TOPOLOGICAL STRUCTURE OF PARTIALLY HYPERBOLIC ATTRACTORS

by

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Contents

Introduction

Since the 60's that hyperbolic sets play an important role in the development of the Theory of Dynamical Systems. A hyperbolic set is a compact invariant set over which the tangent bundle splits into two invariant subbundles, one of them contracting and the other one expanding. In this notes we are concerned with discrete time dynamical systems (smooth transformations of a compact manifold). In the last decades an increasing emphasis has been put on the dynamics of partially hyperbolic sets, i.e. compact invariant sets for which the tangent bundle has a dominated splitting into two invariant subbundles having contracting/expanding behavior in one direction. Precise definitions of all these objects will be given in the next section.

In this context, a special role has been played by the horseshoes, which were introduced by Smale in the sixties, and as shown in [**Sm1**], always appear near a transverse homoclinic point associated to some hyperbolic periodic point of saddle type, i.e. a point whose orbit asymptotically approaches that saddle point, both in the past and in the future. Horseshoes can be used to show that transverse homoclinic points are accumulated by periodic points, but the usefulness of these objects goes far beyond this issue. Horseshoes are Cantor sets which are, in dynamical terms, topologically conjugated to full shifts. A special interest lies in the horseshoes that appear when one unfolds a homoclinic tangency. Knowing how *fat* these horseshoes are can have several implications in the dynamical behavior after the homoclinic bifurcation. In this setting we mention the *thickness*, which has been used by Newhouse [**Ne**] to prove the existence of infinitely many sinks, and the *Hausdorff dimension*, which has been used by Moreira, Palis, Takens, Yoccoz to study the prevalence of hyperbolicity after the unfolding of a homoclinic tangency; see [**MY, PT, PY1, PY2**].

One interesting subject is the Lebesgue measure (volume) of horseshoes. As shown by Bowen in [**Bo1**], there are C^1 diffeomorphisms with hyperbolic horseshoes of positive volume. On the other hand, Bowen has proved in [**Bo2**] that a basic set (locally maximal hyperbolic set with a dense orbit) of a C^2 diffeomorphism which attracts a set with positive volume, necessarily attracts a neighborhood of itself. In particular, the unstable manifolds through points of this set must be contained in it, and consequently C^2 diffeomorphisms have no horseshoes with positive volume.

For diffeomorphisms whose differentiability is higher than one, Alves and Pinheiro prove in [**AP1**] the nonexistence of horseshoe-like sets with positive volume in a context of sets with some partially hyperbolic structure. Going back to context of hyperbolic sets, it is shown in [**AP1**] that the above mentioned result in [**Bo2**] still holds without the local maximality assumption, i.e. a transitive hyperbolic set which attracts a set with positive volume necessarily attracts a neighborhood of itself. It is also proven in [**AP1**] that there are no proper transitive hyperbolic sets with positive volume for diffeomorphisms whose differentiability is higher than one. Similar results for sets with nonempty interior had already been obtained by Abdenur, Bonatti, Díaz in [**ABD**] and Fisher in [**Fi**]. On the other hand, as described in [**ABD**], there exist (non-transitive) hyperbolic sets with positive volume which do not attract neighborhoods of themselves; see [**Fi**] for a detailed construction. The results in [**AP1**] give also a description of the limit set of almost every point in a hyperbolic set with positive volume: there is a finite number of basic sets for which the ω-limit set of Lebesgue almost every point is contained in one of these basic sets. In the partially hyperbolic context these ω-limit sets are contained in the closure of finitely many hyperbolic periodic points. In the conservative setting, Bochi and Viana show in [**BV**] that a hyperbolic set for a volume preