline{\mathcal{N}_{y_0}^*}$.

In the second case, we have $(0, y_0) \in \mathcal{L}$ and $\mathcal{M}_{y_0}^* \subset \mathcal{M}_P^*$. Let n be the largest integer such that $l_1 = (0, y_0/n) \in \mathcal{L}$, then $\mathcal{L}$ is generated by l_1 and $l_2 = (\rho n/y_0, y_2)$ for some $y_2 \in \mathbf{R}$. Since $l_2^* = (y_0/\rho n, 0) \in \mathcal{M}_{y_0}^* \subset \mathcal{M}_P^*$ it follows that $\rho n/y_0 = P/m$ for some $m \in \mathbf{Z}$. Therefore $l_2 = (P/m, y_2) \in \mathcal{L}$, implying condition III. □

3.1.8. the structure of $\mathcal{L}$ and $\mathcal{L}^*$. In this section we characterize those lattices whose holohedry contains σ, and obtain a special form for theis generators. The constraints on $\mathcal{L}$ placed by σ play a crucial role in the remainder of the proof. This is in strong contrast with the $(n+1)$-dimensional case, where we do not have so much information about $\mathcal{L}$, and where, from this point on, we are concerned with lattices and translations in $\mathbf{R}^{n+1}$. Thus, in chapter 4 we will follow a different approach with calculations that are less intuitive.

LEMMA 3.5. *If $\sigma \in \mathbf{J}$ then there are generators for $\mathcal{L}$ of one of the forms below, with $\alpha \neq 0 \neq \beta$,*

$$\mathcal{L} = \{(\alpha, 0), (0, \beta)\}_{\mathbf{Z}} \quad or \quad \mathcal{L} = \{(\alpha/2, \beta/2), (0, \beta)\}_{\mathbf{Z}}.$$

PROOF. Let (a, b) be any element of $\mathcal{L}$. As $\sigma \in \mathbf{J}$ then $\sigma\mathcal{L} = \{\sigma l : l \in \mathcal{L}\} = \mathcal{L}$ and so $(a, -b) \in \mathcal{L}$. Thus, $(2a, 0)$ and $(0, 2b)$ belong to $\mathcal{L}$.

Let $(\alpha, 0)$ and $(0, \beta)$ be the smallest elements of $\mathcal{L}$ in the coordinate axes and let $\mathcal{M} \subset \mathcal{L}$ be the submodule generated by them. If $\mathcal{L} \neq \mathcal{M}$, then it follows that $(\alpha/2, \beta/2)$ lies in $\mathcal{L}$, since $(a, b) \in \mathcal{L} - \mathcal{M}$ implies $(2a, 0)$ and $(0, 2b)$ lie in $\mathcal{L}$. □

From now on, we refer to $\mathcal{L} = \{(\alpha, 0), (0, \beta)\}_{\mathbf{Z}}$ as the lattice in *standard position* and to $\mathcal{L} = \{(\alpha/2, \beta/2), (0, \beta)\}_{\mathbf{Z}}$ as the lattice in *diamond position*. See the examples in figure 10.

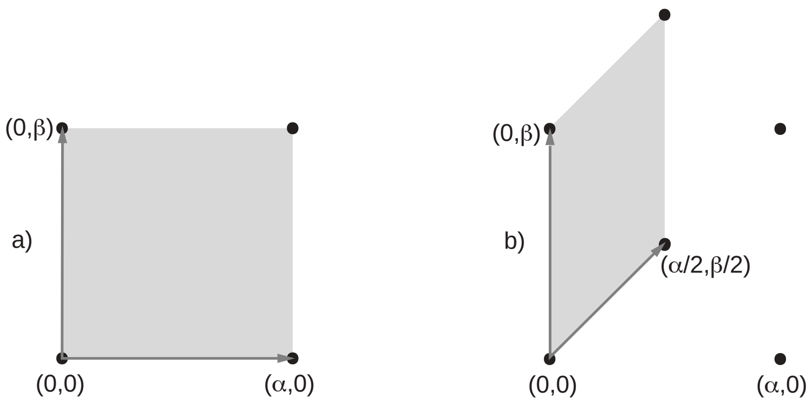

FIGURE 10. In a) we show the generators for a lattice in standard position and, in b), for a lattice in diamond position. Both are square lattices and, thus, $\alpha = \beta$.

LEMMA 3.6. *If $\sigma \in \mathbf{J}$ then either $v_\sigma = (0, v_2)$ or $v_\sigma = (\alpha/2, v_2)$ for some v_2, $0 \leq v_2 < \beta$ where α and β were defined in Lemma 3.5. For the lattice in diamond position, only the first possibility occurs.*

PROOF. Recall that $v_\sigma = (v_1, v_2)$ either belongs to the interior of the fundamental cell of $\mathcal{L}$ or it lies in one of the segments tl_i, with $t \in [0, 1[$ and $i = 1$ or 2. For the generators in Lemma 3.5, this implies $v_2 \in [0, \beta[$ and either $v_1 \in [0, \alpha[$, or $v_1 \in [0, \alpha/2[$, respectively. Moreover, $(v_\sigma, \sigma)^2 = (v_\sigma + \sigma v_\sigma, I) = ((2v_1, 0), I) \in \Gamma$, and thus $(2v_1, 0) \in \mathcal{L}$ or, equivalently, $2v_1 = n\alpha$ for some $n \in \mathbf{Z}$. For the lattice in diamond position, this implies $v_1 = 0$. For the lattice in standard position there is also the possibility $v_1 = \alpha/2$, see figure 11. □

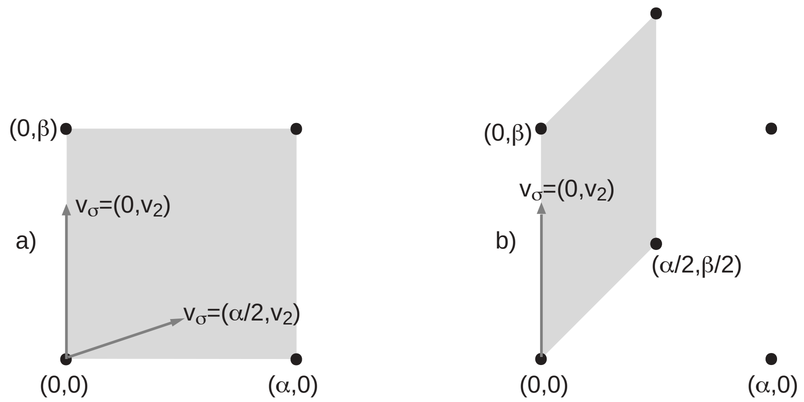

FIGURE 11. A lattice in standard position allows $v_\sigma = (0, v_2)$ and $v_\sigma = (\alpha/2, v_2)$, in a). In b), $v_\sigma = (0, v_2)$ is the only possibility for a lattice in diamond position.

In chapter 4 we adopt a better definition of $(v_\delta, \delta) \in \Gamma$, which avoids an analogue of Lemma 3.6 for the general $(n+1)$-dimensional case.

The relation $\mathcal{L}^* = \mathcal{M}_P^* \cup \mathcal{N}_{y_0}^* \cup \mathcal{N}_\sigma^*$ imposes further restrictions on the generators of $\mathcal{L}$.

LEMMA 3.7. *If $\mathcal{L}^* = \mathcal{M}_P^* \cup \mathcal{N}_{y_0}^* \cup \mathcal{N}_\sigma^*$ and if $\sigma \in \mathbf{J}$ then α, defined in Lemma 3.5, satisfies $\alpha = 2P/n$ for some integer n.*

PROOF. In the two cases of Lemma 3.5 we have $k = (2/\alpha, 0) \in \mathcal{L}^*$. Since $k \in \mathcal{M}_{y_0}^*$, then either $k \in \mathcal{M}_P^*$ or $k \in \mathcal{N}_\sigma^*$. By Lemma 3.6, we have $< k, \sigma v_\sigma + (0, y_0) >$ is either 0 or 1, and therefore $k \notin \mathcal{N}_\sigma^*$ and the result follows. $\square$

3.1.9. from $\mathcal{L}^*$ to $\mathcal{L}$.

In the next lemma we obtain more information about the generators of $\mathcal{L}^*$ in a form that will also be suitable for use in section 3.2. Lemma 3.8 provides the structure for the remainder of the proof.

LEMMA 3.8. *Let $\mathcal{L}^* = \mathcal{M}^* \cup \mathcal{N}_{y_0}^* \cup \mathcal{N}_\sigma^*$, where $\mathcal{M}^*$ is a module, and suppose $\mathcal{L}^* =< l_1^*, l_2^* >_{\mathbf{Z}}$ and $l_1^* \in \mathcal{M}_{y_0}^*$. Then either $l_1^* \in \mathcal{M}^*$ or $l_1^* \in \mathcal{N}_\sigma^*$. These two cases can be subdivided as follows:*

1. $l_1^* \in \mathcal{M}^*$ *and either*
   (i) $l_2^* \in \mathcal{M}^*$, *or*
   (ii) $l_2^*, l_1^* + l_2^* \in \mathcal{N}_\sigma^*$, *or*
   (iii) $(0, y_0) \in \mathcal{L}$
2. $l_1^* \in \mathcal{N}_\sigma^*$ *and either*
   (i) $l_2^* \in \mathcal{N}_\sigma^*$ *and* $l_1^* + l_2^* \in \mathcal{M}^*$, *or*
   (ii) $l_2^* \in \mathcal{M}^*$ *and* $l_1^* + l_2^* \in \mathcal{N}_\sigma^*$, *or*
   (iii) $(0, y_0) \in \mathcal{L}$

Moreover, $\mathcal{L}^* = \mathcal{M}^*$ *in case 1i),* $\mathcal{L}^* = \overline{\mathcal{N}^*_{y_0}}$ *in cases 1iii) and 2iii) and* $\mathcal{L}^* = \overline{\mathcal{N}^*_\sigma}$ *in the remaining cases.*

Proof. Any two of the three elements l_1^*, l_2^* and $l_1^* + l_2^*$ generate $\mathcal{L}^*$ and, by hypothesis, $l_1^* \in \mathcal{M}^* \cup \mathcal{N}^*_\sigma$.

If either l_2^* or $l_1^* + l_2^*$ belongs to $\mathcal{N}^*_{y_0}$ then $\mathcal{L}^* = \overline{\mathcal{N}^*_{y_0}}$, thus $(0, y_0) \in \mathcal{L}$ and we have cases 1iii) and 2iii).

Now suppose neither l_2^* nor $l_1^* + l_2^*$ belong to $\mathcal{N}^*_{y_0}$. Then $l_1^*, l_2^*, l_1^* + l_2^* \in (\mathcal{M}^* \cup \mathcal{N}^*_\sigma)$ and so at least two of them belong to the same set, either $\mathcal{M}^*$ or $\mathcal{N}^*_\sigma$. Therefore, $\mathcal{L}^* = \mathcal{M}^* \cup \overline{\mathcal{N}^*_\sigma}$ and it follows that either $\mathcal{L}^* = \mathcal{M}^*$ or $\mathcal{L}^* = \overline{\mathcal{N}^*_\sigma}$. Case 1i) is equivalent to $\mathcal{L}^* = \mathcal{M}^*$. If $\mathcal{L}^* = \overline{\mathcal{N}^*_\sigma}$, we recall that by the properties (2) of $\mathcal{N}^*_\sigma$, only two of the elements l_1^*, l_2^* and $l_1^* + l_2^*$ can belong to $\mathcal{N}^*_\sigma$ and the three possible cases are 1ii), 2i) and 2ii). □

Lemma 3.9. *Suppose* $\sigma \in \mathbf{J}$ *and* $\mathcal{L}^* = \mathcal{M}^*_P \cup \mathcal{N}^*_{y_0} \cup \mathcal{N}^*_\sigma$. *Then, for the rectangular lattices* $\mathcal{L} = \{(\alpha, 0), (0, \beta)\}_{\mathbf{Z}}$ *one of the conditions I, II or IV of Theorem 3.1 holds.*

Proof. By Lemma 3.7 we have $\alpha = 2P/n$ and $\mathcal{L}^* = \{l_1^*, l_2^*\}_{\mathbf{Z}}$ where $l_1^* = (n/(2P), 0)$ and $l_2^* = (0, 1/\beta)$. Since $l_1^* \in \mathcal{M}^*_{y_0}$, we can use Lemma 3.8 with $\mathcal{M}^* = \mathcal{M}^*_P$.

We have $l_1^* \in \mathcal{M}^*_P$ if and only if n is even and $(2P/n, 0) \in \mathcal{L}$. Therefore $(P, 0) \in \mathcal{L}$ and in case 1 of Lemma 3.8 we obtain condition I.

Suppose now that $l_1^* \notin \mathcal{M}^*_P$, *i.e.*, n is odd, and thus one of the two cases 2ii) or 2iii) of Lemma 3.8 holds. Let $v_\sigma = (v_1, v_2)$. By Lemma 3.6, either $v_1 = 0$ or $v_1 = P/n$. Since $l_1^* \in \mathcal{N}^*_\sigma$ then $v_1 n/(2P) + 1/2 \in \mathbf{Z}$ and $v_1 = P/n$. Thus $((P/n, v_2), \sigma) \in \Gamma$ and $((P/n, v_2), \sigma)^n = ((P, v_2), \sigma) \in \Gamma$. Therefore case 2iii) of Lemma 3.8 implies condition IV.

In case 2ii) we have $l_1^* + l_2^* \in \mathcal{N}^*_\sigma \Leftrightarrow \frac{1}{\beta}(y_0 - v_2) \in \mathbf{Z}$. Then $(0, y_0 - v_2) \in \mathcal{L}$ and $((0, y_0 - v_2), I) \cdot ((P, v_2), \sigma) = ((P, y_0), \sigma) \in \Gamma$, and condition II holds. □

Lemma 3.10. *Suppose* $\sigma \in \mathbf{J}$ *and* $\mathcal{L}^* = \mathcal{M}^*_P \cup \mathcal{N}^*_{y_0} \cup \mathcal{N}^*_\sigma$. *Then, for the lattices* $\mathcal{L} = \{(\alpha/2, \beta/2), (0, \beta)\}_{\mathbf{Z}}$ *one of the conditions I, II or IV of Theorem 3.1 holds.*

Proof. The dual lattice has the form $\mathcal{L}^* = \{l_1^*, l_2^*\}_{\mathbf{Z}}$ with $l_1^* = (n/P, 0) \in \mathcal{M}^*_{y_0}$ and $l_2^* = (-n/(2P), 1/\beta)$. For $\mathcal{M}^* = \mathcal{M}^*_P$ Lemma 3.8 holds.

First notice that $l_1^* \in \mathcal{M}^*_P$ and $l_1^* \notin \mathcal{N}^*_\sigma$, as $v_\sigma = (0, v_2)$ by Lemma 3.6, and we are in case 1 of Lemma 3.8. Case 1iii) implies condition IV because $(P, n\beta/2) \in \mathcal{L}$.

Case 1i) holds if and only if n is even, thus l_2^* lies in $\mathcal{M}_P^*$ implying $(P,0) \in \mathcal{L}$ and condition I holds.

If n is odd (case 1ii)) then $\frac{1}{\beta}(y_0 - v_2) + \frac{1}{2} \in \mathbf{Z} \Leftrightarrow y_0 - v_2 = m\beta + \frac{\beta}{2}$ for some $m \in \mathbf{Z}$. Thus $(P/n, y_0 - v_2) \in \mathcal{L}$ and condition II follows because

$$((P/n, y_0 - v_2), I) \cdot ((0, v_2), \sigma) = ((P/n, y_0), \sigma) \in \Gamma$$

and $((P/n, y_0), \sigma)^n = ((P, y_0), \sigma) \in \Gamma$. □

Thus we have proved Theorem 3.1.

3.2. Projection and Symmetry - Parity on $\Pi_{y_0}(X_\Gamma)$

The description of the full symmetry of a projected pattern is completed with its nontrivial orthogonal part. For one spatial dimension this takes the form of parity.

3.2.1. parity of $\Pi_{y_0}(X_\Gamma)$ related to $\mathcal{L}^*$.

PROPOSITION 3.3. *All functions in $\Pi_{y_0}(X_\Gamma)$ are invariant for the reflection in the point x_0 if and only if one of the conditions holds:*

A. $-\sigma \in \mathbf{J}$ *and for each* $k \in \mathcal{L}^*$ *either* $< k, (0, y_0) > \in \mathbf{Z} - \{0\}$ *or* $< k, -\sigma v_{-\sigma} + (2x_0, 0) > \in \mathbf{Z}$,

B. $-I \in \mathbf{J}$ *and for each* $k \in \mathcal{L}^*$ *either* $< k, (0, y_0) > \in \mathbf{Z} - \{0\}$ *or* $< k, -v_{-I} + (2x_0, y_0) > \in \mathbf{Z}$,

C. $\pm\sigma \in \mathbf{J}$ *and for each* $k \in \mathcal{L}^*$ *either* $< k, (0, y_0) > \in \mathbf{Z} - \{0\}$ *or* $< k, -\sigma v_{-\sigma} + (2x_0, 0) > \in \mathbf{Z}$ *or* $< k, \sigma v_\sigma + (0, y_0) > + \frac{1}{2} \in \mathbf{Z}$.

For the proof of Proposition 3.3 note that a function $g : \mathbf{R} \longrightarrow \mathbf{R}$ is invariant under reflection in a point x_0 if and only if $g(x) = g(2x_0 - x)$ for all $x \in \mathbf{R}$. Therefore, since $-\mathcal{L}_1^* = \mathcal{L}_1^*$, the projection $\Pi_{y_0}(f)(x) = \sum_{k_1 \in \mathcal{L}_1^*} D(k_1)\omega_{k_1}(x)$ is invariant under reflection in a point x_0 if and only if, for each $k_1 \in \mathcal{L}_1^*$,

$$D(k_1) - \omega_{k_1}(-2x_0)D(-k_1) = 0. \tag{4}$$

This expression unfolds into two conditions, in the $(n+1)$-dimensional case, since the analogue of $-\mathcal{L}_1^* = \mathcal{L}_1^*$ may not hold.

Consider the submodules of $\mathcal{L}^*$:

$$\mathcal{M}_{-\sigma}^* = \{k \in \mathcal{L}^* : < k, -\sigma v_{-\sigma} + (2x_0, 0) > \in \mathbf{Z}\}$$
$$\mathcal{M}_{-I}^* = \{k \in \mathcal{L}^* : < k, -v_{-I} + (2x_0, y_0) > \in \mathbf{Z}\},$$

with the conventions $\mathcal{M}_{-\sigma}^* = \{(0,0)\}$ if $-\sigma \notin \mathbf{J}$ and $\mathcal{M}_{-I}^* = \{(0,0)\}$ if $-I \notin \mathbf{J}$. Then Proposition 3.3 may be rewritten as:

Proposition 3.4. *All functions in* $\Pi_{y_0}(X_\Gamma)$ *are invariant for the reflection in the point* x_0 *if and only if one of the conditions holds:*

A. $-\sigma \in \mathbf{J}$ *and* $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_{-\sigma}$,
B. $-I \in \mathbf{J}$ *and* $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_{-I}$,
C. $\pm\sigma \in \mathbf{J}$ *and* $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_{-\sigma} \cup \mathcal{N}^*_{\sigma}$.

Proposition 4.2, the analogue of the above proposition for the general $(n+1)$-dimensional case, has a different third condition. This condition will imply a reformulation of the subsequent proofs. In the two-dimensional case presented along this chapter, this condition is simplified because $\sigma v_{-\sigma} - v_{-\sigma} \in \mathcal{L}$. Thus, the set $\mathcal{N}^*$, as defined in the next chapter, is here the empty set.

3.2.2. proof.

Proof — sufficiency. The difference $D(k_1) - \omega_{k_1}(-2x_0)D(-k_1)$ may be written as

$$\sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} C(k_1,k_2)G(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy. \tag{5}$$

In each case we compute the expression of $G(k_1,k_2)$ in a form that makes it easier to use the conditions on $\mathcal{L}^*$. If $-\sigma \in \mathbf{J}$ then, using $-\sigma\mathcal{L}^* = \mathcal{L}^*$ and writing $k = (k_1,k_2)$, we get

$$G(k_1,k_2) = 1 - \omega_{k_1}(-2x_0)\omega_k(\sigma v_{-\sigma}) \tag{6}$$

and if $-I \in \mathbf{J}$ holds then

$$G(k_1,k_2) = 1 - \omega_{k_1}(-2x_0)\omega_k(v_{-I})\omega_{k_2}(-y_0). \tag{7}$$

If $\pm\sigma \in \mathbf{J}$ then $-I \in \mathbf{J}$ and we obtain

$$G(k_1,k_2) = \frac{1}{2}\left(1 - \omega_{k_1}(-2x_0)\omega_k(\sigma v_{-\sigma})\right)\left(1 + \omega_k(-\sigma v_\sigma)\omega_{k_2}(-y_0)\right) \tag{8}$$

because $\omega_k(v_{-I}) = \omega_k(v_{-\sigma})\omega_k(-\sigma v_\sigma)$ and $\omega_k(\sigma v_{-\sigma}) = \omega_k(v_{-\sigma})$.

With the expressions above it is easy to check that the conditions on $\mathcal{L}^*$ ensure that for each $k \in \mathcal{L}^*$ either $\int_0^{y_0}\omega_{k_2}(y)dy = 0$ or $G(k_1,k_2) = 0$ and, therefore, that (4) is valid for all $k \in \mathcal{L}^*$. □

The statement $\omega_k(\sigma v_{-\sigma}) = \omega_k(v_{-\sigma})$, used in the proof above, is equivalent to $\sigma v_{-\sigma} - v_{-\sigma} \in \mathcal{L}$. For the $(n+1)$-dimensional case this may not hold, yelding an extra case in Proposition 4.1.

PROOF — NECESSITY. Suppose (4) holds for the real and imaginary parts of I_k, for all $k = (k_1, k_2) \in \mathcal{L}^*$, *i.e.*

$$S'(k) = \sum_{\delta \in \mathrm{J}^+(k)} D'(\delta, k) - \omega_{k_1}(-2x_0) \sum_{\delta \in \mathrm{J}^-(k)} D'(\delta, k) = 0,$$

where

$$\mathrm{J}^+(k) = \{\delta \in \mathbf{J} \,:\, \delta k|_1 = k_1\} \quad \text{and} \quad \mathrm{J}^-(k) = \{\delta \in \mathbf{J} \,:\, \delta k|_1 = -k_1\}\,.$$

Note that $S'(k)$ has the form (5) above.

The proof is divided in lemmas. In Lemma 3.2 we have characterized $\mathrm{J}^+(k)$ and a similar result (Lemma 3.11 below) holds for $\mathrm{J}^-(k)$. Lemma 3.12 below describes the consequences of $S'(k) = 0$ for the subsets of $\mathcal{L}^*$. The result will follow by Lemma 3.13. □

3.2.3. the sets $\mathbf{J}^+$ and $\mathbf{J}^-$.

LEMMA 3.11. *Let* $\mathbf{J}^- = \mathbf{J} \cap \{-I, -\sigma\}$. *The set* $\mathrm{J}^-(k)$ *satisfies:*

1. $\mathrm{J}^-(k) = \{\delta \in \mathbf{J} \,:\, \delta k = -k \vee \delta k = -\sigma k\}$.
2. *For any* $k \in \mathcal{L}^*$, $\mathbf{J}^- \subset \mathrm{J}^-(k)$ *and* $\mathrm{J}^-(0,0) = \mathbf{J}$.
3. *For any* $k \in \mathcal{L}^*$, $k = (k_1, k_2) \neq (0,0)$, *if* $\delta \in \mathrm{J}^-(k) - \mathbf{J}^-$ *then* $\delta k = -(k_1, -|\delta| k_2)$ *where* $|.|$ *is the determinant.*
4. *If* $\mathbf{J}^+ = \{I\}$ *and* $k \neq (0,0)$ *then, for any* $k \in \mathcal{L}^*$, $\mathrm{J}^-(k) - \mathbf{J}^-$ *contains at most one element.*
5. *If* $\mathbf{J}^+ = \{I, \sigma\}$ *then for any* $k \in \mathcal{L}^*$, $k \neq (0,0)$, *either* $\mathrm{J}^-(k) = \mathbf{J}^-$ *or* $\mathrm{J}^-(k) - \mathbf{J}^-$ *contains exactly two elements,* δ *and* $\sigma\delta$.

The proof of Lemma 3.11 is similar to that of Lemma 3.1 and is omitted. See figure 4, in subsection 3.1.3, and figure 12, below, for examples where $\mathrm{J}^+(k) \neq \mathbf{J}^+$ and $\mathrm{J}^-(k) \neq \mathbf{J}^-$.

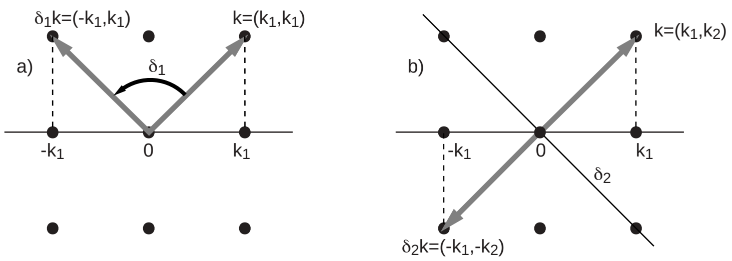

FIGURE 12. a) In a square lattice, if $\mathbf{J}$ contains the rotation δ_1 by $\pi/2$, then $\delta_1 \in \mathrm{J}^-(k)$ for the elements $k = (k_1, k_1) \in \mathcal{L}^*$. b) For all the lattices, the reflection δ_2, in the line perpendicular to k, belongs to $\mathrm{J}^-(k)$. However, neither δ_1 nor δ_2 lies in $\mathbf{J}^-$.

3.2.4. the sets $\mathcal{O}^*$ and $\mathcal{P}^*$. Let

$$\mathcal{O}^* = \left\{k \in \mathcal{L}^* : \mathrm{J}^+(k) = \mathbf{J}^+ \text{ and } \mathrm{J}^-(k) = \mathbf{J}^-\right\}.$$

For $\delta \in \mathbf{J} - \mathbf{J}^+$, let $\mathcal{P}^*_{\delta,+} = \{k \in \mathcal{L}^* : \delta k = (k_1, -|\delta| k_2)\}$ and for $\delta \in \mathbf{J} - \mathbf{J}^-$ let $\mathcal{P}^*_{\delta,-} = \{k \in \mathcal{L}^* : \delta k = -(k_1, -|\delta| k_2)\}$.

If δ is a rotation then for $k \in \mathcal{P}^*_{\delta,\pm}$ we have either $\delta k = (k_1, -k_2)$ or $\delta k = -(k_1, -k_2)$, *i.e.*, k belongs either to the line fixed by $\sigma\delta$ or to the line fixed by $-\sigma\delta$. Therefore each $\mathcal{P}^*_{\delta,\pm}$ is the intersection of those lines with $\mathcal{L}^*$. Similarly, if δ is a reflection then $\mathcal{P}^*_{\delta,\pm}$ is the intersection of $\mathcal{L}^*$ with a line fixed either by δ or by $-\delta$.

Let $\mathcal{P}^* = \bigcup_{\delta \in \mathbf{J}-\mathbf{J}^+} \mathcal{P}^*_{\delta,+} \cup \bigcup_{\delta \in \mathbf{J}-\mathbf{J}^-} \mathcal{P}^*_{\delta,-}$, where unions over an empty index set are conventioned to be $\{(0,0)\}$. Then

$$\mathcal{L}^* = \mathcal{O}^* \cup \bigcup_{\delta \in \mathbf{J}-\mathbf{J}^+} \mathcal{P}^*_{\delta,+} \cup \bigcup_{\delta \in \mathbf{J}-\mathbf{J}^-} \mathcal{P}^*_{\delta,-} \tag{9}$$

where any two sets in the right hand side either are disjoint or coincide. Although we have redefined $\mathcal{P}^*$, Lemma 3.3 still holds and in particular $\mathcal{M}^*_{y_0} \cap \mathcal{P}^* = \{(0,0)\}$.

LEMMA 3.12. *Suppose $S'(k) = 0$ for some $k \in \mathcal{L}^*$, then:*
1. *if $\mathbf{J}^- = \emptyset$ then $k \in \left(\mathcal{N}^*_{y_0} \cup \mathcal{N}^*_\sigma \cup \mathcal{P}^*\right)$.*
2. *If $-\sigma \in \mathbf{J}^-$ then $k \in \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_{-\sigma} \cup \mathcal{N}^*_\sigma \cup \mathcal{P}^*\right)$.*
3. *If $\mathbf{J}^- = \{-I\}$ then $\mathbf{J}^+ = \{I\}$ and $k \in \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_{-I} \cup \mathcal{P}^*\right)$.*

PROOF. If $\mathbf{J}^- = \emptyset$ then $S'(k) = \sum_{\delta \in \mathbf{J}^+} D'(\delta, k) = S(k) = 0$ for $k \in \mathcal{O}^*$. By Lemma 3.2, either $k \in \mathcal{N}^*_{y_0} \cup \mathcal{N}^*_\sigma$ if $\mathbf{J}^+ = \{I, \sigma\}$, or $k \in \mathcal{N}^*_{y_0}$, if $\mathbf{J}^+ = \{I\}$ and 1 follows by (9).

For the remaining cases we write, for $k \in \mathcal{O}^*$, $S'(k)$ in the form (5) and use the expressions for $G(k) = G(k_1, k_2)$ obtained in the proof of sufficiency.

If $\mathbf{J}^- = \{-\sigma\}$ then $\mathbf{J}^+ = \{I\}$ and $G(k)$ has the form (6). Thus $k \in \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_{-\sigma}$ proving 2 with $\mathcal{N}^*_\sigma = \emptyset$.

If $\mathbf{J}^- = \{-I, -\sigma\}$ then we find that $G(k)$ has the form (8) and so either $k_2 y_0 \in \mathbf{Z} - \{0\}$ or $G(k_1, k_2) = 0$ which implies $k \in \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_{-\sigma} \cup \mathcal{N}^*_\sigma$.

Finally, if $\mathbf{J}^- = \{-I\}$ then $\mathbf{J}^+ = \{I\}$ and $G(k)$ has the form (7) implying $k \in \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_{-I}$. □

LEMMA 3.13. *If $S'(k) = 0$ for all $k \in \mathcal{L}^*$ then one of the following conditions holds:*

A. $\mathbf{J}^- = \{-\sigma\}$ *and* $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_{-\sigma}$.
B. $\mathbf{J}^- = \{-I\}$ *and* $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_{-I}$.
C. $\mathbf{J}^- = \{-I, -\sigma\}$ *and* $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_{-\sigma} \cup \mathcal{N}^*_\sigma$.

Proof. Since $S'(k) = 0$ for all $k \in \mathcal{L}^*$, we follow the cases of Lemma 3.12.

Case 3 implies condition B by Lemma 3.3, with $\mathcal{M}^* = \mathcal{M}^*_{-I}$ and with $\mathcal{N}^*_\sigma = \emptyset$. Similarly, case 2 implies either condition A or condition C by Lemma 3.3 with $\mathcal{M}^* = \mathcal{M}^*_{-\sigma}$.

It remains to study case 1, where we have $\mathcal{M}^*_{y_0} \subset (\mathcal{N}^*_\sigma \cup \mathcal{P}^*)$, and therefore $\mathcal{M}^*_{y_0} \subset (\mathcal{N}^*_\sigma \cup \{(0,0)\})$.

Suppose $\mathcal{N}^*_\sigma \neq \emptyset$ and thus $\sigma \in \mathbf{J}$ and $\mathcal{M}^*_{y_0} \neq \{(0,0)\}$. If $k \neq (0,0) \in \mathcal{M}^*_{y_0}$, then $2k \in \mathcal{M}^*_{y_0}$. By (2) then $2k \notin \mathcal{N}^*_\sigma$, a contradiction. Therefore $\mathcal{N}^*_\sigma = \emptyset$ and $\mathcal{M}^*_{y_0} = \{(0,0)\}$. Moreover, $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{P}^*$, and by Lemma 3.3, it follows that $\mathcal{L}^* = \overline{\mathcal{N}^*_{y_0}}$. As in the proof of Lemma 3.4, this imposes $\mathcal{M}^*_{y_0} \neq \{(0,0)\}$, a contradiction. Therefore, this case never happens: if $S'(k) = 0$ for all $k \in \mathcal{L}^*$ then $\mathbf{J}^- \neq \emptyset$. □

3.2.5. parity of $\Pi_{y_0}(X_\Gamma)$ related to $\mathcal{L}$.

Theorem 3.2. *All functions in $\Pi_{y_0}(X_\Gamma)$ are invariant for the reflection in the point x_0 if and only if one of the following conditions holds:*

I. $((2x_0, 0), -\sigma) \in \Gamma$,
II. $((2x_0, y_0), -I) \in \Gamma$,
III. $(0, y_0) \in \mathcal{L}$ *and for some* $y_1 \in \mathbf{R}$ $((2x_0, y_1), -\sigma) \in \Gamma$,
IV. $(0, y_0) \in \mathcal{L}$ *and for some* $y_1 \in \mathbf{R}$ $((2x_0, y_1), -I) \in \Gamma$.

We now illustrate the conditions in Theorem 3.2. As in the previous examples, the patterns correspond to functions $f : \mathbf{R}^2 \longrightarrow \{0,1\}$, where $f(x,y) = 0$ in the white regions. After projection we obtain a function whose value for each $x \in \mathbf{R}$ is the width of the black region above it.

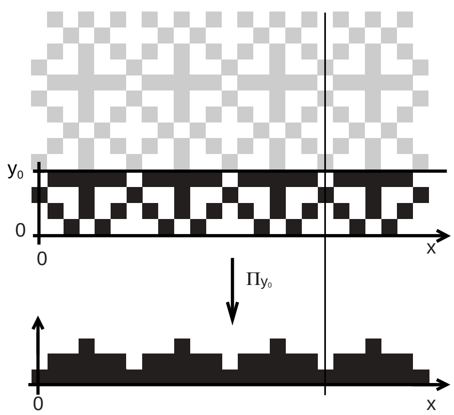

Figure 13. The invariance of the pattern under the reflection in the vertical line ensures an analogous symmetry after projection, for all y_0, by condition I of Theorem 3.2.

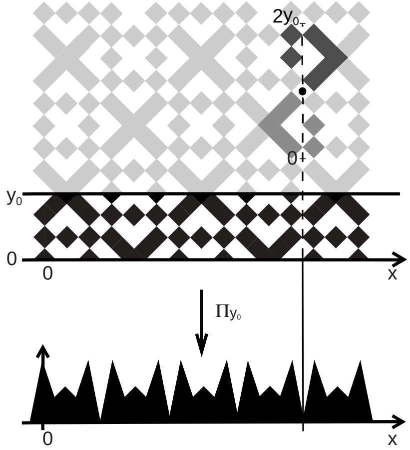

FIGURE 14. By condition II of Theorem 3.2, the rotation of π, around the black dot, acts as a reflection for the projected pattern.

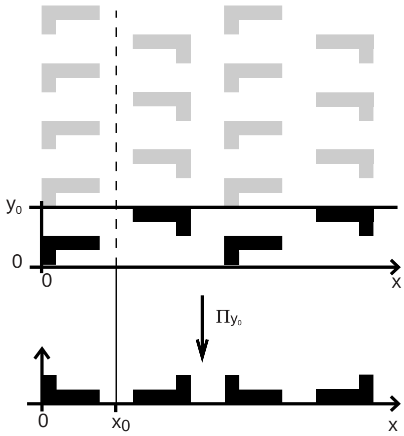

FIGURE 15. The glide reflection in the dashed line acts as a reflection for the projected function, by condition III of Theorem 3.2.

3.2.6. structure of the proof. A direct calculation of the integrals shows that the conditions are sufficient, see section 4.2 for details in the general case. The proof that the conditions are necessary occupies the rest of this section and is divided in lemmas according to the cases of Proposition 3.3. In Lemma 3.14 below, we show that conditions A and B of Proposition 3.3 imply the conditions of Theorem 3.2.

Condition C of Proposition 3.3 yelds a dual lattice with a structure similar to that in the proof of Theorem 3.1. Since $\sigma \in \mathbf{J}$ then Lemma 3.5 shows that $\mathcal{L}$ is either a lattice in standard position (square or rectangular) or a lattice in diamond position (square, centered rectangular or hexagonal). Lemma 3.6 restricts the form of v_σ for each type of

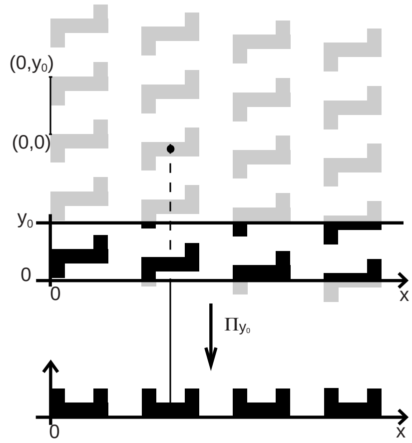

FIGURE 16. If $(0, y_0) \in \mathcal{L}$ then, by condition IV of Theorem 3.2, the rotation of π around the black dot, acts as a reflection for the projected pattern. This example is different from figure 14 because the translation associated to $-I$ does not have any relation to y_0.

lattice. In Lemma 3.15 below we obtain analogous restrictions to the form of $v_{-\sigma}$ and Lemma 3.16 scales the possible lattices according to the value of $v_{-\sigma}$. Thus, one of the following conditions must hold, with $\alpha = 2(2x_0 - u_1)/n$:

a) $\mathcal{L} = \{(\alpha, 0), (0, \beta), \}_{\mathbf{Z}}$, either $v_\sigma = (0, v_2)$ or $v_\sigma = (\alpha/2, v_2)$ and either $v_{-\sigma} = (u_1, 0)$ or $v_{-\sigma} = (u_1, \beta/2)$;

b) $\mathcal{L} = \{(\alpha/2, \beta/2), (0, \beta)\}_{\mathbf{Z}}$, $v_\sigma = (0, v_2)$ and either $v_{-\sigma} = (0, 0)$ or $v_{-\sigma} = (u_1, \beta/2)$ or $v_{-\sigma} = (u_1, \beta)$.

The result is proved for these cases in Lemmas 3.17 and 3.18.

3.2.7. the simple cases.

LEMMA 3.14.

1. *If* $-\sigma \in \mathbf{J}$ *and* $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_{-\sigma}$ *then either I or III of Theorem 3.2 holds.*
2. *If* $-I \in \mathbf{J}$ *and* $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_{-I}$ *then either II or IV of Theorem 3.2 holds.*

PROOF. Let $\xi = -\sigma$ in case 1 and $\xi = -I$ in case 2 and $(u_1, u_2) = v_\xi$. Since both $\mathcal{M}^*_\xi$ and $\overline{\mathcal{N}^*_{y_0}}$ are modules, then from $\mathcal{L}^* = \overline{\mathcal{N}^*_{y_0}} \cup \mathcal{M}^*_\xi$ it follows that either $\mathcal{L}^* = \overline{\mathcal{N}^*_{y_0}}$ or $\mathcal{L}^* = \mathcal{M}^*_\xi$.

In case 1 if $\mathcal{L}^* = \mathcal{M}^*_{-\sigma}$ then $-\sigma v_{-\sigma} + (2x_0, 0) \in \mathcal{L}$ and

$$(v_{-\sigma}, -\sigma) \cdot (\sigma v_{-\sigma} - (2x_0, 0), I) = ((2x_0, 0), -\sigma) \in \Gamma,$$

i.e., condition I is satisfied.

In case 2 if $\mathcal{L}^* = \mathcal{M}^*_{-I}$ then $-v_{-I} + (2x_0, y_0) \in \mathcal{L}$ and

$$((2x_0, y_0) - v_{-I}, I) \cdot (v_{-I}, -I) = ((2x_0, y_0), -I) \in \Gamma,$$

i.e., condition II holds.

Now suppose $\mathcal{L}^* = \overline{\mathcal{N}^*_{y_0}}$ and thus $(0, y_0) \in \mathcal{L}$. For an integer $n \neq 0$ and some $y_2 \in \mathbf{R}$, we choose generators for $\mathcal{L}$ and $\mathcal{L}^*$ as follows:

$$\begin{aligned} \mathcal{L} &= \{l_1 = (0, y_0/n), l_2 = (n\rho/y_0, y_2)\}_{\mathbf{Z}} \\ \mathcal{L}^* &= \{l_1^* = (-y_2/\rho, n/y_0), l_2^* = (y_0/n\rho, 0)\}_{\mathbf{Z}}. \end{aligned}$$

Then $l_2^* \in \mathcal{M}^*_{y_0} \subset \mathcal{M}^*_{\xi}$.

In case 1 we have $\langle l_2^*, -\sigma v_{-\sigma} + (2x_0, 0)\rangle = y_0/n\rho(2x_0 - u_1) = m \in \mathbf{Z}$ and in case 2 we have $\langle l_2^*, -av_{-I} + (2x_0, y_0)\rangle = y_0/n\rho(2x_0 - u_1) = m \in \mathbf{Z}$.

In both cases it follows that $ml_2 = (2x_0 - u_1, my_0) \in \mathcal{L}$ and thus,

$$((2x_0 - u_1, my_2), I) \cdot ((u_1, u_2), \xi) = ((2x_0, y_1), \xi) \in \Gamma,$$

where $y_1 = my_2 + u_2$, implying condition III in case 1 and condition IV in case 2. □

3.2.8. the structure of $\mathcal{L}$ and $\mathcal{L}^*$. Now we obtain restrictions on $v_{-\sigma}$ like those on v_σ of Lemma 3.6.

LEMMA 3.15. *If $-\sigma \in \mathbf{J}$, then:*
for $\mathcal{L} = \{(\alpha, 0), (0, \beta)\}_{\mathbf{Z}}$ then either $v_{-\sigma} = (u_1, 0)$ or $v_{-\sigma} = (u_1, \beta/2)$,
for $\mathcal{L} = \{(\alpha/2, \beta/2), (0, \beta)\}_{\mathbf{Z}}$ then either $v_{-\sigma} = (0, 0)$ or $v_{-\sigma} = (u_1, \beta/2)$ or $v_{-\sigma} = (u_1, \beta)$. Moreover, if $v_{-\sigma} = (u_1, \beta)$ then $((u_1, 0), -\sigma) \in \Gamma$.

PROOF. Let $v_{-\sigma} = (u_1, u_2)$, with $(v_{-\sigma}, -\sigma)^2 = ((0, 2u_2), I) \in \Gamma$. For the lattice in standard position, we have $u_2 \in [0, \beta[$ with $(0, 2u_2) \in \mathcal{L}$. Therefore, either $u_2 = 0$ or $u_2 = \beta/2$.

For the lattice in diamond position, $u_2 \in [0, 3\beta/2[$ with $(0, 2u_2) \in \mathcal{L}$ and thus u_2 is either 0 or $\beta/2$ or β. In the fundamental cell, only the origin has zero second coordinate. If $u_2 = \beta$ then $((0, -\beta), I) \cdot ((u_1, \beta), -\sigma) = ((u_1, 0), -\sigma) \in \Gamma$. □

There are no analogue for the previous lemma in the $(n+1)$-dimensional case, due to a better definition of the elements $(v_\delta, \delta) \in \Gamma$, as told in page 53 for Lemma 3.6.

LEMMA 3.16. *If $-\sigma \in \mathbf{J}$ and $\mathcal{L}^* = \mathcal{M}^*_{-\sigma} \cup \mathcal{N}^*_{y_0} \cup \mathcal{N}^*_{\sigma}$, then for $v_{-\sigma} = (u_1, u_2)$ and for α defined in Lemma 3.5 we have $\alpha = 2(2x_0 - u_1)/n$ for some integer n.*

PROOF. As in the proof of Lemma 3.7, using $\mathcal{M}^*_{-\sigma}$ instead of $\mathcal{M}^*_P$, we obtain $k = (2/\alpha, 0) \in \mathcal{L}^*$ with $k \notin \mathcal{N}^*_{y_0} \cup \mathcal{N}^*_\sigma$. Therefore $\langle (2/\alpha, 0), -\sigma v_{-\sigma} + (2x_0, 0)\rangle \in \mathbf{Z}$ or, equivalently, $2(2x_0 - u_1)/\alpha = n \in \mathbf{Z}$ and the result follows. □

LEMMA 3.17. *Suppose* $\pm\sigma \in \mathbf{J}$ *and* $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_{-\sigma} \cup \mathcal{N}^*_\sigma$. *If* $\mathcal{L} = \{(\alpha, 0), (0, \beta)\}_\mathbf{Z}$, *then one of the conditions of Theorem 3.2 holds.*

PROOF. Writing $\mathcal{L}^* = \{l_1^* = (1/\alpha, 0), l_2^* = (0, 1/\beta)\}_\mathbf{Z}$ then $l_1^* \in \mathcal{M}^*_{y_0}$ and by Lemma 3.8, with $\mathcal{M}^* = \mathcal{M}^*_{-\sigma}$, we have either $l_1^* \in \mathcal{M}^*_{-\sigma}$ or $l_1^* \in \mathcal{N}^*_\sigma$. Let $v_\sigma = (v_1, v_2)$ and $v_{-\sigma} = (u_1, u_2)$, with $\alpha = 2(2x_0 - u_1)/n$ for some $n \in \mathbf{Z}$, as in Lemma 3.16.

If $l_1^* \in \mathcal{M}^*_{-\sigma}$, then n is even and so $(2x_0 - u_1, 0) \in \mathcal{L}$ implying $((2x_0 - u_1, 0), I) \cdot ((u_1, u_2), -\sigma) = ((2x_0, u_2), -\sigma) \in \Gamma$.

By Lemma 3.8 there are three possibilities:

1i) $l_2^* \in \mathcal{M}^*_{-\sigma} \Leftrightarrow u_2/\beta \in \mathbf{Z}$. Then $u_2 = 0$, by Lemma 3.15, and thus condition I holds.

1ii) Since $l_2^* \in \mathcal{N}^*_\sigma$, we have $\frac{1}{\beta}(y_0 - v_2) + \frac{1}{2} \in \mathbf{Z}$ and therefore, $y_0 - v_2 + \frac{\beta}{2} = m\beta$, for some $m \in \mathbf{Z}$. From $l_2^*, l_1^* + l_2^* \in \mathcal{N}^*_\sigma$ it follows that $v_1/\alpha \in \mathbf{Z}$ and by Lemma 3.6 either $v_1 = \alpha/2$, a contradiction, or else $v_1 = 0$.

Since $ml_2 = (0, y_0 - v_2 + u_2) \in \mathcal{L}$, then condition II follows because $((0, y_0 - v_2 + u_2), I) \cdot ((0, v_2), \sigma) \cdot ((2x_0, u_2), -\sigma) = ((2x_0, y_0), -I) \in \Gamma$.

1iii) $(0, y_0) \in \mathcal{L}$ and condition III follows.

If $l_1^* \notin \mathcal{M}^*_{-\sigma}$ then n is odd, $v_1/\alpha + 1/2 \in \mathbf{Z}$ and, by Lemma 3.6, $v_1 = \alpha/2$. Lemma 3.8 divides this into three cases:

2i) $l_1^* + l_2^* \in \mathcal{M}^*_{-\sigma} \Leftrightarrow n/2 + 1/\beta u_2 \in \mathbf{Z}$ and so $u_2 = \beta/2$. Since $l_2^* \in \mathcal{N}^*_\sigma$, then $(y_0 - v_2)/\beta + 1/2 \in \mathbf{Z}$. Condition II follows from $((2x_0, y_0), -I) \in \Gamma$ since

$((0, y_0 - v_2 + u_2), I) \cdot ((2x_0 - u_1, v_2), \sigma) \cdot ((u_1, u_2), -\sigma) = ((2x_0, y_0), -I)$.

2ii) If $l_2^* \in \mathcal{M}^*_{-\sigma}$ and $l_1^* + l_2^* \in \mathcal{N}^*_\sigma$, then both u_2/β and $(y_0 - v_2)/\beta$ are integers. Then II follows since $u_2 = 0$, $(0, y_0 - v_2) \in \mathcal{L}$ and

$((0, y_0 - v_2), I) \cdot ((2x_0 - u_1, v_2), \sigma) \cdot ((u_1, 0), -\sigma) = ((2x_0, y_0), -I) \in \Gamma$.

2iii) $(0, y_0) \in \mathcal{L}$ and condition IV holds, because

$((2x_0 - u_1, v_2), \sigma) \cdot ((u_1, u_2), -\sigma) = ((2x_0, v_2 - u_2), -I) \in \Gamma$.

□

3.2.9. from $\mathcal{L}^*$ to $\mathcal{L}$.

LEMMA 3.18. *If* $\pm\sigma \in \mathbf{J}$ *and* $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_{-\sigma} \cup \mathcal{N}^*_\sigma$ *for the lattice in diamond position* $\mathcal{L} = \{(\alpha/2, \beta/2), (0, \beta)\}_{\mathbf{Z}}$*, then one of the conditions of Theorem 3.2 holds.*

PROOF. We use Lemma 3.6 to have $v_\sigma = (0, v_2)$. Let $v_{-\sigma} = (u_1, u_2)$, then either $v_{-\sigma} = (0,0)$ or u_2 is either $\beta/2$ or β, by Lemma 3.15. We also have $\alpha = 2(2x_0 - u_1)/n$ by Lemma 3.16.

The dual lattice is $\mathcal{L}^* = \{l_1^* = (2/\alpha, 0), l_2^* = (-1/\alpha, 1/\beta)\}_{\mathbf{Z}}$. Since $l_1^* \in \mathcal{M}^*_{y_0}$, and $l_1^* \notin \mathcal{N}^*_\sigma$ then, by Lemma 3.8 with $\mathcal{M}^* = \mathcal{M}^*_{-\sigma}$, we have $l_1^* \in \mathcal{M}^*_{-\sigma}$ and there are three possible cases:

1i) $l_2^* \in \mathcal{M}^*_{-\sigma} \Leftrightarrow u_2/\beta - n/2 \in \mathbf{Z}$.

If n is even then $\frac{n}{2}(\alpha, 0) = (2x_0 - u_1, 0) \in \mathcal{L}$ and, by Lemma 3.15, we have either $u_2 = 0$ or $u_2 = \beta$ and $((u_1, 0), -\sigma) \in \Gamma$. Thus $((2x_0 - u_1, 0), I) \cdot ((u_1, 0), -\sigma) = ((2x_0, 0), -\sigma) \in \Gamma$ and condition I follows.

If n is odd then $u_2 = \beta/2$, by Lemma 3.15. Since $n(\alpha/2, \beta/2) = ((2x_0 - u_1, nu_2) \in \mathcal{L}$ and $((u_1, nu_2), -\sigma) \in \Gamma$. Condition I holds because $((2x_0 - u_1, nu_2), I) \cdot ((u_1, nu_2), -\sigma) = ((2x_0, 0), -\sigma) \in \Gamma$.

1ii) Since $l_2^* \in \mathcal{N}^*_\sigma$, then $\frac{1}{\beta}(y_0 - v_2) \pm \frac{1}{2} \in \mathbf{Z}$. We only need to discuss the cases where 1i does not hold, that are, by Lemma 3.15, either $((u_1, 0), -\sigma) \in \Gamma$ (and n is odd) or $((u_1, \beta/2), -\sigma) \in \Gamma$ (and n is even).

For n odd, let $m = y_0/\beta - v_2/\beta - 1/2 \in \mathbf{Z}$. Then $m(0, \beta) + (\alpha/2, \beta/2) = (\alpha/2, y_0 - v_2) \in \mathcal{L}$ and $((\alpha/2, y_0 - v_2), I) \cdot ((0, v_2), \sigma) = ((\alpha/2, y_0), \sigma) \in \Gamma$ and so $((\alpha/2, y_0), \sigma)^n \cdot ((u_1, 0), -\sigma) = ((2x_0, y_0), -I) \in \Gamma$, implying condition II.

If n is even, let $m = y_0/\beta - v_2/\beta + 1/2 \in \mathbf{Z}$. Then $y_0 - v_2 - u_2 = m\beta$ for some $m \in \mathbf{Z}$ and
$((2x_0 - u_1, y_0 - v_2 + u_2), I) \cdot ((0, v_2), \sigma) \cdot ((u_1, u_2), -\sigma) = ((2x_0, y_0), -I) \in \Gamma$, implying condition II.

1iii) If $(0, y_0) \in \mathcal{L}$ then condition III follows since $n(\alpha/2, \beta/2) = (2x_0 - u_1, n\beta/2) \in \mathcal{L}$ and thus $((2x_0 - u_1, n\beta/2), I) \cdot ((u_1, u_2), -\sigma) = ((2x_0, u_2 + n\beta/2), -\sigma) \in \Gamma$. □

Thus we have completed the proof of Theorem 3.2.

3.3. Restriction of Invariant Functions and Symmetry

Another way of lowering the dimension of a two-dimensional pattern is to restrict it to a line. The results for the restriction are simplifications of those in sections 3.1 and 3.2 and allow us to compare the two methods.

Let Φ_r, $r \in \mathbf{R}$, be the operator that maps $f(x,y)$ to its restriction to the line $\{(x,r) : x \in \mathbf{R}\}$ given by $\Phi_r(f)(x) = f(x,r)$. For $f \in X_\Gamma$ we have, formally,

$$\Phi_r(f)(x) = \sum_{k \in \mathcal{L}^*} C(k)\omega_k(x,r) = \sum_{k_1 \in \mathcal{L}_1^*} \omega_{k_1}(x) \sum_{k_2:(k_1,k_2) \in \mathcal{L}^*} C(k_1,k_2)\omega_{k_2}(r).$$

The expansion is similar to that of the projection, with a term $\omega_{k_2}(r)$ appearing where before we had $\int_0^{y_0} \omega_{k_2}(y)dy$. The condition $\omega_{k_2}(r) = 0$ is never verified and the analogue of expression (3) is

$$\omega_{k_2}(r) - \omega_{k_2}(2r)\omega_{-k_2}(r) = 0$$

that we use to describe the periodicity on $\Phi_r(X_\Gamma)$.

3.3.1. periodicity of $\Phi_r(X_\Gamma)$.

PROPOSITION 3.5. *All functions in $\Phi_r(X_\Gamma)$ have a common period $P \in \mathbf{R} - \{0\}$ if and only if, for each $k = (k_1,k_2) \in \mathcal{L}^*$, one of the following conditions holds:*

A. $k_1 P \in \mathbf{Z}$,

B. $(v_\sigma, \sigma) \in \Gamma$ *and* $< k, \sigma v_\sigma + (0,2r) > + \frac{1}{2} \in \mathbf{Z}$.

Note that if either $(P,0) \in \mathcal{L}$ or $((P,2r),\sigma) \in \Gamma$, with $P \neq 0$, then the functions in $\Phi_r(X_\Gamma)$ have period P.

The proposition can be rewritten in terms of subsets of $\mathcal{L}^*$ by redefining $\mathcal{N}_\sigma^*$ as $\mathcal{N}_\sigma^* = \left\{k \in \mathcal{L}^* : < k, \sigma v_\sigma + (0,2r) > + \frac{1}{2} \in \mathbf{Z}\right\}$. In this setting it states that $P \neq 0$ is a commont period of all functions in $\Phi_r(X_\Gamma)$ if and only if either $\mathcal{L}^* = \mathcal{M}_P^*$ or $\sigma \in \mathbf{J}$ and $\mathcal{L}^* = \mathcal{M}_P^* \cup \mathcal{N}_\sigma^*$. The proof is similar to that of Proposition 3.1 and is omitted.

THEOREM 3.3. *All functions in $\Phi_r(X_\Gamma)$ have a common period $P \neq 0$ if and only if one of the following conditions holds:*

I. $(P,0) \in \mathcal{L}$,

II. $((P,2r),\sigma) \in \Gamma$.

PROOF. A direct calculation shows that the conditions are sufficient. For the necessity, if all functions in $\Phi_r(X_\Gamma)$ have a common period $P \neq 0$ then, by Proposition 3.5, either $\mathcal{L}^* = \mathcal{M}_P^*$ and condition I holds, or $\sigma \in \mathbf{J}$ and $\mathcal{L}^* = \mathcal{M}_P^* \cup \mathcal{N}_\sigma^*$. In the second case, by Lemmas 3.5, 3.6 and 3.7, it follows that $\mathcal{L}^*$ has one of the forms below:

$$\mathcal{L}^* = \left\{ \left(\frac{n}{2P}, 0\right), \left(0, \frac{1}{\beta}\right) \right\}_{\mathbf{Z}} \qquad \text{or} \qquad \mathcal{L}^* = \left\{ \left(\frac{n}{P}, 0\right), \left(-\frac{n}{2P}, \frac{1}{\beta}\right) \right\}_{\mathbf{Z}},$$

where n is the greatest integer such that $\left(\frac{2P}{n}, 0\right) \in \mathcal{L}$. The proof is a simplification of those of Lemmas 3.9 and 3.10 with y_0 replaced by $2r$ and with no conditions involving $\mathcal{N}_{y_0}^*$. □

3.3.2. parity of $\Phi_r(X_\Gamma)$. The proof of Proposition 3.3 may be adapted and simplified to yeld:

PROPOSITION 3.6. *All functions in $\Phi_r(X_\Gamma)$ are invariant for the reflection in the point x_0 if and only if one of the conditions holds:*

A. $(v_{-\sigma}, -\sigma) \in \Gamma$ *and* $< k, -\sigma v_{-\sigma} + (2x_0, 0) > \in \mathbf{Z}$ *for all* $k \in \mathcal{L}^*$,
B. $(v_{-I}, -I) \in \Gamma$ *with* $< k, -v_{-I} + (2x_0, 2r) > \in \mathbf{Z}$ *for all* $k \in \mathcal{L}^*$,
C. *both* $(v_{-\sigma}, -\sigma)$ *and* (v_σ, σ) *belong to* Γ *and for each* $k \in \mathcal{L}^*$ *either* $< k, -\sigma v_{-\sigma} + (2x_0, 0) > \in \mathbf{Z}$ *or* $< k, \sigma v_\sigma + (0, 2r) > +\frac{1}{2} \in \mathbf{Z}$.

THEOREM 3.4. *All functions in $\Phi_r(X_\Gamma)$ are invariant for the reflection in the point x_0 if and only if one of the following conditions holds:*

I. $((2x_0, 0), -\sigma) \in \Gamma$,
II. $((2x_0, 2r), -I) \in \Gamma$.

PROOF. A direct calculation shows that the conditions are sufficient. For the necessity, we follow the cases of Proposition 3.6:

A. $-\sigma v_{-\sigma} + (2x_0, 0) \in \mathcal{L}^*$. For $v_{-\sigma} = (u_1, u_2)$ then $(0, -2u_2) \in \mathcal{L}$,

$$((2x_0, 0), -\sigma) = ((0, -2u_2), I) \cdot ((2x_0 - u_1, u_2), I) \cdot ((u_1, u_2), -\sigma) \in \Gamma.$$

B. $(v_{-I}, -I) \in \Gamma$. For $v_{-I} = (u_1, u_2)$ then $-v_{-I} + (2x_0, 2r) \in \mathcal{L}$,

$$((2x_0, 2r), -I) = ((2x_0 - u_1, 2r - u_2), I) \cdot ((u_1, u_2), -I) \in \Gamma.$$

C. both $(v_{-\sigma}, -\sigma)$ and (v_σ, σ) belong to Γ and $\mathcal{L}^* = \mathcal{M}^*_{-\sigma} \cup \mathcal{N}^*_\sigma$, where $\mathcal{M}^*_{-\sigma} = \{k \in \mathcal{L}^* : < k, -\sigma v_{-\sigma} + (2x_0, 0) > \in \mathbf{Z}\}$ and we use $2r$ instead of y_0 in the definition of $\mathcal{N}^*_\sigma$. The proof is completed using Lemmas 3.17 and 3.18. □

3.4. Characterization of Wallpaper Groups

Let Γ be a wallpaper group and let $\Sigma(y_0)$ and $\Sigma(r)$ be the subgroups of $\mathbf{E}$ that fix, respectively, the space $\Pi_{y_0}(X_\Gamma)$ and $\Phi_r(X_\Gamma)$. In sections 3.1 – 3.3 we started from $\Sigma(y_0)$ and $\Sigma(r)$ and obtained information on Γ. In this section we present these results from a different point of view, showing how these groups depend on Γ, in particular on its lattice $\mathcal{L}$, and on the parameters y_0 and r.

This study for the general $(n + 1)$-dimensional case, is made only for periodicity, in chapter 5.

The results are presented in tables meant to be read from left to right: from the presence of some element in Γ and for some values of y_0 or of r the tables provide information on $\Sigma(y_0)$ or $\Sigma(r)$. Note that each Σ may contain translations, $(a, 1)$, or reflections, $(b, -1)$, with $a, b \in \mathbf{R}$.

Conditions not applicable to the restriction operator are indicated in the tables by "n.a.". In many cases the hypothesis restricts both the lattice and the wallpaper group. We use the standard notation for these groups - a complete description of this notation can be obtained in Armstrong [**1**] and on pages 156 and 157 there is a helpful illustration of the standard action of the seventeen wallpaper groups.

If $((a, b), I) \in \Gamma$ is a generator of $\mathcal{L}$ then (a, b) is the smallest element of $\mathcal{L}$ in its direction, indicated as "gen." in the tables. We say $((a, b), I)$ and $((c, d), I)$ are generators when $\mathcal{L} = \{(a, b), (c, d)\}_{\mathbf{Z}}$. Recall that if $\det \delta = -1$, the element (v_δ, δ) is either a reflection, for $\delta v_\delta = -v_\delta$, or, otherwise, a glide reflection and since $\delta \in \mathbf{O}(2)$ then $\delta^2 = I$.

3.4.1. all wallpaper groups.

original group Γ contains	validity set		image group Σ contains
	y_0	r	
$((a, 0), I)$	$\mathbf{R}^+$	$\mathbf{R}$	$(a, 1)$
$((a, b), I)$ gen. $((0, c), I)$ gen.	$\{nc : n \in \mathbf{N}\}$	n.a.	$(a, 1)$

Figures 6 and 7 illustrate, respectively, the first and the second case of this table. Consider either $r = 0$ or $r = y_0$ as examples of restrictions in those figures.

Since $\mathcal{L}$ is the translation subgroup of Γ then the condition $((a, 0), I) \in \Gamma$ (or $((0, c), I) \in \Gamma$) restricts only the position of the lattice. We discuss this in section 3.5, below.

3.4.2. wallpaper groups with a glide reflection.

original group Γ contains	validity set y_0	validity set r	image group Σ contains
$((a,b),\sigma)$, $((2a,0),I)$ gen. $((0,c),I)$ gen.	$\{b+nc : n \in \mathbf{N_0}\}$	$\left\{\frac{b}{2}+n\frac{c}{2} : n \in \mathbf{Z}\right\}$	$(a,1)$
	$\mathbf{R}^+$	$\mathbf{R}$	$(2a,1)$
$((a,b),\sigma)$, $((2a,0),I)$ gen. $((a,c),I)$ gen.	$\{b+2nc : n \in \mathbf{N_0}\}$	$\left\{\frac{b}{2}+nc : n \in \mathbf{Z}\right\}$	$(a,1)$
	$\mathbf{R}^+$	$\mathbf{R}$	$(2a,1)$
	$\{2nc : n \in \mathbf{N}\}$	n.a.	$(a,1)$
$((a,b),-\sigma)$, $((c,0),I)$ gen. $((0,2b),I)$ gen.	$\{2nb : n \in \mathbf{N}\}$	n.a.	$(a,-1)$
	$\mathbf{R}^+$	$\mathbf{R}$	$(c,1)$
$((a,b),-\sigma)$, $((0,2b),I)$ gen. $((c,b),I)$ gen. $((a+c,0),-\sigma)$	$\{2nb : n \in \mathbf{N}\}$	n.a.	$(a,-1)$ $(c,1)$
	$\mathbf{R}^+$	$\mathbf{R}$	$(2c,1)$ $(a+c,-1)$

The first line of the table is illustrated in figure 8: for the shown projection width y_0, the projected strip contains the pattern and its copy by the glide reflection, and the projected pattern has half the period. The same holds to the restriction to the line at the center of this strip, or the dashed line in the figure.

To obtain this table we use the results of sections 3.1 to 3.3 and consider the effect of a glide in restricting $\mathcal{L}$ and Γ. We group the cases according to the orthogonal component of the glide.

For the first two cases, suppose $((d,b),\sigma) \in \Gamma$ for some d, thus $((2d,0),I) \in \Gamma$ and $\left(\left(\frac{2d}{m},0\right),I\right)$ is a generator for some $m \in \mathbf{N}$. If $m = 2m'$, then $\left(\left(\frac{d}{m'},0\right),I\right)^{m'} = ((d,0),I) \in \Gamma$ and $((d,b),\sigma)$ is a reflection and not a glide, up to elements of $\mathcal{L}$. We treat this case separately in the next section.

For m odd $\left(\left(\frac{d}{m},b\right),\sigma\right) = \left(\left(-\frac{2d}{m},0\right),I\right)^{\frac{m-1}{2}} \cdot ((d,b),\sigma) \in \Gamma$ and so $((a,b),\sigma) \in \Gamma$, for $a = \frac{d}{m}$, and $((2a,0),I)$ is a generator.

By Lemma 3.5 the missing generator is either $((0,c),I)$ or $((a,c),I)$. The first case corresponds to wallpaper groups with either rectangular or square lattices in standard position and, at least, a glide. These are the groups pg, p2mg, p2gg and p4gm. The group p4mm also contains a glide but not at this particular relative position to the lattice.

In the second case we have either a centred rectangular or an hexagonal or a square lattice in diamond position and, at least, one glide reflection. The wallpaper groups with these characteristics are cm, c2mm, p4mm, p4gm, p3m1, p31m and p6mm.

Now suppose $((a,b),-\sigma) \in \Gamma$ and thus $((0,2b),I)$ generates the lattice together with either $((c,0),I)$ or $((c,b),I)$. In the first case both $(a,-1)$ and $(c,1)$ lie in Σ and then $(a+mc,-1)\in\Sigma$ for all $m\in\mathbf{Z}$.

The last case of the table is somewhat more complex. In the wallpaper groups with centred rectangular or hexagonal lattices, reflections and glide reflections arise together on parallel lines, see figure 17, a). This is also valid for wallpaper groups with square lattices, but only for those glide reflections and reflections on lines parallel to the diagonal of the square, as in figure 17, b).

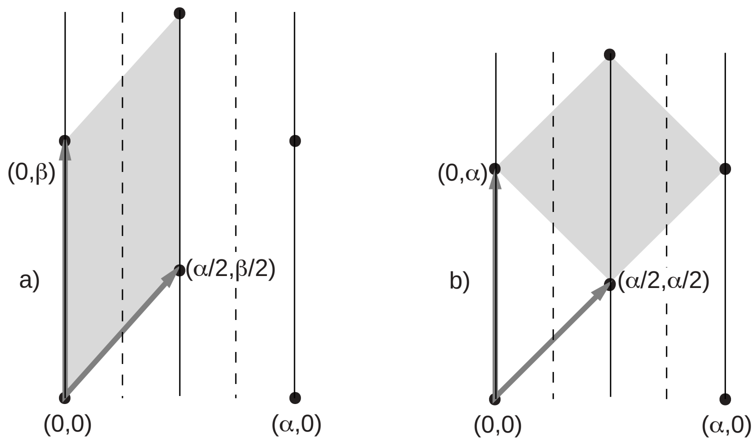

FIGURE 17. Lattices in diamond position. There are reflections on the black continuous lines and glide reflections on the dashed lines, both associated to $-\sigma$. In b) we present the particular case of a square lattice.

When these lines are horizontal, *i.e.*, if σ is the orthogonal component, only the glide is relevant for our study. For $-\sigma$, as can be see on Theorem 3.2, both glides and reflections play an important role. Formally, if $((a,b),-\sigma)\in\Gamma$ and $\mathcal{L}=\{(0,2b),(c,b)\}_{\mathbf{Z}}$ then

$$((a+c,0),-\sigma)=((a,b),-\sigma)\cdot((c,b),I)^{-1}\in\Gamma.$$

From the period $(2c,1)$, we get $(a+(2m+1)c,-1)\in\Sigma$ for all $m\in\mathbf{Z}$.

3.4.3. wallpaper groups with a reflection.

original group Γ contains	validity set		image group Σ contains
	y_0	r	
$((a,0),-\sigma)$, $((c,0),I)$ gen. $((0,b),I)$ gen.	$\mathbf{R}^+$	$\mathbf{R}$	$(a,-1)$ $(c,1)$

A reflection restricts the lattice associated to the wallpaper group. This table presents the case where the lattice is rectangular or square in standard position, see figure 18, and is relevant for the wallpaper groups having one of these lattices and a reflection: pm, p2mm, p2mg and p4mm. The group p4gm has a glide but with a different direction. For the centred rectangular, hexagonal and square lattices in diamond position, $((2x_0', 0), -\sigma)$ appears on table 3.4.2 as $2x_0' = a + c$.

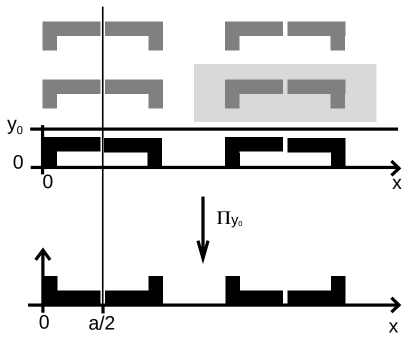

FIGURE 18. A pattern in a rectangular lattice (a cell of this lattice in light gray) with a reflection in the black line.

3.4.4. wallpaper groups with a rotation of order two.

original group Γ contains	validity set		image group Σ contains
	y_0	r	
$((a,b),-I)$	b	$\frac{b}{2}$	$(a,-1)$
$((a,b),-I)$	b	$\frac{b}{2}$	$(a,-1)$
$((c,d),I)$ gen.	$\{ne : n \in \mathbf{N}\}$	n.a.	$(a,-1)$
$((0,e),I)$ gen.			$(c,1)$

Wallpaper groups containing a rotation of order two are p2, p2mm, p2mg, p2gg, c2mm, p4, p4mm, p4gm, p6 and p6mm.

In the first case of the table the projected function is not periodic, see figure 19. This is the only situation where the projection exhibits symmetry without being invariant under a group with a lattice in $\mathbf{R}$.

3.5. Equivariant Approach

We discuss some properties of a model whose solutions have domain $\mathbf{R}$ but are a projection of Γ-invariant solutions of an $\mathbf{E}(2)$-equivariant equation, where Γ is a wallpaper group.

A similar study for the $(n+1)$-dimensional case is presented in section seccao-periodos-equivariante. The general case is more complex since there are many ways of being periodic for a function $\Pi_{y_0}(f)$ posed

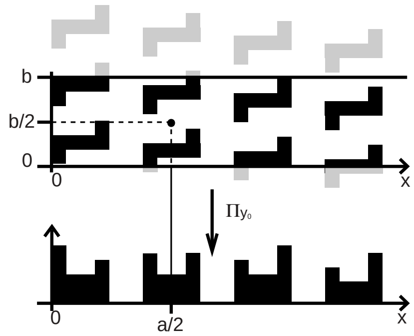

FIGURE 19. A pattern with a rotation of π around the black dot. The nonperiodic projection is invariant by the reflection in the vertical line $x = a/2$.

in a n-dimensional domain, with $n \neq 1$. Thus we state Proposition 5.1 about $\tilde{\mathcal{L}}$-periodic projected functions, where $\tilde{\mathcal{L}}$ is a n-dimensional lattice, and analyse their orbits.

3.5.1. $\mathbf{E}(2)$-equivariance. Let X be a space of functions $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$ and let $\mathcal{P} : X \times \mathbf{R} \longrightarrow X$ be a one parameter family of $\mathbf{E}(2)$-equivariant operators, *i.e.*, $\mathcal{P}(\gamma \cdot f, \lambda) = \gamma \cdot \mathcal{P}(f, \lambda)$ for $\gamma \in \mathbf{E}(2)$. Suppose $\mathcal{P}(f_0, \lambda) = 0$ and f_0 is a Γ-invariant function with $\Gamma \subset \mathbf{E}(2)$. The group Γ is called the isotropy subgroup of f_0 denoted as Γ_{f_0}. It is also called the group of symmetries of the pattern f_0.

3.5.2. orbits of solutions. All the elements in the *orbit of* f_0,

$$\mathbf{E}(2) \cdot f_0 = \{\gamma \cdot f_0 : \gamma \in \mathbf{E}(2)\},$$

are also solutions of $\mathcal{P} = 0$ and, see Golubitsky et al. [**10**], each function $\gamma \cdot f_0$ has isotropy subgroup

$$\Gamma_{\gamma \cdot f_0} = \gamma \cdot \Gamma_{f_0} \cdot \gamma^{-1} = \{\gamma \cdot (u, \xi) \cdot \gamma^{-1} : (u, \xi) \in \Gamma_{f_0}\}.$$

LEMMA 3.19. *Let Γ be a subgroup of $\mathbf{E}(2)$ with translation subgroup $\mathcal{L} \cong \{(u, I) : u \in \mathcal{L}\}$. For $\gamma = (v, \delta) \in \mathbf{E}(2)$, the translation subgroup of $\gamma \cdot \Gamma \cdot \gamma^{-1}$ is $\delta\mathcal{L} \cong \{(\delta u, I) : u \in \mathcal{L}\}$.*

We postpone the proof to chapter 5, Lemma 5.1 in page 109, since it is the same as in the $(n+1)$-dimensional case.

Suppose Γ is a wallpaper group with lattice $\mathcal{L}$. By Lemma 3.19, the solution $\gamma \cdot f_0 = (v, \delta) \cdot f_0$ is $(\delta\mathcal{L})$-periodic. The solution orbit $\mathbf{E}(2) \cdot f_0 = \{\gamma \cdot f_0 : \gamma \in \mathbf{E}(2)\}$ corresponds to the orbit $\mathbf{O}(2)\mathcal{L} = \{\delta\mathcal{L} : \delta \in \mathbf{O}(2)\}$ in the space of all the possible lattices for wallpaper groups.

Our results concern spaces of functions $X_\Gamma \subset X_{\mathcal{L}}$. When looking for solutions of a $\mathbf{E}(2)$-equivariant problem we should consider the orbits $\mathbf{E}(2) \cdot X_{\mathcal{L}} = \{X_{\delta\mathcal{L}} : \delta \in \mathbf{O}(2)\}$, instead of a single space $X_{\mathcal{L}}$.

3.5.3. the projected orbits of solutions. If we project solutions of $\mathcal{P} = 0$, instead of working with $\Pi_{y_0}(X_{\mathcal{L}})$ we will study

$$\{\Pi_{y_0}(X_{\delta\mathcal{L}}) : \delta \in \mathbf{O}(2)\}$$

and the results concerning the lattice $\mathcal{L}$ must be reformulated for the orbit $\mathbf{O}(2)\mathcal{L}$.

Let l be any element of $\mathcal{L}$. We may rotate $\mathcal{L}$ moving l to a horizontal vector $(\|l\|, 0) \in \delta\mathcal{L}$, where $\|l\|$ is the usual norm of $\mathbf{R}^2$. By Theorem 3.1, for all $l \in \mathcal{L}$ there is some $\delta \in \mathbf{O}(2)$ such that $\Pi_{y_0}(X_{\delta\mathcal{L}})$ is a set of functions with period $\|l\|$.

COROLLARY 3.1. *The projection $\Pi_{y_0}(f_0)$ of an $\mathcal{L}$-periodic solution f_0 of $\mathcal{P} = 0$ lies in a family of solutions $\{\Pi_{y_0}(\gamma \cdot f_0) : \gamma \in \mathbf{E}(2)\}$ where for each $l \in \mathcal{L}$ we can find a function with period $\|l\|$.*

Moreover, the orbit of the projected solution $\Pi_{y_0}(f_0)$ have non-periodic functions, $\Pi_{y_0}(\gamma \cdot f_0)$, for the elements $\gamma = (v, \delta) \in \mathbf{E}(2)$ such that $\delta\mathcal{L}$ intersects the horizontal axis $\{(x, 0) : x \in \mathbf{R}\}$ only in the origin.

This result does not depend on y_0 and also holds for $\{\Phi_r(\gamma \cdot f_0) : \gamma \in \mathbf{E}(2)\}$, $r \in \mathbf{R}$.

3.5.4. $\mathbf{E}(1)$-equivariance. When looking for solutions of a $\mathbf{E}(1)$-equivariant system $\tilde{\mathcal{P}} = 0$, with $\tilde{\mathcal{P}} : Y \times \mathbf{R} \longrightarrow Y$ and Y a space of functions $g : \mathbf{R} \longrightarrow \mathbf{R}$, we do not expect a result such as Corollary 3.1.

If we have a solution g_0 with minimum period P, then the translation subgroup of Σ that leaves g_0 invariant is $\{P\}_{\mathbf{Z}}$. By Lemma 3.19 and since $\delta \in \{1, -1\} = \mathbf{O}(1)$, the solutions on $\mathbf{E}(1) \cdot g_0$ have translation subgroups $\delta\{P\}_{\mathbf{Z}} = \{P\}_{\mathbf{Z}}$. Therefore, on an $\mathbf{E}(1)$-equivariant approach all the solutions on an orbit have the same periods.

CHAPTER 4

Projection and Symmetry

In this chapter we state and prove our main result, Theorem 4.1, below. This theorem relates the symmetry of the functions f in the space X_Γ to the symmetry of the projected functions $\Pi_{y_0}(f)$ in the space $\Pi_{y_0}(X_\Gamma)$.

At the end we present a section with an analogous result for the restriction

4.1. Theorem for the Projection

For $\alpha \in \mathbf{O}(n)$, we define the elements of $\mathbf{O}(n+1)$:
$\sigma = \begin{pmatrix} Id_n & 0 \\ 0 & -1 \end{pmatrix}$, $\alpha_+ = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ and $\alpha_- = \sigma\alpha_+ = \begin{pmatrix} \alpha & 0 \\ 0 & -1 \end{pmatrix}$.

For simplicity of notation we write (v_+, α_+) for (v_{α_+}, α_+) and (v_-, α_-) for (v_{α_-}, α_-).

Recall that Γ is a crystallographic subgroup of $\mathbf{E}(n+1) \cong \mathbf{R}^{n+1} \ltimes \mathbf{O}(n+1)$ with lattice $\mathcal{L}$. For $\delta \in \mathbf{O}(n+1)$, we will use the notation $(v, \delta) \in \Gamma$, with $v \in \mathbf{R}^{n+1}$ or, alternatively, $((a, b), \delta) \in \Gamma$, with $a \in \mathbf{R}^n$ and $b \in \mathbf{R}$.

THEOREM 4.1. *All functions in $\Pi_{y_0}(X_\Gamma)$ are invariant under the action of $(v_\alpha, \alpha) \in \mathbf{R}^n \ltimes \mathbf{O}(n)$ if and only if one of the following conditions holds:*

I. $((v_\alpha, 0), \alpha_+) \in \Gamma$,
II. $((v_\alpha, y_0), \alpha_-) \in \Gamma$,
III. $(0, y_0) \in \mathcal{L}$ *and either* $((v_\alpha, y_1), \alpha_+) \in \Gamma$ *or* $((v_\alpha, y_1), \alpha_-) \in \Gamma$, *for some* $y_1 \in \mathbf{R}$.

4.1.1. interpretation of the Theorem.

Let $(x, y) \in \mathbf{R}^{n+1}$ with $x \in \mathbf{R}^n$ and $y \in \mathbf{R}$. Theorem 4.1 refers to elements of Γ whose orthogonal component acts separately in the subspaces $x = 0$, along the direction of projection, and $y = 0$, where the projected functions are defined. In fact, both α_+ and α_- leave those subspaces invariant. Thus, Theorem 4.1 states that the symmetry of $\Pi_{y_0}(X_\Gamma)$ depends strongly on the relation between the elements of Γ and the direction of the projection.

After projection, and under some extra conditions we discuss below, both α_+ and α_- ensure a symmetry with orthogonal component $\alpha \in \mathbf{O}(n)$.

The conditions of Theorem 4.1 also concern the translations associated to either α_+ or α_- and the projection width y_0.

I. The elements of Γ whose orthogonal component fixes the one-dimensional subspace $x = 0$, *i.e.*, those elements with orthogonal part α_+ and translation $(v_\alpha, 0)$, act effectivelly in the subspace $y = 0$. This symmetry remains after the projection, whatever the projection width is.

II. The elements with orthogonal part α_- will contribute to the symmetry of $\Pi_{y_0}(X_\Gamma)$ if the associated translation is (v_α, y_0), *i.e.*, if it has the last component equal to the width of the projection. If the width of the projection changes then these elements lose their influence upon the symmetry group of the projected functions.

III. If $\mathcal{L}$ has one element in the direction of the projection and whose norm is the projection width, *i.e.*, if $(0, y_0) \in \mathcal{L}$, then all the elements with orthogonal component either α_+ or α_- will induce some symmetry for all the projected functions.

Under the conditions described, the translation that remains after projection corresponds to the n first components of the translation in the relevant elements of Γ.

For $n = 1$ and $\alpha = -1$ we obtain the conditions of Theorem 3.2, since in this case $\alpha_+ = -\sigma$ and $\alpha_- = -Id_2$. In the case $\alpha = 1$ we have $\alpha_+ = Id_2$ and $\alpha_- = \sigma$ yelding the conditions of Theorem 3.1. Here the two conditions asking for $(0, y_0) \in \mathcal{L}$ are grouped into one.

4.1.2. reformulation of the Theorem for symmetry groups. Theorem 4.1 states that the elements in Γ that effectivelly contribute to the symmetry of $\Pi_{y_0}(X_\Gamma)$ are those in the subgroup Γ' given by:

$$\Gamma' = \left\{ \left((v_\alpha, y_\alpha), \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \right) \in \Gamma : \right.$$
$$\left. : (y_\alpha = 0 \wedge \beta = 1) \vee (y_\alpha = y_0 \wedge \beta = -1) \vee (0, y_0) \in \mathcal{L} \right\}.$$

Let Σ be the largest subgroup of $\mathbf{E}(n) \cong \mathbf{R}^n \ltimes \mathbf{O}(n)$ that fixes all the elements in $\Pi_{y_0}(X_\Gamma)$. Thus, Theorem 4.1 defines the group homomorphism

$$\begin{array}{ccc} \Gamma' & \longrightarrow & \Sigma \\ \left((v_\alpha, y_\alpha), \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \right) & \longmapsto & (v_\alpha, \alpha) \end{array}$$

whose kernel is given by the elements in Γ' with $v_\alpha = 0$ and $\alpha = Id_n$.

4.2. Proof of Sufficiency in the Theorem

Each one of the conditions I, II and III of Theorem 4.1 is sufficient by Lemma 4.1, below. Its proof is easy to carry out since the only tools needed are basic properties of the integrals. This proof was omitted in chapter 3, we include it here for completeness.

LEMMA 4.1. *All functions in* $\Pi_{y_0}(X_\Gamma)$ *are invariant under the action of* $(v_\alpha, \alpha) \in \mathbf{R}^n \ltimes \mathbf{O}(n)$ *if one of the following conditions holds:*

I. $((v_\alpha, 0), \alpha_+) \in \Gamma$,
II. $((v_\alpha, y_0), \alpha_-) \in \Gamma$,
III. $(0, y_0) \in \mathcal{L}$ *and either* $((v_\alpha, y_1), \alpha_+) \in \Gamma$ *or* $((v_\alpha, y_1), \alpha_-) \in \Gamma$*, for some* $y_1 \in \mathbf{R}$ *.*

PROOF. Let f be any function in X_Γ and recall that $(v_\delta, \delta) \cdot f(x, y) = f((v_\delta, \delta)^{-1} \cdot (x, y))$ and that $(v_\delta, \delta)^{-1} = (-\delta^{-1} v_\delta, \delta^{-1})$, where (v_δ, δ) is an element of the Euclidean group of dimension $n + 1$.

If condition I holds then

$$f(x, y) = ((v_\alpha, 0), \alpha_+) \cdot f(x, y) = f(\alpha^{-1} x - \alpha^{-1} v_\alpha, y)$$

and so $\Pi_{y_0}(f)(x) = \int_0^{y_0} f(x, y) dy$ equals

$$\int_0^{y_0} f(\alpha^{-1}x - \alpha^{-1}v_\alpha, y) dy = \Pi_{y_0}(f)(\alpha^{-1}x - \alpha^{-1}v_\alpha) = (v_\alpha, \alpha) \cdot \Pi_{y_0}(f)(x).$$

Similarly, if condition II holds then

$$\Pi_{y_0}(f)(x) = \int_0^{y_0} f(\alpha^{-1}x - \alpha^{-1}v_\alpha, y_0 - y) dy$$

which, for $z = y_0 - y$, equals

$$\int_0^{y_0} f(\alpha^{-1}x - \alpha^{-1}v_\alpha, z) dz = (v_\alpha, \alpha) \cdot \Pi_{y_0}(f)(x).$$

If III holds then

$$\Pi_{y_0}(f)(x) = \int_0^{y_0} f(\alpha^{-1}x - \alpha^{-1}v_\alpha, \pm(y - y_1)) dy$$

which either equals $\int_{-y_1}^{y_0 - y_1} f(\alpha^{-1}x - \alpha^{-1}v_\alpha, z) dz$ or $\int_{y_1 - y_0}^{y_1} f(\alpha^{-1}x - \alpha^{-1}v_\alpha, z) dz$. Since f has period $(0, y_0)$, both integrals equal

$$\int_0^{y_0} f(\alpha^{-1}x - \alpha^{-1}v_\alpha, z) dz = (v_\alpha, \alpha) \cdot \Pi_{y_0}(f)(x).$$

□

4.3. Symmetry of $\Pi_{y_0}(X_\Gamma)$ Related to Γ and $\mathcal{L}^*$

Proving that the conditions of Theorem 4.1 are necessary will take the remainder of this chapter (excluding the last section which concerns the restriction). First, in Proposition 4.1 below we establish an equivalence between the (v_α, α)-invariance of all the functions in $\Pi_{y_0}(X_\Gamma)$ and a set of properties of the group Γ and of the dual lattice $\mathcal{L}^*$, as was done in Propositions 3.1 and 3.3 for the two-dimensional case. These properties will then impose restrictions on Γ and $\mathcal{L}$. More specifically, they will imply the presence of some particular elements in Γ, as in Theorem 4.1.

4.3.1. structure of $\mathcal{L}^*$ due to σ. The reflection σ fixes the subspace perpendicular to the direction of projection. Although there is no explicit reference to σ in Theorem 4.1, the cases when $\mathbf{J}$ contains this reflection will be treated separately. The simultaneous presence of (v_σ, σ) and of (v_+, α_+) in a group Γ imposes strong restrictions on $\mathcal{L}^*$. One of these restrictions is fundamental for the formulation and the proof of Proposition 4.1 and is presented in next Lemma.

LEMMA 4.2. *If both $(v_\sigma, \sigma) \in \Gamma$ and $(v_+, \alpha_+) \in \Gamma$ then $2(\sigma v_+ - v_+) \in \mathcal{L}$.*

PROOF. By definition of v_- and since $\sigma\alpha_+ = \alpha_- \in \mathbf{J}$ then $(v_\sigma, \sigma) \cdot (v_+, \alpha_+) = (v_\sigma + \sigma v_+, \alpha_-) \in \Gamma$ implies

$$v_\sigma + \sigma v_+ - v_- \in \mathcal{L}. \tag{10}$$

Moreover, $(v_+, \alpha_+) \cdot (v_\sigma, \sigma) = (v_+ + \alpha_+ v_\sigma, \alpha_-)$ implies

$$v_+ + \alpha_+ v_\sigma - v_- \in \mathcal{L}. \tag{11}$$

Subtracting expressions in (10) and (11), we obtain

$$v = v_\sigma + \sigma v_+ - v_+ - \alpha_+ v_\sigma \in \mathcal{L}.$$

As $\sigma\mathcal{L} = \{\sigma l : l \in \mathcal{L}\} = \mathcal{L}$ then $v - \sigma v = 2(\sigma v_+ - v_+) + (Id_{n+1} - \alpha_+ - \sigma + \alpha_-)v_\sigma$ also belongs to $\mathcal{L}$. A direct calculation shows that $-\alpha_+ - \sigma + \alpha_- = -Id_{n+1}$ and so $v - \sigma v = 2(\sigma v_+ - v_+)$ or, which is equivalent,

$$2 < k, \sigma v_+ - v_+ > \in \mathbf{Z} \quad \text{for all } k \in \mathcal{L}^*.$$

□

In the two-dimensional case $v_+ = v_{Id_2} = (0, 0)$ for $\alpha = 1$, and $v_+ = v_{-\sigma}$ for $\alpha = -1$. In the second case

$$(v_{-\sigma}, -\sigma)^2 = (v_{-\sigma} - \sigma v_{-\sigma}, Id_2) \in \Gamma$$

because $(-\sigma)^2 = Id_2$. Thus, $\sigma v_+ - v_+$ is always an element of $\mathcal{L}$, for $n = 1$.

4.3.2. how symmetry restricts the dual lattice $\mathcal{L}^*$.

PROPOSITION 4.1. *All functions in* $\Pi_{y_0}(X_\Gamma)$ *are invariant under the action of* $(v_\alpha, \alpha) \in \mathbf{R}^n \ltimes \mathbf{O}(n)$ *if and only if one of the following conditions holds:*

A. $(v_+, \alpha_+) \in \Gamma$ *and for each* $k \in \mathcal{L}^*$ *either* $< k, (0, y_0) > \in \mathbf{Z} - \{0\}$ *or* $< k, v_+ - (v_\alpha, 0) > \in \mathbf{Z}$,
B. $(v_-, \alpha_-) \in \Gamma$ *and for each* $k \in \mathcal{L}^*$ *either* $< k, (0, y_0) > \in \mathbf{Z} - \{0\}$ *or* $< k, v_- - (v_\alpha, y_0) > \in \mathbf{Z}$,
C. *both* $(v_\sigma, \sigma) \in \Gamma$ *and* $(v_+, \alpha_+) \in \Gamma$. *Moreover, if* $< k, \sigma v_+ - v_+ > \in \mathbf{Z}$ *then one of the conditions Ci), Cii) or Ciii) below holds and, if* $< k, \sigma v_+ - v_+ > +\frac{1}{2} \in \mathbf{Z}$, *one of the conditions Ci) or Civ) holds:*
   (i) $< k, (0, y_0) > \in \mathbf{Z} - \{0\}$,
   (ii) $< k, v_+ - (v_\alpha, 0) > \in \mathbf{Z}$,
   (iii) $< k, v_\sigma - (0, y_0) > +\frac{1}{2} \in \mathbf{Z}$,
   (iv) $< k, v_- - (v_\alpha, y_0) > \in \mathbf{Z}$ *and either* $< k, v_\sigma - (0, y_0) > +\frac{1}{4} \in \mathbf{Z}$ *or* $< k, v_\sigma - (0, y_0) > -\frac{1}{4} \in \mathbf{Z}$.

Note the increase in complexity compared to the corresponding propositions in the two-dimensional case, since condition C has one extra case. If $n = 1$, then $< k, \sigma v_+ - v_+ > +1/2 \in \mathbf{Z}$ never holds, which simplifies both the statement and the proof.

4.3.3. subsets of the dual lattice. As in chapter 3, a more concise formulation of this result is possible using the subsets of $\mathcal{L}^*$ that we proceed to define. Let $\mathcal{M}^*$, $\mathcal{M}^*_+$ and $\mathcal{M}^*_-$ be the modules

$$\begin{aligned} \mathcal{M}^* &= \{k \in \mathcal{L}^* : < k, \sigma v_+ - v_+ > \in \mathbf{Z}\}, \\ \mathcal{M}^*_+ &= \{k \in \mathcal{L}^* : < k, v_+ - (v_\alpha, 0) > \in \mathbf{Z}\} \text{ and} \\ \mathcal{M}^*_- &= \{k \in \mathcal{L}^* : < k, v_- - (v_\alpha, y_0) > \in \mathbf{Z}\}, \end{aligned}$$

and let

$$\begin{aligned} \mathcal{N}^* &= \left\{k \in \mathcal{L}^* : < k, \sigma v_+ - v_+ > +\frac{1}{2} \in \mathbf{Z}\right\}, \\ \mathcal{N}^*_{y_0} &= \{k \in \mathcal{L}^* : < k, (0, y_0) > \in \mathbf{Z} - \{0\}\}, \\ \mathcal{N}^*_\sigma &= \left\{k \in \mathcal{L}^* : < k, v_\sigma - (0, y_0) > +\frac{1}{2} \in \mathbf{Z}\right\} \text{ and} \\ \mathcal{N}^*_{\tilde\sigma} &= \left\{k \in \mathcal{L}^* : < k, v_\sigma - (0, y_0) > \pm\frac{1}{4} \in \mathbf{Z}\right\}. \end{aligned}$$

The last four sets are not modules. The smallest modules generated by each of them are, respectively,

$$\overline{\mathcal{N}^*} = \mathcal{N}^* \cup \mathcal{M}^*,$$

which equals $\mathcal{L}^*$ under the conditions of Lemma 4.2,

$$\overline{\mathcal{N}^*_{y_0}} = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_{y_0}, \quad \overline{\mathcal{N}^*_{\sigma}} = \mathcal{N}^*_{\sigma} \cup \mathcal{M}^*_{\sigma} \quad \text{and} \quad \overline{\mathcal{N}^*_{\tilde{\sigma}}} = \mathcal{N}^*_{\tilde{\sigma}} \cup \overline{\mathcal{N}^*_{\sigma}},$$

where all the unions are disjoint and $\mathcal{M}^*_{y_0}$ and $\mathcal{M}^*_{\sigma}$ are the modules

$$\mathcal{M}^*_{y_0} = \{k \in \mathcal{L}^* : < k, (0, y_0) >= 0\} \text{ and}$$
$$\mathcal{M}^*_{\sigma} = \{k \in \mathcal{L}^* : < k, v_\sigma - (0, y_0) > \in \mathbf{Z}\}\,.$$

We summarize below some properties of $\mathcal{N}^*_{\sigma}$ and $\mathcal{N}^*_{\tilde{\sigma}}$ that will be used in the sequel.

PROPERTIES OF $\mathcal{N}^*_{\sigma}$ AND $\mathcal{N}^*_{\tilde{\sigma}}$. *Let $m_1, m_2 \in \mathbf{Z}$.*

1. *If $g_1, g_2 \in \mathcal{N}^*_{\sigma}$ then*

$$m_1 g_1 + m_2 g_2 \in \begin{cases} \mathcal{M}^*_{\sigma} & \text{if } m_1 + m_2 \text{ even} \\ \mathcal{N}^*_{\sigma} & \text{if } m_1 + m_2 \text{ odd} \end{cases}.$$

2. *If $g_1, g_2 \in \mathcal{N}^*_{\tilde{\sigma}}$ then*

$$m_1 g_1 + m_2 g_2 \in \begin{cases} \overline{\mathcal{N}^*_{\sigma}} & \text{if } m_1 + m_2 \text{ even} \\ \mathcal{N}^*_{\tilde{\sigma}} & \text{if } m_1 + m_2 \text{ odd} \end{cases}.$$

4.3.4. reformulation of the Proposition. Proposition 4.1 may therefore be written the following way:

PROPOSITION 4.2. *All functions in $\Pi_{y_0}(X_\Gamma)$ are invariant under the action of $(v_\alpha, \alpha) \in \mathbf{R}^n \ltimes \mathbf{O}(n)$ if and only if one of the following conditions holds:*

A. $(v_+, \alpha_+) \in \Gamma$ *and* $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+$,
B. $(v_-, \alpha_-) \in \Gamma$ *and* $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_-$,
C. *both* (v_σ, σ) *and* (v_+, α_+) *belong to* Γ *and, moreover,*

$$\mathcal{M}^* \subset \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+ \cup \mathcal{N}^*_{\sigma}\right) \text{ and } \mathcal{N}^* \subset \left(\mathcal{N}^*_{y_0} \cup \left(\mathcal{M}^*_- \cap \mathcal{N}^*_{\tilde{\sigma}}\right)\right).$$

When condition C holds it follows that $\mathcal{L}^* = \mathcal{N}^* \cup \mathcal{M}^*$, by Lemma 4.2. Condition C is simpler in Propositions 3.2 and 3.4, since $\mathcal{N}^* = \emptyset$ for $n = 1$.

4.3.5. structure for the proof of the Proposition. There are three main steps in the proof of Proposition 4.1. First, in Lemma 4.3, we write the (v_α, α)-invariance of the projection of $f \in X_\Gamma$ as a relation between the width y_0 of the projection band, the projection of the dual lattice $\mathcal{L}^*$ and the coefficients of the formal Fourier expansion of f in waves. Second, we prove that the conditions A, B and C are sufficient by making the restrictions they imply in $\mathcal{L}^*$ and in the Fourier coefficients explicit. Finally we conclude that the conditions of Proposition 4.1 are also necessary by the (v_α, α)-invariance of the projection of I_k, whose real and imaginary components are the simplest Γ-invariant functions. This last part is divided into lemmas.

The main tools used in this proof are properties of waves and of Fourier coefficients, due to the symmetries in Γ and to the symmetry $(v_\alpha, \alpha) \in \mathbf{R}^n \ltimes \mathbf{O}(n)$, together with properties of the modules and subsets of $\mathcal{L}^*$ defined above.

Recall that if $f \in X_\Gamma$ then

$$f(x,y) = \sum_{k \in \mathcal{L}^*} \omega_k(x,y) C(k)$$

and

$$\Pi_{y_0}(f)(x) = \sum_{k_1 \in \mathcal{L}_1^*} \omega_{k_1}(x) D(k_1),$$

where

$$\mathcal{L}_1^* = \{k_1 : (k_1, k_2) \in \mathcal{L}^*\}$$

and

$$D(k_1) = \sum_{k_2 : (k_1,k_2) \in \mathcal{L}^*} C(k_1, k_2) \int_0^{y_0} \omega_{k_2}(y) dy.$$

For any $k, v \in \mathbf{R}^{n+1}$ and any $\delta \in \mathbf{O}(n+1)$ the wave $\omega_{\delta k}(v)$ satisfies

$$\omega_{\delta k}(v) = \omega_k(\delta^{-1} v) \tag{12}$$

due to the orthogonality of δ, and this property will be used often in this chapter.

4.3.6. symmetry of $\Pi_{y_0}(X_\Gamma)$ related to $\mathcal{L}_1^*$. With the above notation and for $\alpha \mathcal{L}_1^* = \{\alpha k_1 : k_1 \in \mathcal{L}_1^*\}$, we have:

LEMMA 4.3. *Let $f \in X_\Gamma$ and $(v_\alpha, \alpha) \in \mathbf{R}^n \ltimes \mathbf{O}(n)$. The projection $\Pi_{y_0}(f)(x)$ is (v_α, α)-invariant if and only if for each $k_1 \in \mathcal{L}_1^*$ the following conditions hold:*

1. *if $k_1 \in \mathcal{L}_1^* \cap \alpha \mathcal{L}_1^*$ then $D(k_1) = \omega_{k_1}(-v_\alpha) D(\alpha^{-1} k_1)$,*
2. *if $k_1 \notin \mathcal{L}_1^* \cap \alpha \mathcal{L}_1^*$ then $D(k_1) = 0$.*

Proof. Notice first that the equality

$$\Pi_{y_0}(f)(x) = (v_\alpha, \alpha) \cdot \Pi_{y_0}(f)(x) = \Pi_{y_0}(f)(\alpha^{-1}x - \alpha^{-1}v_\alpha)$$

is equivalent to

$$\sum_{k_1 \in \mathcal{L}_1^*} \omega_{k_1}(x) D(k_1) = \sum_{k_1 \in \mathcal{L}_1^*} \omega_{k_1}(\alpha^{-1}x)\omega_{k_1}(-\alpha^{-1}v_\alpha) D(k_1), \tag{13}$$

where the right hand side equals, by property (12),

$$\sum_{k_1 \in \mathcal{L}_1^*} \omega_{\alpha k_1}(x)\omega_{\alpha k_1}(-v_\alpha) D(k_1)$$

and, for $\tilde{k}_1 = \alpha k_1$,

$$= \sum_{\tilde{k}_1 \in \alpha\mathcal{L}_1^*} \omega_{\tilde{k}_1}(x)\omega_{\tilde{k}_1}(-v_\alpha) D(\alpha^{-1}\tilde{k}_1).$$

Thus, by the unicity of the Fourier expansion, expression (13) is valid for all $x \in \mathbf{R}^n$ if and only if, for any $k_1 \in \mathcal{L}_1^*$, the conditions hold. □

If $n = 1$ then $\alpha\mathcal{L}_1^* = \mathcal{L}_1^*$ and the first condition of Lemma 4.3 above, must hold. For $\alpha = 1$ this condition is equivalent to $D(k_1)(1 - \omega_{k_1}(-v_\alpha)) = 0$, see conditions A' and B' in page 42 of chapter 3. If $\alpha = -1$ then the first condition of Lemma 4.3 becomes $D(k_1) = \omega_{k_1}(-v_\alpha)D(-k_1)$, *i.e.*, equation (4) in page 55 of chapter 3.

4.4. Proof of Sufficiency in the Proposition

We treat each case separately. In each case we prove that $D(k_1) - \omega_{k_1}(-v_\alpha)D(\alpha^{-1}k_1)$, in the form

$$\sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} C(k_1,k_2)G(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy,$$

is zero, with an explicit computation of $G(k_1, k_2)$, as in the corresponding proofs in chapter 3.

Proof — sufficiency of condition A. Suppose condition A happens. Since $\alpha_+ = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ belongs to $\mathbf{J}$ then $\alpha_+\mathcal{L}^* = \{\alpha_+ k : k \in \mathcal{L}^*\} = \mathcal{L}^*$, which implies $\alpha\mathcal{L}_1^* = \mathcal{L}_1^*$. Therefore, for any $f \in X_\Gamma$, the projection $\Pi_{y_0}(f)(x) = \sum_{k_1\in\mathcal{L}_1^*}\omega_{k_1}(x)D(k_1)$ is (v_α, α)-invariant if and only if condition 1 of Lemma 4.3 is valid for all $k_1 \in \mathcal{L}_1^*$.

The (v_+, α_+)-invariance of f implies $C(k) = \omega_k(-v_+)C(\alpha^{-1}k)$ for all its Fourier coefficients and so

$$D(\alpha^{-1}k_1) = \sum_{k_2:(\alpha^{-1}k_1,k_2)\in\mathcal{L}^*} C(\alpha^{-1}k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy$$

$$= \sum_{k_2:(\alpha^{-1}k_1,k_2)\in\mathcal{L}^*} \omega_k(v_+)C(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy.$$

This equals

$$\sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} \omega_k(v_+)C(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy$$

because $\{k_2 : (k_1,k_2)\in\mathcal{L}^*\} = \{k_2 : (\alpha^{-1}k_1,k_2)\in\mathcal{L}^*\}$.

Thus, for all $k_1 \in \mathcal{L}_1^*$,

$$D(k_1) - \omega_{k_1}(-v_\alpha)D(\alpha^{-1}k_1) =$$

$$= \sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} (1-\omega_k(v_+ - (v_\alpha,0)))\,C(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy$$

which is zero because either $\int_0^{y_0}\omega_{k_2}(y)dy = 0$, if $< k,(0,y_0) > \in \mathbf{Z} - \{0\}$, or $1-\omega_k(v_+ - (v_\alpha,0)) = 0$ for $< k, v_+ - (v_\alpha,0) > \in \mathbf{Z}$. □

Proof — sufficiency of condition B. If condition B is valid then $\alpha_- = \begin{pmatrix} \alpha & 0 \\ 0 & -1 \end{pmatrix} \in \mathbf{J}$ and so $\alpha\mathcal{L}_1^* = \mathcal{L}_1^*$ as in the previous case. Moreover,

$$D(\alpha^{-1}k_1) = \sum_{k_2:(\alpha^{-1}k_1,-k_2)\in\mathcal{L}^*} C(\alpha_{-1}k_1,-k_2)\int_0^{y_0}\omega_{-k_2}(y)dy$$

$$= \sum_{k_2:(\alpha^{-1}k_1,-k_2)\in\mathcal{L}^*} \omega_k(v_-)C(k_1,k_2)\int_0^{y_0}\omega_{-k_2}(y)dy.$$

As $\{k_2 : (k_1,k_2)\in\mathcal{L}^*\} = \{k_2 : (\alpha^{-1}k_1,-k_2)\in\mathcal{L}^*\}$ and using the property

$$\int_0^{y_0}\omega_{-k_2}(y)dy = \omega_{k_2}(-y_0)\int_0^{y_0}\omega_{k_2}(y)dy, \tag{14}$$

the above expression becomes

$$D(\alpha^{-1}k_1) = \sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} \omega_k(v_-)\omega_{k_2}(-y_0)C(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy.$$

Therefore, for all $k_1 \in \mathcal{L}_1^*$,

$$D(k_1) - \omega_{k_1}(-v_\alpha)D(\alpha^{-1}k_1) =$$

$$= \sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} (1-\omega_k(v_- - (v_\alpha,y_0)))\,C(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy.$$

All the terms of the summation are zero because either $\int_0^{y_0}\omega_{k_2}(y)dy = 0$, if $< k,(0,y_0) > \in \mathbf{Z} - \{0\}$, or $1-\omega_k(v_- - (v_\alpha,y_0)) = 0$ for $< k, v_- - (v_\alpha,y_0) > \in \mathbf{Z}$. □

PROOF — SUFFICIENCY OF CONDITION C. When C happens then $\sigma = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix} \in \mathbf{J}$ and so $(k_1, -k_2) \in \mathcal{L}^*$ if $(k_1, k_2) \in \mathcal{L}^*$. Thus $D(k_1)$ is

$$\frac{1}{2} \sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} \left(C(k_1,k_2) \int_0^{y_0} \omega_{k_2}(y)dy + C(k_1,-k_2) \int_0^{y_0} \omega_{-k_2}(y)dy \right)$$

$$= \frac{1}{2} \sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} \left(1 + \omega_k(v_\sigma)\omega_{k_2}(-y_0)\right) C(k_1,k_2) \int_0^{y_0} \omega_{k_2}(y)dy,$$

by property (14), and $D(\alpha^{-1}k_1)$ equals

$$\frac{1}{2} \sum_{k_2:(\alpha^{-1}k_1,k_2)\in\mathcal{L}^*} \left(C(\alpha^{-1}k_1,k_2) \int_0^{y_0} \omega_{k_2}(y)dy + C(\alpha^{-1}k_1,-k_2) \int_0^{y_0} \omega_{-k_2}(y)dy \right)$$

$$= \frac{1}{2} \sum_{k_2:(\alpha^{-1}k_1,k_2)\in\mathcal{L}^*} \left(\omega_k(v_+) + \omega_k(v_-)\omega_{k_2}(-y_0)\right) C(k_1,k_2) \int_0^{y_0} \omega_{k_2}(y)dy$$

by the invariance of f under the action of (v_+, α_+) and (v_-, α_-), as $\alpha_- = \sigma\alpha_+ \in \mathbf{J}$.

The condition 1 of Lemma 4.3 is valid for all $k_1 \in \mathcal{L}_1^*$ because the terms of the summation in expression

$$D(k_1) - \omega_{k_1}(-v_\alpha) D(\alpha^{-1}k_1) = \sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} G(k_1,k_2) C(k_1,k_2) \int_0^{y_0} \omega_{k_2}(y)dy,$$

where

$$G(k_1,k_2) = 1 + \omega_k(v_\sigma)\omega_{k_2}(-y_0) - \omega_{k_1}(-v_\alpha)\left(\omega_k(v_+) + \omega_k(v_-)\omega_{k_2}(-y_0)\right),$$

vanish. If c1) happens then $\int_0^{y_0} \omega_{k_2}(y)dy = 0$, otherwise, we show below that $G(k_1,k_2) = 0$ if k verifies any other condition of case C.

First notice that the hypotheses of Lemma 4.2 are valid and expression (10) implies

$$\omega_k(v_-) = \omega_k(v_\sigma)\omega_k(\sigma v_+). \tag{15}$$

If $< k, \sigma v_+ - v_+ > \in \mathbf{Z}$ then $\omega_k(\sigma v_+ - v_+) = 1$ and $G(k_1,k_2)$ equals, using (15),

$$1 + \omega_k(v_\sigma)\omega_{k_2}(-y_0) - \omega_{k_1}(-v_\alpha)\omega_k(v_+)\left(1 + \omega_k(\sigma v_+ - v_+)\omega_k(v_\sigma)\omega_{k_2}(-y_0)\right)$$

$$= \left(1 - \omega_k(v_+ - (v_\alpha, 0))\right)\left(1 + \omega_k(v_\sigma - (0, y_0))\right) = 0$$

because either $1 - \omega_k(v_+ - (v_\alpha, 0)) = 0$, by condition Cii), or $1 + \omega_k(v_\sigma - (0, y_0)) = 0$, by Ciii).

If $< k, \sigma v_+ - v_+ > + \frac{1}{2} \in \mathbf{Z}$ then $\omega_k(\sigma v_+)\omega_k(-v_+) = -1$ and

$$\begin{aligned} \omega_{k_1}(-v_\alpha)\omega_k(v_+) &= -\omega_{k_1}(-v_\alpha)\omega_k(\sigma v_+) \\ &= -\omega_{k_1}(-v_\alpha)\omega_k(v_-)\omega_k(-v_\sigma), \text{ by expression (15)} \\ &= -\omega_k(v_- - (v_\alpha, y_0))\omega_k(-v_\sigma + (0, y_0)). \end{aligned}$$

Thus $G(k_1, k_2)$ is

$$1 + \omega_k(v_\sigma - (0, y_0)) + \omega_k(v_- - (v_\alpha, y_0)) \left(\omega_k(-v_\sigma + (0, y_0)) - 1\right) = 0$$

because, by condition Civ), $\omega_k(v_\sigma - (0, y_0)) = \pm i$ and $\omega_k(v_- - (v_\alpha, y_0)) = 1$. Notice that we use the property $\omega_k(-v) = \overline{\omega_k(v)}$ in order to obtain this result. □

4.5. Proof of Necessity in the Proposition

PROOF — NECESSITY IN PROPOSITION 4.1. We want to show that if the hypothesis of Proposition 4.1 holds for the simplest invariant functions in X_Γ, the real and imaginary parts of

$$I_k(x, y) = \sum_{\delta \in \mathbf{J}} \omega_{\delta k}(-v_\delta)\omega_{\delta k}(x, y),$$

then one of the three conditions A, B or C must hold.

For $\delta \in \mathbf{O}(n+1)$ and $k \in \mathcal{L}^*$, let $\delta k = (\tilde{k}_1, \tilde{k}_2)$, where $\tilde{k}_1 \in \mathbf{R}^n$ and $\tilde{k}_2 \in \mathbf{R}$. With the notation $\delta k|_1 = \tilde{k}_1$ and $\delta k|_2 = \tilde{k}_2$, the projections of the I_k, with $k \in \mathcal{L}^*$, have the form

$$\Pi_{y_0}(I_k)(x) = \sum_{\delta \in \mathbf{J}} \omega_{\delta k|_1}(x) D'(\delta, k)$$

where

$$D'(\delta, k) = \omega_{\delta k}(-v_\delta) \int_0^{y_0} \omega_{\delta k|_2}(y)dy.$$

This corresponds to a summation over the projection of the orbit $\mathbf{J}k$ given by:

$$\mathbf{J}k|_1 = \{\delta k|_1 : \ \delta \in \mathbf{J}\}.$$

Grouping the terms with the same first n components, we obtain

$$\Pi_{y_0}(I_k)(x) = \sum_{\tilde{k}_1 \in \mathbf{J}k|_1} \omega_{\tilde{k}_1}(x) \sum_{\tilde{k}_2 : (\tilde{k}_1, \tilde{k}_2) \in \mathbf{J}k} D'(\delta, \tilde{k}).$$

In particular, for $k = (k_1, k_2)$, the Fourier coefficient of $\Pi_{y_0}(I_k)$ associated to ω_{k_1} is

$$\sum_{\delta \in \mathrm{J}^{Id}(k)} D'(\delta, k),$$

where $\mathrm{J}^{Id}(k)$ is the subset of $\mathbf{J}$ which preserves k_1:

$$\mathrm{J}^{Id}(k) = \{\delta \in \mathbf{J} : \ \delta k|_1 = k_1\}.$$

Analogously, we define

$$\mathrm{J}^{\alpha}(k) = \left\{\delta \in \mathbf{J} : \ \delta k|_1 = \alpha^{-1} k_1\right\}.$$

Since $\Pi_{y_0}(I_k)$ is (v_α, α)-invariant, by hypothesis, then Lemma 4.3 holds and therefore, for all $k = (k_1, k_2) \in \mathcal{L}^*$, we have:

a. if $k_1 \in \mathcal{L}_1^* \cap \alpha\mathcal{L}_1^*$ then

$$\sum_{\delta \in \mathrm{J}^{Id}(k)} D'(\delta, k) = \omega_{k_1}(-v_\alpha) \sum_{\delta \in \mathrm{J}^{\alpha}(k)} D'(\delta, k),$$

b. if $k_1 \notin \mathcal{L}_1^* \cap \alpha\mathcal{L}_1^*$ then

$$\sum_{\delta \in \mathrm{J}^{Id}(k)} D'(\delta, k) = 0.$$

The rest of the proof is divided in three Lemmas. Although these conditions involve the sets $\mathrm{J}^{Id}(k)$ and $\mathrm{J}^{\alpha}(k)$ for all $k \in \mathcal{L}^*$, we show in Lemmas 4.4, 4.5 and 4.6 below that for this proof we will only need the elements of $\mathbf{J}$ that lie in the following subsets:

$$\mathbf{J}^{Id} = \{Id_{n+1}, \sigma\} \cap \mathbf{J} \quad \text{and} \quad \mathbf{J}^{\alpha} = \left\{\alpha_+^{-1}, \alpha_-^{-1}\right\} \cap \mathbf{J}.$$

In Lemma 4.4 we describe all the possibilities for $\mathbf{J}^{Id}$ and $\mathbf{J}^{\alpha}$ and obtain in each case some consequences for $\mathcal{L}^*$ in terms of the subsets defined before the statement of Proposition 4.2.

In Lemma 4.5 we study the set of all $k \in \mathcal{L}^*$ such that either $\mathrm{J}^{Id}(k) \neq \mathbf{J}^{Id}$ or $\mathrm{J}^{\alpha}(k) \neq \mathbf{J}^{\alpha}$. Finally, conditions A, B and C are obtained in Lemma 4.6. □

4.5.1. the sets $\mathbf{J}^{Id}$ and $\mathbf{J}^{\alpha}$.

PROPERTIES OF $\mathrm{J}^{Id}(k)$ AND $\mathrm{J}^{\alpha}(k)$. *Let* $k \in \mathcal{L}^*$.

1. $\mathrm{J}^{Id}(k) = \{\delta \in \mathbf{J} : \ \delta k = k \vee \delta k = \sigma k\}$ *and* $\mathrm{J}^{\alpha}(k) = \left\{\delta \in \mathbf{J} : \ \delta k = \alpha_+^{-1} k \vee \delta k = \alpha_-^{-1} k\right\}$.
2. $\mathbf{J}^{Id} \subset \mathrm{J}^{Id}(k)$, $\mathbf{J}^{\alpha} \subset \mathrm{J}^{\alpha}(k)$ *and* $\mathrm{J}^{Id}(0,0) = \mathrm{J}^{\alpha}(0,0) = \mathbf{J}$.

PROOF. Property 1, for $\mathrm{J}^{Id}(k)$, follows by orthogonality of $\mathbf{J}$, since any element of the orbit $\mathbf{J}(k_1, k_2)$ whose n first components equal k_1 is of the form $(k_1, \pm k_2)$. For $\mathrm{J}^{\alpha}(k)$, the elements on $\mathbf{J}(k_1, k_2)$ with n first components $\alpha^{-1}k_1$ are of the form $(\alpha^{-1}k_1, \pm k_2)$, by orthogonality of $\mathbf{J}$ and of α.

Property 2 follows directly from the previous one and from the definitions of $\mathbf{J}^{Id}$ and $\mathbf{J}^{\alpha}$. □

In the two-dimensional case $\mathbf{J}^{Id} = \mathbf{J}^+$ and either $\mathbf{J}^{\alpha} = \mathbf{J}^+$, for $\alpha = 1$, or $\mathbf{J}^{\alpha} = \mathbf{J}^-$, for $\alpha = -1$.

4.5.2. the set $\mathcal{O}^*$. Lemma 4.4 describes, under the hypothesis of Proposition 4.1, the structure of the set

$$\mathcal{O}^* = \left\{ k \in \mathcal{L}^* : \; \mathrm{J}^{Id}(k) = \mathbf{J}^{Id} \;\wedge\; \mathrm{J}^{\alpha}(k) = \mathbf{J}^{\alpha} \right\}$$

according to each of the possible cases for $\mathbf{J}^{Id}$ and $\mathbf{J}^{\alpha}$. In its proof we use the definition of $\mathcal{O}^*$ in order to simplify conditions a and b. Applying the properties of the waves we will be able to restate these conditions in terms of the submodules and subsets of $\mathcal{L}^*$ previously defined.

LEMMA 4.4. *Suppose that*

a. *if* $k_1 \in \mathcal{L}_1^* \cap \alpha \mathcal{L}_1^*$ *then*

$$\sum_{\delta \in \mathrm{J}^{Id}(k)} D'(\delta, k) = \omega_{k_1}(-v_\alpha) \sum_{\delta \in \mathrm{J}^{\alpha}(k)} D'(\delta, k) \text{ and}$$

b. *if* $k_1 \notin \mathcal{L}_1^* \cap \alpha \mathcal{L}_1^*$ *then*

$$\sum_{\delta \in \mathrm{J}^{Id}(k)} D'(\delta, k) = 0,$$

for all $k = (k_1, k_2) \in \mathcal{L}^*$. *Then one of the following cases holds:*

1. $\mathbf{J}^{Id} = \{Id_{n+1}\}$, $\mathbf{J}^{\alpha} = \emptyset$ *and* $\mathcal{O}^* \subset \mathcal{N}^*_{y_0}$,
2. $\mathbf{J}^{Id} = \{Id_{n+1}, \sigma\}$, $\mathbf{J}^{\alpha} = \emptyset$ *and* $\mathcal{O}^* \subset \left(\mathcal{N}^*_{y_0} \cup \mathcal{N}^*_{\sigma}\right)$,
3. $\mathbf{J}^{Id} = \{Id_{n+1}\}$, $\mathbf{J}^{\alpha} = \{\alpha_+^{-1}\}$ *and* $\mathcal{O}^* \subset \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_{+}\right)$,
4. $\mathbf{J}^{Id} = \{Id_{n+1}\}$, $\mathbf{J}^{\alpha} = \{\alpha_-^{-1}\}$ *and* $\mathcal{O}^* \subset \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_{-}\right)$,
5. $\mathbf{J}^{Id} = \{Id_{n+1}, \sigma\}$, $\mathbf{J}^{\alpha} = \{\alpha_+^{-1}, \alpha_-^{-1}\}$,
   $(\mathcal{O}^* \cap \mathcal{M}^*) \subset \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_{+} \cup \mathcal{N}^*_{\sigma}\right)$ *and*
   $(\mathcal{O}^* \cap \mathcal{N}^*) \subset \left(\mathcal{N}^*_{y_0} \cup \left(\mathcal{M}^*_{-} \cap \mathcal{N}^*_{\tilde{\sigma}}\right)\right)$.

PROOF. Cases 1 to 5 enumerate all the possibilities for $\mathbf{J}^{Id}$ and $\mathbf{J}^{\alpha}$. This happens because $\mathbf{J}^{Id}$ is a group; if $\alpha_+^{-1}, \alpha_-^{-1} \in \mathbf{J}$ then $\alpha_+ \alpha_-^{-1} = \sigma \in \mathbf{J}$ and if $\sigma \in \mathbf{J}$ then either $\mathbf{J}^{\alpha} = \emptyset$ or $\mathbf{J}^{\alpha}$ has two elements.

In this proof we will use the property

$$(16) \qquad \omega_k(-\sigma v_\sigma) = \omega_k(v_\sigma) \quad \text{if } k \in \mathcal{L}^* \text{ and } (v_\sigma, \sigma) \in \Gamma,$$

and this hods because if $(v_\sigma, \sigma) \in \Gamma$ then

$$(v_\sigma, \sigma) \cdot (v_\sigma, \sigma) = (v_\sigma + \sigma v_\sigma, I) \in \Gamma$$

$$\Rightarrow v_\sigma + \sigma v_\sigma \in \mathcal{L}.$$

If $\mathbf{J}^{\alpha} = \emptyset$ then, for all $k = (k_1, k_2) \in \mathcal{O}^*$, the conditions in the hypothesis of the lemma become $\sum_{\delta \in \mathbf{J}^{Id}} D'(\delta, k) = 0$. Thus, either

$\int_0^{y_0} \omega_{k_2}(y)dy + \omega_{\sigma k}(-v_\sigma) \int_0^{y_0} \omega_{-k_2}(y)dy = 0$ or $\int_0^{y_0} \omega_{k_2}(y)dy = 0$, according to the presence or absence of σ in $\mathbf{J}$. Using (14), the first case is equivalent to

$$(1 + \omega_{\sigma k}(-v_\sigma)\omega_{k_2}(-y_0)) \int_0^{y_0} \omega_{k_2}(y)dy = 0 \Leftrightarrow$$

$$\Leftrightarrow (1 + \omega_k(v_\sigma - (0, y_0))) \int_0^{y_0} \omega_{k_2}(y)dy = 0,$$

by properties (12) and (16). Cases 1 and 2 follow because $\int_0^{y_0} \omega_{k_2}(y)dy = 0$ implies $k \in \mathcal{N}_{y_0}^*$ and $1 + \omega_k(v_\sigma - (0, y_0)) = 0$ implies $k \in \mathcal{N}_\sigma^*$.

In the remaining cases either α_+ or α_- belong to $\mathbf{J}$. Thus, $\alpha\mathcal{L}_1^* = \mathcal{L}_1^*$ and the first condition in the hypothesis of the lemma must be verified for all $k_1 \in \mathcal{L}_1^*$. For $k \in \mathcal{O}^*$ this condition becomes

$$\sum_{\delta \in \mathbf{J}^{Id}} D'(\delta, k) = \omega_{k_1}(-v_\alpha) \sum_{\delta \in \mathbf{J}^\alpha} D'(\delta, k). \tag{17}$$

Recall that $(v_\delta, \delta)^{-1} = (-\delta^{-1}v_\delta, \delta^{-1})$ when writing $D'(\delta, k)$ explicitly, below. In case 3, the condition above is

$$\left(1 - \omega_{k_1}(-v_\alpha)\omega_{\alpha_+^{-1}k}(\alpha_+^{-1}v_+)\right) \int_0^{y_0} \omega_{k_2}(y)dy = 0$$

and, by property (12), is equivalent to

$$(1 - \omega_{k_1}(-v_\alpha)\omega_k(v_+)) \int_0^{y_0} \omega_{k_2}(y)dy = 0.$$

The result follows because $1 - \omega_{k_1}(-v_\alpha)\omega_k(v_+) = 0$ implies $k \in \mathcal{M}_+^*$.

For case 4, condition (17) is equivalent to

$$\int_0^{y_0} \omega_{k_2}(y)dy - \omega_{k_1}(-v_\alpha)\omega_{\alpha_-^{-1}k}(\alpha_-^{-1}v_-) \int_0^{y_0} \omega_{-k_2}(y)dy = 0 \Leftrightarrow$$

$$\Leftrightarrow (1 - \omega_{k_1}(-v_\alpha)\omega_k(v_-)\omega_{k_2}(-y_0)) \int_0^{y_0} \omega_{k_2}(y)dy = 0.$$

Thus, either $k \in \mathcal{N}_{y_0}^*$ or $1 - \omega_{k_1}(-v_\alpha)\omega_k(v_-)\omega_{k_2}(-y_0) = 0$, which implies $k \in \mathcal{M}_-^*$.

Condition (17) is, for case 5, equivalent to

$$\int_0^{y_0} \omega_{k_2}(y)dy + \omega_{\sigma k}(-v_\sigma) \int_0^{y_0} \omega_{-k_2}(y)dy -$$

$$-\omega_{k_1}(-v_\alpha)\left(\omega_{\alpha_+^{-1}k}(\alpha_+^{-1}v_+)\int_0^{y_0}\omega_{k_2}(y)dy + \omega_{\alpha_-^{-1}k}(\alpha_-^{-1}v_-)\int_0^{y_0}\omega_{-k_2}(y)dy\right) = 0,$$

which, by properties (12), (14) and (16), has the form

$$G(k_1, k_2) \int_0^{y_0} \omega_{k_2}(y)dy = 0,$$

where

$$G(k_1,k_2) = 1+\omega_k(v_\sigma)\omega_{k_2}(-y_0)-\omega_{k_1}(-v_\alpha)\left(\omega_k(v_+)+\omega_k(v_-)\omega_{k_2}(-y_0)\right),$$

as in the proof of Proposition 4.1. Therefore, either $k \in \mathcal{N}^*_{y_0}$ or $G(k_1,k_2) = 0$.

In case 5 we are under the conditions of Lemma 4.2 and so $\mathcal{O}^* \subset (\mathcal{M}^* \cup \mathcal{N}^*)$. If $k = (k_1,k_2) \in \mathcal{M}^*$ then $G(k_1,k_2) = 0$ is equivalent, as shown in the proof of Proposition 4.1, to

$$(1-\omega_k(v_+ - (v_\alpha,0)))\,(1+\omega_k(v_\sigma-(0,y_0))) = 0$$

and the result follows. For $k = (k_1,k_2) \in \mathcal{N}^*$, the term $G(k_1,k_2)$ equals, by the proof of Proposition 4.1,

$$1+\omega_k(v_\sigma-(0,y_0))+\omega_k(v_- - (v_\alpha,y_0))\left(\overline{\omega_k(v_\sigma-(0,y_0))}-1\right).$$

Equation $G(k_1,k_2) = 0$ is equivalent, for $\omega_k(v_\sigma - (0,y_0)) = z_1$ and $\omega_k(v_- - (v_\alpha,y_0)) = z_2$, to

$$\frac{1+z_1}{1-\overline{z_1}} = z_2$$

because $z_1 = 1$ is not a solution of $G(k_1,k_2) = 0$. Therefore,

$$\left|\frac{1+z_1}{1-\overline{z_1}}\right| = 1$$

which implies

$$\mathrm{Re}(z_1) = 0 \Leftrightarrow \omega_k(v_\sigma-(0,y_0)) = \pm i \text{ and}$$

$$z_2 = \omega_k(v_- - (v_\alpha,y_0)) = 1,$$

leading to $k \in \left(\mathcal{M}^*_- \cap \mathcal{N}^*_{\tilde{\sigma}}\right)$. □

4.5.3. the set $\mathcal{P}^*$. Let $\mathcal{P}^*$ be the complement of $\mathcal{O}^*$ in $\mathcal{L}^*$:

$$\mathcal{P}^* = \left\{k \in \mathcal{L}^* : \ \mathrm{J}^{Id}(k) \neq \mathbf{J}^{Id} \ \vee \ \mathrm{J}^\alpha(k) \neq \mathbf{J}^\alpha\right\}.$$

In Lemma 4.6 we reformulate the cases of Lemma 4.4 in terms of $\mathcal{L}^*$ instead of $\mathcal{O}^*$. We show that the first two cases of Lemma 4.4 cannot occur because, under their conditions, $\mathcal{P}^*$ is too small to be a complement of $\mathcal{O}^*$. In the remaining cases we show that $\mathcal{P}^*$ can be ignored and, therefore, that $\mathcal{L}^*$ can be substituted for $\mathcal{O}^*$ in the expressions given. Thus, the estimate of the size of $\mathcal{P}^*$ in the next lemma is an essential step.

Lemma 4.5. *$\mathcal{P}^*$ is contained in the union of a finite number of vector subspaces of $\mathbf{R}^{n+1}$ with codimension at least one.*

PROOF. $\mathcal{P}^*$ is the union of the submodules

$$\bigcup_{\delta \in \mathbf{J}-\mathbf{J}^{Id}} \mathcal{M}^*_{\delta,Id} \;\cup \bigcup_{\delta \in \mathbf{J}-\mathbf{J}^{\alpha}} \mathcal{M}^*_{\delta,\alpha}$$

where

$$\mathcal{M}^*_{\delta,Id} = \left\{k \in \mathcal{L}^* : \delta \in \mathrm{J}^{Id}(k)\right\} \text{ and } \mathcal{M}^*_{\delta,\alpha} = \left\{k \in \mathcal{L}^* : \delta \in \mathrm{J}^{\alpha}(k)\right\}.$$

This union is finite because $\mathbf{J}$ is a finite group.

Moreover, for all $\xi \in \mathbf{O}(n+1)$,

$$\mathrm{Fix}(\xi) = \left\{(x,y) \in \mathbf{R}^{n+1} : \xi(x,y) = (x,y)\right\}$$

is a vector subspace of $\mathbf{R}^{n+1}$ and $\mathrm{Fix}(\xi) = \mathbf{R}^{n+1} \Leftrightarrow \xi = Id_{n+1}$.

Let $\delta \in \mathbf{J} - \mathbf{J}^{Id}$. If $k \in \mathcal{M}^*_{\delta,Id}$ then either $\delta k = k$ or $\delta k = \sigma k \Leftrightarrow \sigma\delta k = k$, which implies

$$\mathcal{M}^*_{\delta,Id} \subset \left(\mathrm{Fix}(\delta) \cup \mathrm{Fix}(\sigma\delta)\right).$$

Moreover, neither $\delta = Id_{n+1}$ nor $\sigma\delta = Id_{n+1}$, by the hypothesis $\delta \in \mathbf{J} - \mathbf{J}^{Id}$. Thus, the codimensions of subspaces the $\mathrm{Fix}(\delta)$ and $\mathrm{Fix}(\sigma\delta)$ are at least one.

Analogously, if $\delta \in \mathbf{J} - \mathbf{J}^{\alpha}$ and $k \in \mathcal{M}^*_{\delta,\alpha}$ then either $\delta k = \alpha_+^{-1} k \Leftrightarrow \alpha_+\delta k = k$ or $\delta k = \alpha_-^{-1} k \Leftrightarrow \alpha_-\delta k = k$. Therefore,

$$\mathcal{M}^*_{\delta,\alpha} \subset \left(\mathrm{Fix}(\alpha_+\delta) \cup \mathrm{Fix}(\alpha_-\delta)\right),$$

where both $\mathrm{Fix}(\alpha_+\delta)$ and $\mathrm{Fix}(\alpha_-\delta)$ have codimensions at least one due to the hypothesis $\delta \in \mathbf{J} - \mathbf{J}^{\alpha}$. □

LEMMA 4.6. *Suppose that*

a. *if* $k_1 \in \mathcal{L}_1^* \cap \alpha\mathcal{L}_1^*$ *then*

$$\sum_{\delta \in \mathrm{J}^{Id}(k)} D'(\delta,k) = \omega_{k_1}(-v_\alpha) \sum_{\delta \in \mathrm{J}^{\alpha}(k)} D'(\delta,k) \text{ and}$$

b. *if* $k_1 \notin \mathcal{L}_1^* \cap \alpha\mathcal{L}_1^*$ *then*

$$\sum_{\delta \in \mathrm{J}^{Id}(k)} D'(\delta,k) = 0,$$

for all $k = (k_1,k_2) \in \mathcal{L}^*$. *Then one of the following cases holds:*

A. $\mathbf{J}^{\alpha} = \{\alpha_+^{-1}\}$ *and* $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+$,
B. $\mathbf{J}^{\alpha} = \{\alpha_-^{-1}\}$ *and* $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_-$,
C. $\mathbf{J}^{\alpha} = \{\alpha_+^{-1}, \alpha_-^{-1}\}$,
$\mathcal{M}^* \subset \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+ \cup \mathcal{N}^*_{\sigma}\right)$ *and* $\mathcal{N}^* \subset \left(\mathcal{N}^*_{y_0} \cup \left(\mathcal{M}^*_- \cap \mathcal{N}^*_{\tilde{\sigma}}\right)\right)$.

Notice that the conditions in Lemma 4.6 are the same of Proposition 4.1 as $\delta^{-1} \in \mathbf{J}^\alpha$ is equivalent to $(v_\delta, \delta) \in \Gamma$ for some $v_\delta \in \mathbf{R}^{n+1}$, by definition.

Next proof will follow the same structure as the proof of Lemma 3.3, for the two-dimensional case. However, proving that $\mathcal{Q}^*_{k,h} \cap \mathcal{P}^*$ is a finite set is far more difficult than to prove that $\mathcal{Q}^*_k \cap \mathcal{P}^*$ is finite, the analogous step for the particular case $n = 1$. This is due to the dimension of the spaces involved, via the geometrical description of $\mathcal{P}^*$ and the existence, for each $k \in \mathcal{L}^*$, of an infinite family of sets $\mathcal{Q}^*_{k,h}$, with $h \in \mathcal{M}^*_k$, instead of a unique set $\mathcal{Q}^*_k$.

Proof. At first, we prove that

$$\left(\mathcal{M}^*_{y_0} \cap \mathcal{P}^*\right) - \{(0,0)\} = \emptyset. \tag{18}$$

If $k \in \mathcal{M}^*_{y_0}$ then $k = (k_1, 0)$ for some $k_1 \in \mathbf{R}^n$. If, moreover, $k \in \mathcal{P}^*$ then either $\delta(k_1, 0) = (k_1, 0)$, for some $\delta \in \mathbf{J} - \mathbf{J}^{Id}$, or $\delta(k_1, 0) = (\alpha^{-1}k_1, 0)$, for some $\delta \in \mathbf{J} - \mathbf{J}^\alpha$, by the definition of $\mathcal{P}^*$ and the properties of $\mathrm{J}^{Id}(k)$ and $\mathrm{J}^\alpha(k)$. By orthogonality of δ the first case implies, for $k_1 \neq 0$, either $\delta = I$ or $\delta = \sigma$, which is equivalent to $\delta \in \mathbf{J}^{Id}$. Similarly, for $k_1 \neq 0$, the second case implies $\delta \in \mathbf{J}^\alpha$, by orthogonality of δ and α.

For any element $k \neq (0,0)$ of the dual lattice $\mathcal{L}^*$, let $g \neq (0,0)$ be the smallest element of $\mathcal{L}^*$ in the direction of k. Thus, there are elements $g_1, \dots, g_n \in \mathcal{L}^*$ such that

$$\mathcal{L}^* = \{g, g_1, \dots, g_n\}_{\mathbf{Z}}.$$

Let $\mathcal{M}^*_k$ be the submodule of $\mathcal{L}^*$

$$\mathcal{M}^*_k = \{g_1, g_2, \dots, g_n\}_{\mathbf{Z}}$$

and, given $h \in \mathcal{M}^*_k$, let $\mathcal{Q}^*_{k,h}$ be the set

$$\mathcal{Q}^*_{k,h} = \{k + mh : m \in \mathbf{Z}\}.$$

We claim that there is some $h \in \mathcal{M}^*_k$ such that $\mathcal{Q}^*_{k,h} \cap \mathcal{P}^*$ is a finite set. Lemma 4.5 asserts that $\mathcal{P}^* \subset \bigcup_{i=1}^m H_i$, where each H_i is a codimension one subspace of $\mathbf{R}^{n+1}$. Let $p \in \mathbf{N}$ and consider the subset of $k + \mathcal{M}^*_k$ with p^n elements:

$$W_p = \{k + m_1 g_1 + \cdots + m_n g_n : m_i \in \mathbf{Z}, 1 \leq m_i \leq p\}.$$

Each H_i has at most p^{n-1} elements in W_p and so $W_p \cap \bigcup_{i=1}^m H_i$ has, at most, mp^{n-1} elements. For $p > m$ we have $p^n > mp^{n-1}$ and there is some $h \in \mathcal{M}^*$ such that $k + h \notin \bigcup_{i=1}^m H_i$. For this h, let r be a line containing $\mathcal{Q}^*_{k,h}$. Since for each i, $r \cap H_i$ is either r or a finite set, and r contains at least the element $k + h \notin H_i$, it follows that $\bigcup_{i=1}^m (r \cap H_i)$ is a finite set. The claim is proved because $\mathcal{Q}^*_{k,h} \cap \mathcal{P}^*$ is a subset of $\bigcup_{i=1}^m (r \cap H_i)$.

Let k be any element of $\mathcal{L}^* - \{(0,0)\}$ and choose some $h \in \mathcal{M}_k^*$ such that $\mathcal{Q}_{k,h}^* \cap \mathcal{P}^*$ is a finite set. For simplicity of notation we write $\mathcal{Q}^*$ instead of $\mathcal{Q}_{k,h}^*$.

The intersection $\mathcal{Q}^* \cap \overline{\mathcal{N}_{y_0}^*}$ is either the empty set or a set with only a point or an infinite set of equally spaced points. This happens because $\overline{\mathcal{N}_{y_0}^*}$ is a module and the existence of any two distinct elements of $\mathcal{Q}^* \cap \overline{\mathcal{N}_{y_0}^*}$, $k + m_1 h$ and $k + m_2 h$, implies $(m_2 - m_1)h \in \overline{\mathcal{N}_{y_0}^*}$ and

$$\{k + m_1 h + m(m_2 - m_1)h : m \in \mathbf{Z}\} \subset \left(\mathcal{Q}^* \cap \overline{\mathcal{N}_{y_0}^*}\right).$$

A characteristic period, τ_{y_0}, is given by the smallest difference between two elements of $\mathcal{Q}^* \cap \overline{\mathcal{N}_{y_0}^*}$.

For the set $\mathcal{Q}^* \cap \mathcal{N}_\sigma^*$ there are also the three possible results. Although $\mathcal{N}_\sigma^*$ is not a module, the smallest difference between two elements of $\mathcal{Q}^* \cap \mathcal{N}_\sigma^*$ defines a period $\tau_\sigma \in \mathcal{M}_\sigma^*$, by properties of $\mathcal{N}_\sigma^*$ in page 78. Thus, whenever $\mathcal{Q}^* \cap \mathcal{N}_\sigma^*$ has more than one element, if $k + m_1 h \in \mathcal{N}_\sigma^*$ then

$$\{k + m_1 h + m\tau_\sigma : m \in \mathbf{Z}\} = \mathcal{Q}^* \cap \mathcal{N}_\sigma^*.$$

An analogous construction may be done for the sets $\mathcal{Q}^* \cap \mathcal{M}_+^*$ and $\mathcal{Q}^* \cap \mathcal{M}_-^*$. Thus, if these sets have more than one element we may define, respectively, characteristic periods τ_+ and τ_-.

If the set $\mathcal{Q}^* \cap \left(\mathcal{M}_-^* \cap \mathcal{N}_{\tilde{\sigma}}^*\right)$ has two distinct elements, $k + m_1 h$ and $k + m_2 h$, then $(m_2 - m_1)h \in \left(\mathcal{M}_-^* \cap \overline{\mathcal{N}_\sigma^*}\right)$ and

$$\{k + m_1 h + m(m_2 - m_1)h : m \in \mathbf{Z}\} \subset \left(\mathcal{Q}^* \cap \left(\mathcal{M}_-^* \cap \mathcal{N}_{\tilde{\sigma}}^*\right)\right),$$

by the module structure of $\mathcal{M}_-^*$ and by the properties of $\mathcal{N}_{\tilde{\sigma}}^*$ in page 78. As above, this set has also a period, $\tau_{\tilde{\sigma}}$.

Under the hypothesis of the Lemma, one of the cases 1 to 5 of Lemma 4.4 must happen.

If case 1 happens then $\mathcal{L}^* = \mathcal{N}_{y_0}^* \cup \mathcal{P}^*$, which implies $\mathcal{M}_{y_0}^* \subset \mathcal{P}^*$ and, by (18), $\mathcal{M}_{y_0}^* = \{(0,0)\}$. Moreover, $\mathcal{Q}^* \cap \mathcal{N}_{y_0}^*$ must be an infinite set because $\mathcal{Q}^* \cap \mathcal{P}^*$ is, by construction, finite. Thus, there exists the period τ_{y_0} implying that $\mathcal{Q}^* - \overline{\mathcal{N}_{y_0}^*}$ is either the empty set or an infinite set. Since $\left(\mathcal{Q}^* - \overline{\mathcal{N}_{y_0}^*}\right) \subset (\mathcal{Q}^* \cap \mathcal{P}^*)$ is finite, it follows that $\mathcal{L}^* = \overline{\mathcal{N}_{y_0}^*}$. However, by property 2 of the bases in page 92, in this case $\mathcal{M}_{y_0}^* \neq \{(0,0)\}$ and so case 1 cannot occur.

In case 2, $\mathcal{L}^* = \mathcal{N}_{y_0}^* \cup \mathcal{N}_\sigma^* \cup \mathcal{P}^*$ which, by (18), implies $\mathcal{M}_{y_0}^* \subset (\mathcal{N}_\sigma^* \cup \{(0,0)\})$. Moreover, $\mathcal{M}_{y_0}^* \neq \{(0,0)\}$ due to the existence of σ in $\mathbf{J}$, (see properties 2 and 3 of the bases in page 92). Suppose $\tilde{k} \in \mathcal{M}_{y_0}^*$ and $\tilde{k} \neq (0,0)$. Thus, $\tilde{k} \in \mathcal{N}_\sigma^*$ and $2\tilde{k} \in \mathcal{M}_{y_0}^*$. However, by properties of $\mathcal{N}_\sigma^*$ on page 78, $2\tilde{k} \notin \mathcal{N}_\sigma^*$ and so case 2 is also impossible.

For case 3 we follow the arguments of case 1. As $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+ \cup \mathcal{P}^*$ then $\mathcal{Q}^* \cap \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+\right)$ is an infinite set and at least one of the periods τ_{y_0} or τ_+ must exist. The least common multiple of the existing periods is a period of $\mathcal{Q}^* \cap \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+\right)$ which implies that $\mathcal{Q}^* - \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+\right)$ is the empty set. Therefore $k \in \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+\right)$ and condition A follows by definition of k and because $(0,0) \in \mathcal{M}^*_+$.

In a similar way, with $\mathcal{M}^*_-$ and τ_- instead of $\mathcal{M}^*_+$ and τ_+, we prove that case 4 of Lemma 4.4 leads to condition B.

In case 5 $(\mathcal{Q}^* \cap \mathcal{M}^*) - \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+ \cup \mathcal{N}^*_\sigma\right)$ must be the empty set by the necessary existence of, at least, one of the periods τ_{y_0}, τ_+ or τ_σ and, analogously, $(\mathcal{Q}^* \cap \mathcal{N}^*) - \left(\mathcal{N}^*_{y_0} \cup \left(\mathcal{M}^*_- \cap \mathcal{N}^*_{\tilde{\sigma}}\right)\right)$ is empty due to the least common multiple of the periods τ_{y_0} and $\tau_{\tilde{\sigma}}$. Besides, either $k \in (\mathcal{Q}^* \cap \mathcal{M}^*)$ or $k \in (\mathcal{Q}^* \cap \mathcal{N}^*)$ and, as $(0,0) \notin \mathcal{N}^*$, condition C follows. $\square$

This completes the proof of Propositions 4.1 and 4.2.

4.6. Proof of Necessity in the Theorem

In order to complete the proof of Theorem 4.1 we will show how the conditions A, B and C of the Proposition 4.1 lead to the conclusion that one of the cases I, II and III of the theorem must hold.

Our aim is then to translate the restrictions on the group Γ and its dual lattice $\mathcal{L}^*$ into restrictions concerning Γ and its lattice $\mathcal{L}$. One major tool will be to use the conditions of Propositions 4.2 to obtain restrictions on a basis of $\mathcal{L}^*$. This in turn is used to find a suitable basis for $\mathcal{L}$. The properties of the modules involved will also be extensively used.

We begin by stating some properties of the bases for the lattices. Then we show that the conditions in Propositions 4.2 imply those of Theorem 4.1. Each condition of Propositions 4.2 is treated in a separate lemma.

4.6.1. the lattices $\mathcal{L}$ and $\mathcal{L}^*$.

PROPERTIES OF THE BASES FOR $\mathcal{L}$ AND $\mathcal{L}^*$ AND NOTATION. *Let $\{l_1, \dots, l_{n+1}\}$ be a basis for $\mathcal{L}$ and $\{l^*_1, \dots, l^*_{n+1}\}$ be its dual basis, i.e., a basis for $\mathcal{L}^*$ such that $< l^*_i, l_j >= \delta_{ij}$ for $i, j \in \{1, \dots, n+1\}$.*

For $M = \begin{pmatrix} l_1 \\ \vdots \\ l_{n+1} \end{pmatrix}$ *then* $M^* = (M^{-1})^T = \begin{pmatrix} l^*_1 \\ \vdots \\ l^*_{n+1} \end{pmatrix}$ *and the following properties hold:*

1. *If* $(v_\delta, \delta) \in \Gamma$ *then, for any given real numbers* $r_1, \dots, r_{n+1}$*, we may write* $v_\delta = \sum_{i=1}^{n+1} s_i l_i$ *with* $(s_i - r_i) \in [0,1[$ *for all* $i \in \{1, \dots, n+1\}$.
2. *If* $(0,a) \in \mathcal{L}$ *for some* $a \neq 0$ *then we may choose the basis* $\{l_1, \dots, l_{n+1}\}$ *for* $\mathcal{L}$ *such that*
   (i) $M = \begin{pmatrix} A & B \\ 0 & b \end{pmatrix}$*, where* $b = \frac{a}{m}$ *for some* $m \in \mathbf{Z}$ *,*
$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \text{ with } a_i \in \mathbf{R}^n \text{ for } i \in \{1, \dots, n\} \text{ and}$$
$$B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \text{ with } b_i \in \mathbf{R} \text{ for } i \in \{1, \dots, n\}.$$
   (ii) $M^* = \begin{pmatrix} A^* & 0 \\ -\frac{1}{b} B^T A^* & \frac{1}{b} \end{pmatrix}$*, where* $A^* = (A^{-1})^T = \begin{pmatrix} a_1^* \\ \vdots \\ a_n^* \end{pmatrix}$*,*
   with $< a_i^*, a_j >= \delta_{ij}$ *for* $i, j \in \{1, \dots, n\}$.
   (iii) *The set* $\{a_1, \dots, a_n\}$ *is a basis for a lattice in* $\mathbf{R}^n$ *and* $\{a_1^*, \dots, a_n^*\}$ *is a basis for its dual.*
   (iv) $l_i^* = (a_i^*, 0)$ *for* $i \in \{1, \dots, n\}$ *and*
$$\mathcal{M}_{y_0}^* = \{l_1^*, \dots, l_n^*\}_{\mathbf{Z}}.$$
3. *If* σ *belongs to the holohedry of* $\mathcal{L}$ *then there is some nonzero* $a \in \mathbf{R}$ *such that* $(0,a) \in \mathcal{L}$*. Moreover, each entry* b_i *of the matrix* B *defined above, may be taken to be either zero or* $b/2$.

The item 3 of the properties above leads, for $n = 1$, to the result stated in Lemma 3.5 in page 52 of the previous chapter.

Proof. 1. The set $\{l_1, \dots, l_{n+1}\}$ is a basis for $\mathbf{R}^{n+1}$ and so $v_\delta = \sum_{i=1}^{n+1} s_i l_i$ with $s_i \in \mathbf{R}$ for all $i \in \{1, \dots, n+1\}$. As v_δ is defined up to elements of $\mathcal{L}$ then we may restrict each s_i to an interval $[r_i, r_i + 1[$, where $r_i \in \mathbf{R}$.

2. If $(0,a) \in \mathcal{L}$ for some $a \neq 0$ then there is some $b \neq 0$ such that $(0,b)$ is the smallest element of the lattice $\mathcal{L}$ in the direction of $(0,a)$. Thus $(0,b)$ is a generator and $(0,a) = m(0,b)$ for some $m \in \mathbf{Z}$. Moreover, there are elements $l_1, \dots, l_n$ in $\mathcal{L}$ such that $\mathcal{L} = \{l_1, \dots, l_n, (0,b)\}_{\mathbf{Z}}$. For $l_i = (a_i, b_i)$, with $i \in \{1, \dots, n\}$, and $(0,b) = l_{n+1}$ we obtain the matrix M and it is easy to show that $M(M^*)^T = Id_{n+1}$, where M^* is the matrix defined in 2ii). Property 2iv) follows by the definition of the module $\mathcal{M}_{y_0}^*$.

3. There is some $(c,d) \in \mathcal{L}$ with $d \neq 0$. If $\sigma\mathcal{L} = \mathcal{L}$ then $(c,d) - \sigma(c,d) = (0,2d) \in \mathcal{L}$ and property 2 is valid. For $l_i = (a_i, b_i)$, with $i \in \{1,\dots,n\}$, the elements $l_i - \sigma l_i = (0, 2b_i)$ belong to $\mathcal{L}$ and so $(0,2b_i) = m(0,b)$ for some $m \in \mathbf{Z}$. Therefore $l_i = \left(a_i, \frac{mb}{2}\right)$, which is either $(a_i, 0)$ or $\left(a_i, \frac{b}{2}\right)$ up to multiples of $(0,b) = l_{n+1}$. □

4.6.2. symmetry of $\Pi_{y_0}(X_\Gamma)$ related to Γ and $\mathcal{L}$.

Lemma 4.7. *If $(v_+, \alpha_+) \in \Gamma$ and $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+$ then one of the following conditions of Theorem 4.1 holds:*
I. $((v_\alpha, 0), \alpha_+) \in \Gamma$,
III. $(0, y_0) \in \mathcal{L}$ *and* $((v_\alpha, y_1), \alpha_+) \in \Gamma$ *for some* $y_1 \in \mathbf{R}$.

Proof. If $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+$ then either $\mathcal{L}^* = \overline{\mathcal{N}^*_{y_0}}$ or $\mathcal{L}^* = \mathcal{M}^*_+$. In the second case $< k, v_+ - (v_\alpha, 0) > \in \mathbf{Z}$ for all $k \in \mathcal{L}^*$, *i.e.* , $v_+ - (v_\alpha, 0) \in \mathcal{L}$, and so

$$(-v_+ + (v_\alpha, 0), I) \cdot (v_+, \alpha_+) = ((v_\alpha, 0), \alpha_+) \in \Gamma.$$

If $\mathcal{L}^* = \overline{\mathcal{N}^*_{y_0}}$ then $(0, y_0) \in \mathcal{L}$ and we may use the basis $\left\{l_1^*, \dots, l_{n+1}^*\right\}$ for $\mathcal{L}^*$ having the properties 2 above. As $\mathcal{M}^*_{y_0} \subset \mathcal{M}^*_+$, it follows that $< l_i^*, v_+ - (v_\alpha, 0) > \in \mathbf{Z}$ for all $i \in \{1, \dots n\}$. Now we show that $v_+ - (v_\alpha, y_1) \in \mathcal{L}$ for some $y_1 \in \mathbf{R}$. For any element $k \in \mathcal{L}^*$ and any $y_1 \in \mathbf{R}$,

$$\begin{aligned} < k, v_+ - (v_\alpha, y_1) > &= < k, v_+ - (v_\alpha, 0) > - < k, (0, y_1) > \\ &= m_1 + m_2 < l^*_{n+1}, v_+ - (v_\alpha, 0) > - m_2 \tfrac{m}{y_0} y_1, \end{aligned}$$

with $m_1, m_2 \in \mathbf{Z}$.

Taking, for instance, $y_1 =< l^*_{n+1}, v_+ - (v_\alpha, 0) > \frac{y_0}{m}$ we obtain $< k, v_+ - (v_\alpha, y_1) > \in \mathbf{Z}$. Thus,

$$(-v_+ + (v_\alpha, y_1), I) \cdot (v_+, \alpha_+) = ((v_\alpha, y_1), \alpha_+) \in \Gamma.$$

□

Lemma 4.8. *If $(v_-, \alpha_-) \in \Gamma$ and $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_-$ then one of the following conditions of Theorem 4.1 holds:*
II. $((v_\alpha, y_0), \alpha_-) \in \Gamma$,
III. $(0, y_0) \in \mathcal{L}$ *and* $((v_\alpha, y_1), \alpha_-) \in \Gamma$ *for some* $y_1 \in \mathbf{R}$.

Proof. The proof of this lemma is analogous to the previous one with $v_- - (v_\alpha, y_0)$ instead of $v_+ - (v_\alpha, 0)$ and $y_1 =< l^*_{n+1}, v_- - (v_\alpha, y_0) > \frac{y_0}{m} + y_0$. □

LEMMA 4.9. *If both* (v_σ, σ) *and* (v_+, α_+) *belong to* Γ*, and if both*

$$\mathcal{M}^* \subset \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+ \cup \mathcal{N}^*_\sigma\right) \text{ and } \mathcal{N}^* \subset \left(\mathcal{N}^*_{y_0} \cup \left(\mathcal{M}^*_- \cap \mathcal{N}^*_{\tilde{\sigma}}\right)\right),$$

then one of the following conditions of Theorem 4.1 holds:

I. $((v_\alpha, 0), \alpha_+) \in \Gamma$,

II. $((v_\alpha, y_0), \alpha_-) \in \Gamma$,

III. $(0, y_0) \in \mathcal{L}$ *and either* $((v_\alpha, y_1), \alpha_+) \in \Gamma$ *or* $((v_\alpha, y_1), \alpha_-) \in \Gamma$, *for some* $y_1 \in \mathbf{R}$.

The proof of Lemma 4.9 has three main steps. First we describe some properties of the linear components of (v_σ, σ) and (v_+, α_+), elements of Γ, and restrict the bases of $\mathcal{L}$ and $\mathcal{L}^*$ under the hypothesis of the Lemma. In particular we show that either $v_1 = v_\alpha$ or we may choose $a_1 = 2(v_1 - v_\alpha)$. Afterwards, we prove the result for each of these two cases.

PROOF. Let $v_+ = (v_1, v_2)$ with $v_1 \in \mathbf{R}^n$ and $v_2 \in \mathbf{R}$ and notice that, since $\sigma \in \mathbf{J}$, the bases $\{l_1, \ldots, l_{n+1}\}$ and $\left\{l_1^*, \ldots, l_{n+1}^*\right\}$ have the properties 1 to 3 described on page 91, where, in particular, $l_1 = (a_1, b_1)$ and $(0, b) \in \mathcal{L}$.

We claim that the following properties are also verified:

1. $v_\sigma + \sigma v_\sigma \in \mathcal{L}$.
2. $\sigma v_+ - v_+ = -(0, 2v_2)$. Therefore
   (i) $(0, 4v_2) \in \mathcal{L}$ and
   (ii) if $(0, 2v_2) \in \mathcal{L}$ then $\mathcal{N}^* = \emptyset$.
3. $\mathcal{M}^*_{y_0} \subset \left(\mathcal{M}^*_+ \cup \mathcal{N}^*_\sigma\right)$.
4. Either $v_1 = v_\alpha$ or we may choose $a_1 = 2(v_1 - v_\alpha)$.
5. In both cases of property 4, $l_i^* \in \mathcal{M}^*_+$ for all $i \in \{2, \ldots, n\}$.

We now prove the claims above.

1. This property has already been used and proved, see (16), page 85.

2. Lemma 4.2 is valid and these properties are a consequence of the definition of $\mathcal{M}^*$ and $\mathcal{N}^*$.

3. The sets $\mathcal{M}^*_{y_0}$ and $\mathcal{N}^*$ are disjoint, by 2 and by the property 2iv) of the bases, on page 92. Thus the hypothesis of Lemma 4.9 implies this result.

4. The last property implies that for all $i \in \{1, \ldots, n\}$,

either $< (a_i^*, 0), v_+ - (v_\alpha, 0) > \in \mathbf{Z}$ or $< (a_i^*, 0), v_\sigma - (0, y_0) > + \dfrac{1}{2} \in \mathbf{Z}$.

If $l_i^* \in \mathcal{N}^*_\sigma$ then $2l_i^* \notin \mathcal{N}^*_\sigma$ and so, for all $i \in \{1, \ldots, n\}$,

$$2 < (a_i^*, 0), v_+ - (v_\alpha, 0) > = < a_i^*, 2(v_1 - v_\alpha) > \in \mathbf{Z}.$$

Therefore, $2(v_1 - v_\alpha) = \sum_{i=1}^n m_i a_i$ with $m_i \in \mathbf{Z}$ for all $i \in \{1, \ldots, n\}$ and, if $v_1 \neq v_\alpha$, we may choose $a_1 = \frac{2(v_1 - v_\alpha)}{m}$ for some $m \in \mathbf{Z}$, by the property 2iii) of the bases. If $v_\alpha = \sum_{i=1}^{n} r_i a_i$, with $r_i \in \mathbf{R}$, then, by property 1 of the bases, v_+ may be writen as $\sum_{i=1}^{n+1} s_i l_i$ such that $2\,(r_i - s_i) \in [0, 2[$ for all $i \in \{1, \ldots, n\}$. Thus, $m = 1$ and the result follows.

5. $v_1 - v_\alpha$ is either zero or $a_1/2$. Therefore

$$< l_i^*, v_+ - (v_\alpha, 0) >=< a_i^*, v_1 - v_\alpha >= 0,$$

for $i \in \{2, \ldots, n\}$.

Property 4 divides Lemma 4.9 in two major cases that we consider separately.

Suppose $v_1 = v_\alpha$.

Thus $v_+ - (v_\alpha, 0) = (0, v_2)$ and all the elements $l_1^*, \ldots, l_n^*$ belong to the module $\mathcal{M}_+^*$.

If $l_{n+1}^* \in \mathcal{M}_+^*$ then $\mathcal{L}^* = \mathcal{M}_+^*$ and, as in the proof of Lemma 4.7, $((v_\alpha, 0), \alpha_+) \in \Gamma$, *i.e.*, condition I holds.

If $l_{n+1}^* \in \mathcal{N}_{y_0}^*$ then, by property 2iv) of the bases, on page 92, $\mathcal{L}^* = \overline{\mathcal{N}_{y_0}^*}$, which implies $(0, y_0) \in \mathcal{L}$. Condition III follows since $((v_\alpha, v_2), \alpha_+) \in \Gamma$.

Now suppose that

$$\begin{array}{rcl} l_{n+1}^* & \notin & \left(\mathcal{M}_+^* \cup \mathcal{N}_{y_0}^*\right) \text{ and, consequently,} \\ l_i^* + l_{n+1}^* & \notin & \left(\mathcal{M}_+^* \cup \mathcal{N}_{y_0}^*\right) \text{ for all } i \in \{1, \ldots, n\}. \end{array} \tag{19}$$

If $(0, 2v_2) \in \mathcal{L}$ then we may choose $2v_2 = b$ and the above conditions, as $\mathcal{N}^* = \emptyset$, lead to

$$l_{n+1}^* \in \mathcal{N}_\sigma^* \quad \text{and} \quad l_i^* \in \mathcal{M}_\sigma^* \text{ for all } i \in \{1, \ldots, n\}$$

and, if $v_\sigma = \sum_{i=1}^{n+1} s_i l_i$, with $s_i \in [0, 1[$ for $i \in \{1, \ldots, n\}$, then

$$s_{n+1} - \frac{y_0}{b} + \frac{1}{2} \in \mathbf{Z} \quad \text{and} \quad s_i = 0 \text{ for all } i \in \{1, \ldots, n\}.$$

Therefore, up to multiples of $(0, b)$, we have $v_\sigma = (0, y_0 + b/2) = (0, y_0 + v_2)$ and

$$((0, y_0 + v_2), \sigma) \cdot ((v_\alpha, v_2), \alpha_+) = ((v_\alpha, y_0), \alpha_-) \in \Gamma,$$

i.e., condition II.

If $(0, 2v_2) \notin \mathcal{L}$ then conditions (19) imply

$$l_{n+1}^* \in \mathcal{M}_-^* \quad \text{and} \quad l_i^* \in \mathcal{M}_-^* \text{ for all } i \in \{1, \ldots, n\}.$$

Thus, $\mathcal{L}^* = \mathcal{M}_-^*$ and, as in Lemma 4.8, $((v_\alpha, y_0), \alpha_-) \in \Gamma$, completing the proof in the case $v_1 = v_\alpha$.

Now suppose that $v_1 \neq v_\alpha$ and let $a_1 = 2(v_1 - v_\alpha)$.

Since $l_1^* \notin \mathcal{M}_+^*$, property 3 above implies

$$l_1^* \in \mathcal{N}_\sigma^* \quad \text{and} \quad l_i^* \in \mathcal{M}_\sigma^* \text{ for all } i \in \{2, \dots, n\},$$

which, for $v_\sigma = \sum_{i=1}^{n+1} s_i l_i$, with $s_i \in [0, 1[$ for $i \in \{1, \dots, n\}$, can be writen as

$$s_1 = \frac{1}{2} \quad \text{and} \quad s_i = 0 \text{ for all } i \in \{2, \dots, n\}.$$

Thus, $v_\sigma = l_1/2 + s_{n+1}(0, b)$ and , by property 1, $(a_1, 0) \in \mathcal{L}$, *i.e.*, $b_1 = 0$. As $v_\sigma = (a_1/2, 0) + s_{n+1}(0, b) = v_+ - (v_\alpha, 0) + (0, s_{n+1}b - v_2)$, property 1 allows us to conclude

$$(-\sigma v_+ + (v_\alpha, s_{n+1}b - v_2), \sigma) \in \Gamma.$$

If $l_{n+1}^* \in \mathcal{N}_{y_0}^*$ then, as in the proof of the previous Lemma, $(0, y_0) \in \mathcal{L}$. Moreover,

$$(-\sigma v_+ + (v_\alpha, s_{n+1}b - v_2), \sigma) \cdot (v_+, \alpha_+) = ((v_\alpha, s_{n+1}b - v_2), \alpha_-) \in \Gamma$$

and condition III follows.

Now suppose that $l_{n+1}^* \notin \mathcal{N}_{y_0}^*$ and, consequently, that $l_i^* + l_{n+1}^* \notin \mathcal{N}_{y_0}^*$ for all $i \in \{1, \dots, n\}$. If $l_{n+1}^* \in \mathcal{M}_+^*$ then $< l_{n+1}^*, l_1/2 + (0, v_2) >= v_2/b \in \mathbf{Z}$ and $(0, v_2) \in \mathcal{L}$, since $(0, b) \in \mathcal{L}$. Moreover, as $l_1^* \notin \mathcal{M}_+^*$, we must impose $l_1^* + l_{n+1}^* \in \mathcal{N}_\sigma^*$, which implies $s_{n+1} + y_0/b \in \mathbf{Z}$. Therefore, choosing $s_{n+1} = y_0/b$,

$$((0, v_2), I) \cdot (-\sigma v_+ + (v_\alpha, y_0 - v_2), \sigma) \cdot (v_+, \alpha_+) = ((v_\alpha, y_0), \alpha_-) \in \Gamma.$$

For $(0, 2v_2) \in \mathcal{L}$, the only missing case is $l_{n+1}^* \in \mathcal{N}_\sigma^*$, where $s_{n+1} + y_0/b + 1/2 \in \mathbf{Z}$ and, up to multiples of $(0, b)$, $s_{n+1}b - v_2 = y_0$. Condition II follows because

$$(-\sigma v_+ + (v_\alpha, y_0), \sigma) \cdot (v_+, \alpha_+) = ((v_\alpha, y_0), \alpha_-) \in \Gamma.$$

If $(0, 2v_2) \notin \mathcal{L}$ then both l_{n+1}^* and $l_i^* + l_{n+1}^*$ belong to $\mathcal{M}_-^*$ for $i \in \{1, \dots, n\}$. Thus, as in the previous Lemma, condition II follows. □

4.7. Restriction

In this section we present, for the restriction of functions in X_Γ, a theorem analogous to the one obtained in the previous sections for the projection.

4.7.1. Theorem for the restriction. Recall that, for $r \in \mathbf{R}$, the operator Φ_r maps $f(x,y)$ to its restriction to the affine subspace $\{(x,r) : x \in \mathbf{R}^n\}$ given by $\Phi_r(f)(x) = f(x,r)$.

If $f \in X_\Gamma$ then, formally,

$$\Phi_r(f)(x) = \sum_{k\in\mathcal{L}^*} C(k)\omega_k(x,r) =$$

$$= \sum_{k\in\mathcal{L}^*} C(k)\omega_{k_1}(x)\omega_{k_2}(r) =$$

$$= \sum_{k_1\in\mathcal{L}_1^*} \omega_{k_1}(x) \sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} C(k_1,k_2)\omega_{k_2}(r),$$

where $\mathcal{L}_1^* = \{k_1 : (k_1,k_2) \in \mathcal{L}^*\}$.

THEOREM 4.2. *All functions in $\Phi_r(X_\Gamma)$ are invariant under the action of $(v_\alpha,\alpha) \in \mathbf{R}^n \ltimes \mathbf{O}(n)$ if and only if one of the following conditions holds:*

I. $((v_\alpha,0),\alpha_+) \in \Gamma$,
II. $((v_\alpha,2r),\alpha_-) \in \Gamma$.

4.7.2. notes about the proof. For any function $f \in X_\Gamma$ its restriction, $\Phi_r(f)$, and its projection, $\Pi_{y_0}(f)$, have analogous formal Fourier series. The difference lies on the term $\omega_{k_2}(r)$ of the restriction that corresponds, in the projection, to the integral $\int_0^{y_0} \omega_{k_2}(y)dy$. Thus, the results concerning the restriction Φ_r are similar to the ones proved throughout the previous sections for the projection Π_{y_0}.

We do not present the proof of Theorem 4.2 because it is analogous to the one for the projection, with $\omega_{k_2}(r)$ instead of $\int_0^{y_0} \omega_{k_2}(y)dy$. The condition $\omega_{k_2}(r) = 0$ is never verified and so the sets $\mathcal{N}_{y_0}^*$ and $\mathcal{M}_{y_0}^*$ disappear and we don't have an analogous to the condition $(0,y_0) \in \mathcal{L}$. Moreover, the expression

$$\int_0^{y_0} \omega_{k_2}(y)dy - \omega_{k_2}(y_0)\int_0^{y_0} \omega_{-k_2}(y)dy = 0$$

has the analogous

$$\omega_{k_2}(r) - \omega_{k_2}(2r)\omega_{-k_2}(r) = 0.$$

It follows that $2r$ appears where originally we had the variable y_0.

We now present some intermediate results in order to obtain Theorem 4.2 and make remarks about the proof for the analogous of Lemma 4.6.

PROPOSITION 4.3. *All functions in $\Phi_r(X_\Gamma)$ are invariant under the action of $(v_\alpha, \alpha) \in \mathbf{R}^n \ltimes \mathbf{O}(n)$ if and only if one of the following conditions holds:*

A. $(v_+, \alpha_+) \in \Gamma$ *and* $\mathcal{L}^* = \mathcal{M}_+^*$,
B. $(v_-, \alpha_-) \in \Gamma$ *and* $\mathcal{L}^* = \mathcal{M}_-^*$,
C. *both* (v_σ, σ) *and* (v_+, α_+) *belong to* Γ,

$$\mathcal{M}^* \subset \left(\mathcal{M}_+^* \cup \mathcal{N}_\sigma^*\right) \text{ and } \mathcal{N}^* \subset \left(\mathcal{M}_-^* \cap \mathcal{N}_{\tilde{\sigma}}^*\right).$$

The analogous to Lemma 4.4 is, for $D'(\delta, k) = \omega_{\delta k}(-v_\delta)\omega_{\delta k|_2}(r)$:

LEMMA 4.10. *Suppose that*

a. *if* $k_1 \in \mathcal{L}_1^* \cap \alpha\mathcal{L}_1^*$ *then*

$$\sum_{\delta \in \mathrm{J}^{Id}(k)} D'(\delta, k) = \omega_{k_1}(-v_\alpha) \sum_{\delta \in \mathrm{J}^{\alpha}(k)} D'(\delta, k) \text{ and}$$

b. *if* $k_1 \notin \mathcal{L}_1^* \cap \alpha\mathcal{L}_1^*$ *then*

$$\sum_{\delta \in \mathrm{J}^{Id}(k)} D'(\delta, k) = 0,$$

for all $k = (k_1, k_2) \in \mathcal{L}^*$. *Then one of the following cases holds:*

1. $\mathbf{J}^{Id} = \{Id_{n+1}, \sigma\}$, $\mathbf{J}^\alpha = \emptyset$ *and* $\mathcal{O}^* \subset \mathcal{N}_\sigma^*$,
2. $\mathbf{J}^{Id} = \{Id_{n+1}\}$, $\mathbf{J}^\alpha = \{\alpha_+^{-1}\}$ *and* $\mathcal{O}^* \subset \mathcal{M}_+^*$,
3. $\mathbf{J}^{Id} = \{Id_{n+1}\}$, $\mathbf{J}^\alpha = \{\alpha_-^{-1}\}$ *and* $\mathcal{O}^* \subset \mathcal{M}_-^*$,
4. $\mathbf{J}^{Id} = \{Id_{n+1}, \sigma\}$, $\mathbf{J}^\alpha = \{\alpha_+^{-1}, \alpha_-^{-1}\}$, $(\mathcal{O}^* \cap \mathcal{M}^*) \subset \left(\mathcal{M}_+^* \cup \mathcal{N}_\sigma^*\right)$ *and* $(\mathcal{O}^* \cap \mathcal{N}^*) \subset \left(\mathcal{M}_-^* \cap \mathcal{N}_{\tilde{\sigma}}^*\right)$.

In a similar way, we may state a lemma analogous to Lemma 4.6. Under the conditions for the restriction, in the proof of Lemma 4.6, property (18) concerning the set $\mathcal{M}_{y_0}^*$ does not hold. By Lemma 4.10 above, the paragraph corresponding to case 1 disappear and in the paragraph corresponding to case 2 (see page 90), the dual lattice is $\mathcal{L}^* = \mathcal{N}_\sigma^* \cup \mathcal{P}^*$. In this paragraph the arguments concerning $\mathcal{M}_{y_0}^*$ and $\mathcal{N}_{y_0}^*$ must be replaced by: let $\tilde{k} \notin \mathcal{P}^*$ and, thus, $\tilde{k} \in \mathcal{N}_\sigma^*$. However both $2\tilde{k} \notin \mathcal{P}^*$ and $2\tilde{k} \notin \mathcal{N}_\sigma^*$, by definition of $\mathcal{P}^*$ and the properties of $\mathcal{N}_\sigma^*$, and so this case is not possible.

CHAPTER 5

Periodicity

In this chapter we discuss the periodicity of the projection of Γ-invariant functions. Since we assume the original functions are at least $\mathcal{L}$-periodic, we are interested in repetitive structures that remain after projection. Analogously, we may ask if there is some periodicity left after restricting those functions to a hyperplane. In section 5.1 we present results concerning periodicity of the functions after projection and after restriction.

Gomes [**11**] describes an experiment where observations can be interpreted either as solutions in a two-dimensional model or as the projection of solutions in a three-dimensional one. We make in this chapter some considerations on possible methods for detecting a projection in this kind of experiments.

The projection of lattices has a physical interpretation in the context of quasicrystals, where the aim is to obtain quasiperiodic projections of periodic structures in higher dimension. One step in the construction of nonperiodic projections considers the intersection of a lattice with a subspace. In section 5.2 we relate projections and restrictions of lattices and compare the results obtained with the ones usually refered in the theoretical models of quasicrystals.

Finally, in section 5.3, we describe the conditions ensuring that the projections of $\mathcal{L}$-periodic functions are still $\tilde{\mathcal{L}}$-periodic functions, for a lattice $\tilde{\mathcal{L}} \in \mathbf{R}^n$. After that, we approach the projection and restriction of repetitive patterns with the tools of equivariant theory and propose a way of detecting a projection in $\mathbf{E}(m)$-equivariant models.

5.1. Periodicity of the Projection

In this section we state necessary and sufficient conditions for the existence of a period $P \in \mathbf{R}^n$ for all the functions in $\Pi(X_\Gamma)$, by means of a corollary of Theorem 4.1. Similarly we present a corollary of Theorem 4.2 for the existence of periods in the space of restricted functions.

After summarizing the results obtained, we suggest how they may be used in interpreting experimental observations.

5.1.1. periodic projection. A period of a function in $\Pi_{y_0}(X_\Gamma)$ is a symmetry $(P, Id_n) \in \mathbf{R}^n \ltimes \mathbf{O}(n)$. For $v_\alpha = P$ and $\alpha = Id_n$, Theorem 4.1 leads the following corollary:

COROLLARY 5.1. *All functions in $\Pi_{y_0}(X_\Gamma)$ are periodic with period $P \in \mathbf{R}^n - \{0\}$ if and only if one of the following conditions holds:*

I. $(P, 0) \in \mathcal{L}$,

II. $((P, y_0), \sigma) \in \Gamma$,

III. $(0, y_0) \in \mathcal{L}$ *and either* $(P, y_1) \in \mathcal{L}$ *or* $((P, y_1), \sigma) \in \Gamma$, *for some* $y_1 \in \mathbf{R}$.

Condition I is well known and independent on the width of the projected band. Condition II ensures that the projected functions have period P when the projection width is y_0 but also implies the existence of the period $2P$ for the projected functions whatever the projection width is because

$$((P, y_0), \sigma) \cdot ((P, y_0), \sigma) = ((2P, 0), Id_{n+1}) \in \Gamma. \tag{20}$$

The case when $((P, y_1), Id_{n+1}) \in \Gamma$ in condition III has different consequences. For the width y_0 the projected functions have period P but may be nonperiodic for the remaining widths of the projection. Moreover, as we state below, the restrictions of the original functions may also be nonperiodic. In section 5.2 we study this case, taking into account only the lattice $\mathcal{L}$, the set of the periods of the original functions.

5.1.2. periodic restriction. Similarly, we can reformulate Theorem 4.2 for the periodicity in the space of restricted functions $\Phi_r(X_\Gamma)$.

COROLLARY 5.2. *All functions in $\Phi_r(X_\Gamma)$ are periodic with period $P \in \mathbf{R}^n - \{0\}$ if and only if one of the following conditions holds:*

I. $(P, 0) \in \mathcal{L}$,

II. $((P, 2r), \sigma) \in \Gamma$.

As described in section 4.7, the results for the restriction are analogous to the ones for the projection without the conditions where $(0, y_0) \in \mathcal{L}$ and with $2r$ instead of y_0 in the remaining ones.

5.1.3. summary. The next table summarizes the results obtained above for the periodicity of transformed functions. In the first column we list the elements in Γ that will ensure, for the values of y_0 and r described, the existence of some period for all the functions in $\Pi_{y_0}(X_\Gamma)$ and $\Phi_c(X_\Gamma)$. Here, "n.a." means that this case does not occur for the restriction and we assume $P \neq 0$.

original group Γ contains	validity set		period
	y_0	r	
$((P,0), Id_{n+1})$	$\mathbf{R}^+$	$\mathbf{R}$	P
$((P,a),\sigma)$ and $((0,b), Id_{n+1})$	$\{a+mb : m \in \mathbf{N_0}\}$ $\cup\{mb : m \in \mathbf{N}\}$	$\{\frac{a}{2}+m\frac{b}{2} : m \in \mathbf{Z}\}$	P
	$\mathbf{R}^+$	$\mathbf{R}$	$2P$
$((P,a), Id_{n+1})$ and $((0,b), Id_{n+1})$	$\{mb : m \in \mathbf{N}\}$	n.a.	P

Table 1.

The first row of the table describes the case when all the periods of the transformed functions, either by a projection or by a restriction, are only due to conditions I of the above corollaries. This is an intuitive result, very easy to understand if we look at lattices $\mathcal{L}$ instead of $\mathcal{L}$-periodic functions. We will discuss the projection and restriction of lattices in section 5.2 with some examples.

The second row presents a case where all the conditions of the corollaries happen but for different values of the parameters y_0 and r. The characteristic of Γ that defines this situation is the presence of σ in $\mathbf{J}$ associated to a *glide reflection*, *i.e.*,

$$((P,a),\sigma) \in \Gamma \quad \text{with } (P,0) \notin \mathcal{L}.$$

Although there is no explicit reference, $((2P,0), Id_{n+1})$ belongs to Γ, by (20). Moreover, the condition $((0,b), Id_{n+1}) \in \Gamma$ is a consequence of $\sigma \in \mathbf{J}$, by the properties of the bases in page 91. This element is written in the table only to introduce the necessary notation.

In the third row of the table we describe the case when the periodicity of the projected functions is due to $(0,y_0) \in \mathcal{L}$. If we compare this case to the previous one it becomes clear that the existence of the glide reflection modifies all the expected results.

For each function $f \in X_\Gamma$ and for any of the functions $\Pi_{y_0}(f)$ and $\Phi_r(f)$ the set of all its periods is a module over $\mathbf{Z}$. Thus $P = 0$ is always a period (since $((0,0), Id_{n+1}) \in \Gamma$ always holds), even for the case indicated as "n.a." on table 1.

5.1.4. how to detect a projection - I. Consider an experiment where patterns are observed on a thin layer.

An exemple of such an experiment is described by Gomes ([**11**] and references therein) for chemical reactions in a thin layer of gel. These patterns are usually described as a two-dimensional phenomenon, due to the small width of the layer. Gomes presents a different approach for a particular pattern observed during these experiments: the black-eye pattern (see figure 1 in page 16), not typical in the context of a two-dimensional model. Gomes obtains a black-eye pattern projecting into a plane a slice of a three-dimensional repetitive pattern.

Suppose we observe a planar periodic pattern in a layer of gel whose thickness y_0 is much smaller than its area. Our aim is to understand if this observation is due to a three-dimensional repetitive structure, via the projection $\Pi_{y_0}(f) : \mathbf{R}^2 \longrightarrow \mathbf{R}$, for some solution f, or if it is intrinsic to the approximate two-dimensional nature of the problem.

Suppose we can change the thickness of the layer. If the pattern remains unchanged, *i.e.*, if the periods remain with some eventual variation on the intensity, we cannot distinguish, by this method, if this pattern is a solution of a two-dimensional problem or the projection of a higher dimension solution. The last case would correspond to the first row of table 1, where the period of the projection does not depend on y_0.

Projected patterns with some period described by the second and third rows of table 1 would be detected varying the thickness: a continuous variation of y_0 would yield sudden variations of the period. For the third row of table 1, arbitrarily small perturbations of the thickness would destroy periodicity. For the case on the second row of table 1, as y_0 varies, the period will drop suddenly to half its value - an increase in symmetry.

5.1.5. some questions. Interesting questions arise fom this way of handling y_0 as a parameter. For instance, what can we experimentally observe if a pattern only holds for a specific value of y_0, as in the case of the third row of table 1? How is the bifurcation problem for the projection with parameter y_0?

Other ways of detecting projections could be developed, using the information in table 1, for experiments where a restriction to a plane can be observed.

5.2. Projection of Lattices

Even when a repetitive pattern loses periodicity after projection, some semblance of symmetry may remain yelding a quasiperiodic pattern - an ordered pattern with no periods. The quasiperiodic structure associated to *quasicrystals*, aperiodic crystals breaking the crystallographic restricion, may be obtained theoretically by a method called *canonical projection*. This is the projection of that portion of an *integral lattice*, a lattice where all the elements have integer squared lengh, between two parallel affine subspaces. If the intersection of the lattice with each affine subspace has, at most, one element then, for some distances between the affine subspaces, we obtain a quasiperiodic pattern.

See Senechal [**22**] for a definition of the terms involved and a formal description of the canonical projection.

5.2.1. the periodic projection of lattices. Corollaries 5.1 and 5.2 above, are stated for all the functions in a given space. We now use an intuitive approach in order to apply these results to lattices. We can think of a lattice as a pattern defined by a Dirac delta peak in each $l \in \mathcal{L} \subset \mathbf{R}^{n+1}$, and follow the function perspective, or simply compare the canonical projection and the restrictions described by Senechal [**22**] to the operators we have defined for the projection and the restriction. These are analogous transformations. Moreover, we may use the corollaries only to point out some possible results for lattices and, then, try these considerations directly in the lattices

$$\mathcal{L} = \{l_1, \dots, l_{n+1}\}_{\mathbf{Z}} = \left\{ \sum_{i=1}^{n+1} m_i l_i, m_i \in \mathbf{Z} \right\}$$

and confirm the empirical hypotheses.

Given a lattice $\mathcal{L} \in \mathbf{R}^{n+1}$, we may study its projection or restriction supposing that Γ is the symmetry group of $\mathcal{L}$ in the Corollaries 5.1 and 5.2 above.

We present some examples of projection of lattices in $\mathbf{R}^2$, one for each row of table 1. We look at the intersection of the lattice with the subspace $y = 0$ in order to compare these results with the ones concerning integral lattices. Figure 4 explains the periodicity of the projected lattice when $(0, y_0) \in \mathcal{L}$.

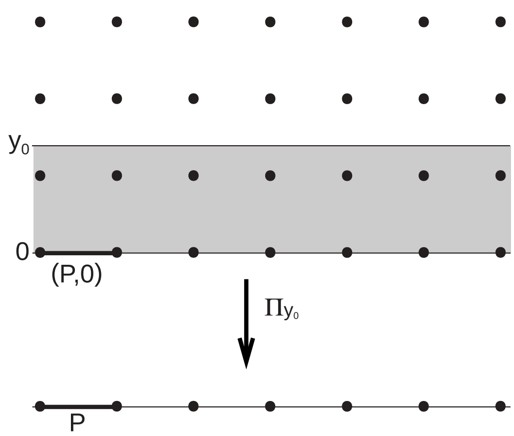

FIGURE 1. $(P, 0) \in \mathcal{L}$ ensures projections with period P for all y_0. The intersection with the line $y = 0$ also has this period.

The third row of table 1 describes the case when the restriction $\Phi_0(f)$ has no periods and the projection is periodic for some values of y_0. This happens when there is an element in $\mathcal{L}$ of the form $(0, a)$.

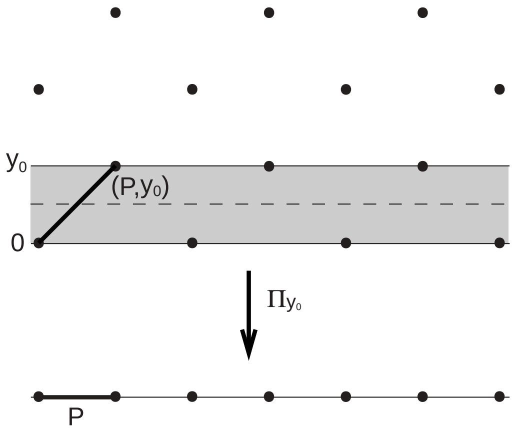

FIGURE 2. The glide reflection in the dashed line acts as a translation by P, after the projection for particular values of y_0. The restriction to $y = 0$ and all the projections have period $2P$.

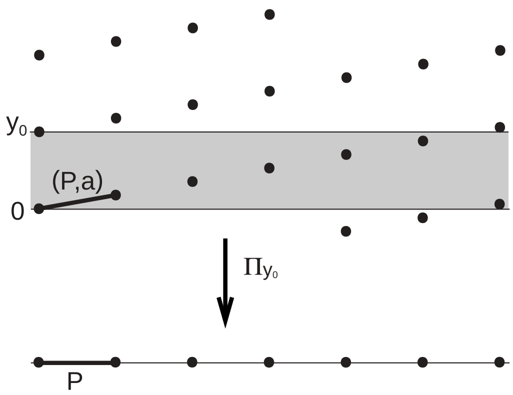

FIGURE 3. $(0, y_0) \in \mathcal{L}$ ensures projections with period P for the projection width y_0 even if the restriction to $y = 0$ is nonperiodic.

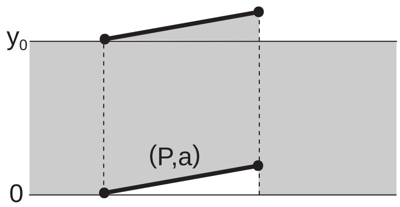

FIGURE 4. If $(0, y_0) \in \mathcal{L}$ then projecting a band with width y_0 is equivalent to the projection of the cell defined by $(0, y_0)$ and (P, a).

If $(0, a) \in \mathcal{L}$, for some $a \neq 0$, then the projection of all the lattice $\mathcal{L}$,

$$\{\tilde{l}_1 : (\tilde{l}_1, \tilde{l}_2) \in \mathcal{L}\},$$

is a lattice in $\mathbf{R}^n$. This is very easy to show, for example, using the property 2 of the bases in page 92. This result is valid even if the restriction

$$\{\tilde{l_1} : (\tilde{l_1}, 0) \in \mathcal{L}\},$$

a submodule of the lattice $\mathcal{L}$, is the origin alone.

Moreover, notice that, in this case, the projection of the part of the lattice lying between the hyperplanes $y = 0$ and $y = y_0$ is periodic if $y_0 \geq a > 0$, where $(0, a)$ is the smallest element of $\mathcal{L}$ in its direction.

5.2.2. a remark on integral lattices. The previous results apparently contradict the conditions for the canonical projection of a lattice. By Proposition 2.17 in Senechal, [**22**, page 57], the projection of the portion of the lattice between $y = 0$ and $y = y_0$ is nonperiodic if the restriction to $y = 0$ is the origin. Analogously, if we project all the lattice, by Proposition 2.15 in Senechal, [**22**, page 55], we expect a nonperiodic pattern if the restriction of the lattice to $y = 0$ is the origin.

However these propositions in Senechal [**22**] are for integral lattices. They are compatible with of Corollaries 5.1 and 5.2: we claim that if the restriction of $\mathcal{L}$ to $y = 0$ is $\{(0, 0)\}$ and the projection is periodic, *i.e.*, if we are in the third case of table 1, then $\mathcal{L}$ is not an integral lattice.

To prove the claim, let (b, c) be any element of $\mathcal{L}$ and $(0, a) \in \mathcal{L}$. If $\mathcal{L}$ is an integral lattice then $a^2 \in \mathbf{Z}$ and $\mathcal{L} \subset \mathcal{L}^*$, see Senechal [**22**, section 2.2]. Thus, $< (0, a), (b, c) > \in \mathbf{Z}$ which implies $ac \in \mathbf{Z}$ and $c/a = p/q \in \mathbf{Q}$. Therefore, $q(b, c) - p(0, a) = (qb, 0) \in \mathcal{L}$ and it follows that the restriction of $\mathcal{L}$ to the subspace $y = 0$ is not the origin alone.

5.3. Equivariant Approach

5.3.1. $\tilde{\mathcal{L}}$-periodic projection. Corollary 5.1 states necessary and sufficient conditions such that $\Pi_{y_0}(X_\Gamma)$ is a set of $\tilde{l}$-periodic functions for some $\tilde{l} \in \mathbf{R}^n$. Let $\tilde{\mathcal{L}}$ be a lattice in $\mathbf{R}^n$. We now ask what are the necessary and sufficient conditions such that the elements in $\Pi_{y_0}(X_\Gamma)$ are $\tilde{l}$-periodic for all $\tilde{l} \in \tilde{\mathcal{L}}$, *i.e.*, $\tilde{\mathcal{L}}$-periodic functions. Proposition 5.1, below, answers this question.

We begin with functions whose set of periods is a lattice and now we study how to preserve this structure after projection.

PROPOSITION 5.1. *Let $\tilde{\mathcal{L}}$ be a lattice in $\mathbf{R}^n$ that may be written as*

$$\tilde{\mathcal{L}} = \{a_1, \ldots, a_n\}_{\mathbf{Z}},$$

for some $a_1, \dots, a_n \in \mathbf{R}^n$*, and let* M *be the matrix of the basis of* $\mathcal{L}$*. All functions in* $\Pi_{y_0}(X_\Gamma)$ *are* $\tilde{\mathcal{L}}$*-periodic if and only if one of the following conditions holds:*

1. $((c, y_0), \sigma) \notin \Gamma$ *for all* $c \in \mathbf{R}^n$*,* $(0, y_0) \notin \mathcal{L}$ *and* M *may be written in one of the forms below*

$$\text{either} \quad M = \begin{pmatrix} a_1 & 0 \\ \vdots & \vdots \\ a_n & 0 \\ a & b \end{pmatrix} \quad \text{or} \quad M = \begin{pmatrix} \frac{a_1}{m_1} & \frac{b}{m_1} \\ \vdots & \vdots \\ \frac{a_n}{m_n} & \frac{b}{m_n} \\ 0 & b \end{pmatrix}$$

for some $a \in \mathbf{R}^n$*,* $b \in \mathbf{R}$ *and some* n *nonzero integers* $m_1, \dots, m_n$*.*

If σ *belongs to the holohedry of* $\mathcal{L}$*, then* $m_i = 1$ *or* 2*, for all* $i \in \{1, \dots, n\}$*.*

2. $((a_1, y_0), \sigma) \in \Gamma$*,* $(0, y_0) \in \mathcal{L}$ *and* M *may be written as*

$$M = \begin{pmatrix} a & b \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \\ 0 & y_0/m \end{pmatrix}$$

for some $m \in \mathbf{N}$ *and* $b_2, \dots, b_n \in \mathbf{R}$*, where* (a, b) *is either* (a_1, b_1)*, for some* $b_1 \in \mathbf{R}$*, or* $(2a_1, 0)$*.*

3. $((c, y_0), \sigma) \notin \Gamma$ *for all* $c \in \mathbf{R}^n$*,* $(0, y_0) \in \mathcal{L}$ *and* M *may be written as in the previous condition with* $(a, b) = (a_1, b_1)$*, for some* $b_1 \in \mathbf{R}$*.*

4. $((a_1, y_0), \sigma) \in \Gamma$*,* $(0, y_0) \notin \mathcal{L}$ *and* M *may be written as*

$$M = \begin{pmatrix} \frac{2a_1}{m_1} & \frac{b}{m_1} \\ \frac{a_2}{m_2} & \frac{b}{m_2} \\ \vdots & \vdots \\ \frac{a_n}{m_n} & \frac{b}{m_n} \\ 0 & b \end{pmatrix}$$

for some $b \in \mathbf{R}$ *and* $m_i = 1$ *or* 2*, for all* $i \in \{1, \dots, n\}$*.*

PROOF. Let

$$\widetilde{M} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix},$$

with $a_i \in \mathbf{R}^n$ for $i = \{1, \dots, n\}$, be the matrix of the basis of $\tilde{\mathcal{L}}$ and let

$$M = \begin{pmatrix} A & B \\ a & b \end{pmatrix}$$

be the matrix of the basis for $\mathcal{L}$, where A is a n-dimensional matrix,

$$B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix},$$

$a \in \mathbf{R}^n$ and $b, b_1, \ldots, b_n \in \mathbf{R}$.

Recall that

$$\{a_1, \ldots, a_n\}_{\mathbf{R}} = \mathbf{R}^n$$

and, if either $(0, y_0) \in \mathcal{L}$ or σ belongs to the holohedry of $\mathcal{L}$, then, by the properties of the bases in subsection 4.6.1 (page 91), we may choose $a = 0$.

1. First suppose that $(0, y_0) \notin \mathcal{L}$ and that $((c, y_0), \sigma) \notin \Gamma$ for all $c \in \mathbf{R}^n$. If the functions in $\Pi_{y_0}(X_\Gamma)$ have n linearly independent periods $a_i \in \mathbf{R}^n$ then, by Corollary 5.1, $\mathcal{L}$ must contain the elements $(a_i, 0)$ for $i \in \{1, \ldots, n\}$. Thus, we may write

$$A = \begin{pmatrix} \frac{1}{m_1} a_1 \\ \vdots \\ \frac{1}{m_n} a_n \end{pmatrix} \tag{21}$$

for some integers m_i, such that $(0, m_i b_i) \in \mathcal{L}$ for all $i \in \{1, \ldots, n\}$. If there is some $b_i \neq 0$ then we may choose $a = 0$ and $b_i = b/m_i$.

If σ belongs to the holohedry then $a = 0$ and, for all $i \in \{1, \ldots, n\}$, either $b_i = 0$ or $b_i = b/2$ up to elements of the lattice. Thus, $m_i = 1$ or 2.

By Corollary 5.1, if $\mathcal{L}$ is generated by the row vectors of M built this way, then $\Pi_{y_0}(X_\Gamma)$ is a set of $\tilde{\mathcal{L}}$-periodic functions.

If $B = 0$ then $m_i = 1$ for all $i \in \{1, \ldots, n\}$ because

$$\left(\frac{1}{m_i} a_i, 0\right) \in \mathcal{L} \quad \Longrightarrow \quad \frac{1}{m_i} a_i \in \tilde{\mathcal{L}}.$$

The generator (a, b) does not contribute with periods to $\tilde{\mathcal{L}}$.

3. If $(0, y_0) \in \mathcal{L}$ and $((c, y_0), \sigma) \notin \Gamma$ for all $c \in \mathbf{R}^n$ then, by Corollary 5.1, matrix A may be written as in (21) for some integers m_i. Converselly, by condition III of Corollary 5.1, these generators would imply $\frac{1}{m_i} a_i \in \tilde{\mathcal{L}}$ and so $m_i = 1$ for all $i \in \{1, \ldots, n\}$.

4. Now suppose $(0, y_0) \notin \mathcal{L}$ and $((c, y_0), \sigma) \in \Gamma$ for some $c \in \mathbf{R}^n$, and let c be the smallest element in its direction such that $((c, y_0), \sigma) \in \Gamma$.

By condition II of Corollary 5.1, we have $c \in \tilde{\mathcal{L}}$ and we may choose the elements a_i such that $a_1 = c$.

Recall that

$$((a_1, y_0), \sigma)^2 = ((2a_1, 0), Id_{n+1}) \in \Gamma$$

and that v_δ, the translation associated to σ, is unique up to elements of $\mathcal{L}$. Thus, we ensure the periods $a_2, \ldots, a_n \in \mathbf{R}^n$ for all the functions in $\Pi_{y_0}(X_\Gamma)$ by the first condition of Corollary 5.1, *i.e.*, $(a_i, 0)$ must belong to $\mathcal{L}$ for $i \in \{2, \ldots, n\}$. The result follows as in case 1 of this Proposition.

2. Finally, if $(0, y_0) \in \mathcal{L}$ and $((c, y_0), \sigma) \in \Gamma$ for some $c \in \mathbf{R}^n$, we may choose $a_1 = c$ and follow case 3. As $a_1 \in \tilde{\mathcal{L}}$ is ensured by the existence of $((a_1, y_0), \sigma)$ in Γ, then the first row of matrix M may also be $(2a_1, 0)$. □

An immediate consequence of the previous result is that $\Pi_{y_0}(X_\Gamma)$ is a set of $\tilde{\mathcal{L}}$-periodic functions if either the projection of $\mathcal{L}$,

$$\{\tilde{l_1} : (\tilde{l_1}, \tilde{l_2}) \in \mathcal{L}\},$$

or its restriction to the hyperplane $y = 0$,

$$\{\tilde{l_1} : (\tilde{l_1}, 0) \in \mathcal{L}\},$$

is a lattice $\tilde{\mathcal{L}}'$. Notice that $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}'$ may be different but either $\tilde{\mathcal{L}} \subset \tilde{\mathcal{L}}'$ or $\tilde{\mathcal{L}}' \subset \tilde{\mathcal{L}}$.

A simpler formulation, which will be used in the next subsections, is presented in Corollary 5.3, bellow.

COROLLARY 5.3. *If all the functions in $\Pi_{y_0}(X_\Gamma)$ are $\tilde{\mathcal{L}}$-periodic, for some lattice $\tilde{\mathcal{L}}$ in $\mathbf{R}^n$, then the matrix of the basis for $\mathcal{L}$ has one of the forms*

$$M = \begin{pmatrix} A & 0 \\ a & b \end{pmatrix} \quad \text{or} \quad M = \begin{pmatrix} A & B \\ 0 & b \end{pmatrix},$$

where A, B and a are, respectively, a n-dimensional square matrix, column vector and row vector, and $b \in \mathbf{R}$.

5.3.2. the $\mathbf{E}(n+1)$-equivariant problem. Let X be a space of functions $f : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}$ whose sets of periods are lattices in $\mathbf{R}^{n+1}$. Suppose each function in X has a formal Fourier expansion and its summation commutes with the integral of the projection as the elements in X_Γ defined in chapter 2.

Let

$$\mathcal{A} : X \times \mathbf{R} \longrightarrow X$$

be a one parameter family of $\mathbf{E}(n+1)$-*equivariant* operators, *i.e.*, operators such that

$$\gamma \cdot \mathcal{A}(f, \lambda) = \mathcal{A}(\gamma \cdot f, \lambda) \qquad \forall \gamma \in \mathbf{E}(n+1) \quad \forall f \in X.$$

5.3.3. orbit of a Γ-invariant solution. Suppose that f_0 is a solution of $\mathcal{A} = 0$. By the $\mathbf{E}(n+1)$-equivariance of $\mathcal{A}$, all the functions in the orbit of f_0 by the action of $\mathbf{E}(n+1)$,

$$\mathbf{E}(n+1) \cdot f_0 = \{\gamma \cdot f_0 : \gamma \in \mathbf{E}(n+1)\},$$

are also solutions of $\mathcal{A} = 0$.

If Γ is the symmetry group of f_0 then, see Golubitsky et al. [**10**], each function $\gamma \cdot f_0$, with $\gamma = (v, \delta) \in \mathbf{E}(n+1)$, has symmetry group

$$\begin{aligned} \gamma \cdot \Gamma \cdot \gamma^{-1} &= \{(v,\delta) \cdot (u,\xi) \cdot (-\delta^{-1}v, \delta^{-1}) : (u,\xi) \in \Gamma\} \\ &= \{(v + \delta u - \delta\xi\delta^{-1}v, \delta\xi\delta^{-1}) : (u,\xi) \in \Gamma\}. \end{aligned}$$

LEMMA 5.1. *Let Γ be a subgroup of $\mathbf{E}(n+1)$ with translation subgroup $\mathcal{L} \cong \{(u, I) : u \in \mathcal{L}\}$. For $\gamma = (v, \delta) \in \mathbf{E}(n+1)$, the translation subgroup of $\gamma \cdot \Gamma \cdot \gamma^{-1}$ is $\delta\mathcal{L} \cong \{(\delta u, I) : u \in \mathcal{L}\}$.*

PROOF. For $\gamma = (v, \delta)$, the elements in $\gamma \cdot \Gamma \cdot \gamma^{-1}$ have the form

$$(v + \delta u - \delta\xi\delta^{-1}v, \delta\xi\delta^{-1}) \tag{22}$$

where $(u, \xi) \in \Gamma$. The translations in $\gamma \cdot \Gamma \cdot \gamma^{-1}$ have trivial orthogonal component $\delta\xi\delta^{-1} = Id_{n+1}$ or, equivalently, $\xi = Id_{n+1}$. These elements are, thus, $(v + \delta u - \delta Id_{n+1}\delta^{-1}v, Id_{n+1}) = (\delta u, Id_{n+1})$. □

Suppose that f_0 is a Γ-periodic function, where Γ is a crystallographic group with lattice $\mathcal{L}$ and point group $\mathbf{J}$.

By Lemma 5.1, each group $\gamma \cdot \Gamma \cdot \gamma^{-1}$, with $\gamma = (v, \delta) \in \mathbf{E}(n+1)$, is a crystallographic subgroup of $\mathbf{E}(n+1)$ with lattice $\delta\mathcal{L}$ and, by (22), point group $\delta\mathbf{J}\delta^{-1}$. Thus, each function $\gamma \cdot f_0$, in the orbit of f_0, is $(\delta\mathcal{L})$-periodic.

The orbit of a $\mathcal{L}$-periodic solution f_0 induces an orbit

$$\mathbf{O}(n+1)\mathcal{L} = \{\delta\mathcal{L} : \delta \in \mathbf{O}(n+1)\},$$

in the set of all the lattices in $\mathbf{R}^{n+1}$, and an orbit in the set of $(n+1)$-dimensional crystallographic subgroups whose elements are $\gamma \cdot \Gamma \cdot \gamma^{-1}$, with $\gamma \in \mathbf{E}(n+1)$.

5.3.4. orbit of X_Γ. In the context of $\mathbf{E}(n+1)$-equivariant problems, instead of the space X_Γ of functions, we should consider the orbit

$$\mathbf{E}(n+1) \cdot X_\Gamma = \{X_{\gamma \cdot \Gamma \cdot \gamma^{-1}} : \gamma \in \mathbf{E}(n+1)\}.$$

Each element $X_{\gamma \cdot \Gamma \cdot \gamma^{-1}}$, with $\gamma = (v, \delta) \in \mathbf{E}(n+1)$, is a space of $\gamma \cdot \Gamma \cdot \gamma^{-1}$-invariant functions, where $\gamma \cdot \Gamma \cdot \gamma^{-1}$ is a crystallographic group with latice $\delta\mathcal{L}$ and point group $\delta\mathbf{J}\delta^{-1}$.

5.3.5. $\tilde{\mathcal{L}}$-periodic projected solutions. Until now, our results concern all the functions in a certain function space, as X_Γ or $\Pi_{y_0}(X_\Gamma)$. When we study a particular function, like $f_0 \in X_\Gamma$, we may obtain particular effects. For example, $\Pi_{y_0}(f_0)$ may be periodic even if none of the conditions of Corollary 5.1 are satisfied, only due to some particular structure of f_0. We will classify as *generical* the conclusions about a particular function that would be valid when stated for all the functions in the given function space. For example, generically, a $\mathcal{L}$-periodic function has symmetry group $\mathcal{L}$.

Suppose that $\Pi_{y_0}(f_0)$ is a $\tilde{\mathcal{L}}$-periodic function, where $\tilde{\mathcal{L}}$ is a lattice in $\mathbf{R}^n$ and f_0 is a Γ-invariant solution of $\mathcal{A} = 0$.

Thus, generically, the lattices $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are related as described in Proposition 5.1 and, in particular, the matrix of the basis of $\mathcal{L}$ is

$$\text{either } M = \begin{pmatrix} A & 0 \\ a & b \end{pmatrix} \text{ or } M = \begin{pmatrix} A & B \\ 0 & b \end{pmatrix},$$

for some a, b, A and B, as stated in Corollary 5.3.

5.3.6. orbits after projection. Let f_0 be under the conditions described in the previous subsection. Now we study the periodicity after projection of the functions in $\mathbf{E}(n+1) \cdot f_0$, for $\gamma = (v, \delta)$.

- If

  $$\delta = \begin{pmatrix} \tilde{\delta} & 0 \\ 0 & \pm 1 \end{pmatrix}$$

  for some $\tilde{\delta} \in \mathbf{O}(n)$ then each projected function $\Pi_{y_0}(\gamma \cdot f_0)$ is $(\tilde{\delta}\tilde{\mathcal{L}})$-periodic.
- Suppose δ is not as in the previous item and let $\delta\mathcal{L}$ be generated by the rows of a matrix

  $$\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{a} & \tilde{b} \end{pmatrix},$$

  for some $\tilde{a}$, $\tilde{b}$, $\tilde{A}$ and $\tilde{B}$, such that either $\tilde{a} = 0$ or $\tilde{B} = 0$. Thus, by Proposition 5.1, $\Pi_{y_0}(\gamma \cdot f_0)$ may be $\tilde{\mathcal{L}}'$-periodic for some lattice $\tilde{\mathcal{L}}'$ in $\mathbf{R}^n$ that is not related to $\tilde{\mathcal{L}}$ by any orthogonal transformation.
- If the matrix of the basis of $\delta\mathcal{L}$ is

  $$\text{neither } \begin{pmatrix} A & 0 \\ a & b \end{pmatrix} \text{ nor } \begin{pmatrix} A & B \\ 0 & b \end{pmatrix},$$

  for some a, b, A and B, as stated in Corollary 5.3, then generically the projection of the $(\delta\mathcal{L})$-periodic function, $\gamma \cdot f_0$, is not a $\tilde{\mathcal{L}}'$-periodic function for any lattice $\tilde{\mathcal{L}}'$ in $\mathbf{R}^n$.

Thus, the orbit of a $\tilde{\mathcal{L}}$-periodic solution contains functions with very different periodicity properties. In this orbit there is a $(\tilde{\delta}\tilde{\mathcal{L}})$-periodic function for each $\tilde{\delta} \in \mathbf{O}(n)$. There are $\tilde{\mathcal{L}}'$-periodic functions, for other lattices $\tilde{\mathcal{L}}' \in \mathbf{R}^n$, and there are also functions whose sets of periods are not lattices, *i.e.*, are sets whose number of noncolinear generators is either $n+1$ or less than n.

5.3.7. how to detect a projection - II. Suppose we have a model whose solutions have domain $\mathbf{R}^n$. Let $\tilde{\mathcal{L}}$ be a lattice in $\mathbf{R}^n$ and suppose we have a $\tilde{\mathcal{L}}$-periodic solution, g_0. We want to describe two possible orbits of this pattern

a. for a model presupposing that n-dimensional solutions are the projection of solutions in $\mathbf{R}^{n+1}$, or
b. considering a model whose natural domain is $\mathbf{R}^n$.

a. Following the notation introduced in subsections 5.3.2 and 5.3.3, the equation $\mathcal{A} = 0$, where $\mathcal{A}$ is a $\mathbf{E}(n+1)$-equivariant operator, models the problem we study. We look for solutions of this equation that are invariant under crystallographic subgroups of $\mathbf{E}(n+1)$. However, these are intermediate solutions. Final solutions are obtained applying the projection operator Π_{y_0} to the functions satisfying $\mathcal{A} = 0$.

Suppose that the referred solution g_0 is the projection of a $(n+1)$-dimensional function f_0 such that $\mathcal{A}(f_0, \lambda) = 0$ and that f_0 is a Γ-invariant function, where Γ is a crystallographic group with lattice $\mathcal{L}$. As g_0 is $\tilde{\mathcal{L}}$-periodic, its orbit is described in subsection 5.3.6, above.

Thus, for a model with solutions that are the projection of functions posed in $\mathbf{R}^{n+1}$, we may conclude that, generically, the orbit of a $\tilde{\mathcal{L}}$-periodic solution g_0 will contain a $\tilde{\delta}\tilde{\mathcal{L}}$-periodic function, for all $\tilde{\delta} \in \mathbf{O}(n)$, other functions whose periods form different lattices in $\mathbf{R}^n$ and functions whose set of periods is not a lattice in $\mathbf{R}^n$.

b. Now we have a model whose solutions satisfy $\tilde{\mathcal{A}} = 0$, where $\tilde{\mathcal{A}}$ is a $\mathbf{E}(n)$-equivariant operator. Let g_0 be a solution. As described in subsections 5.3.2 and 5.3.3, the orbit of g_0 is $\mathbf{E}(n) \cdot g_0 = \{\tilde{\gamma} \cdot g_0 : \tilde{\gamma} = (\tilde{v}, \tilde{\delta}) \in \mathbf{E}(n)\}$. If g_0 is a $\tilde{\mathcal{L}}$-periodic solution then, by Lemma 5.1, all the elements in its orbit have lattices as set of periods. Moreover, these lattices are related to $\tilde{\mathcal{L}}$ by the orthogonal transformations $\tilde{\delta} \in \mathbf{O}(n)$.

Thus, if we have information about the orbits of $\tilde{\mathcal{L}}$-periodic solutions then we may distinguish between the two models, a and b. Can one achieve this by twiddling initial or boundary conditions?

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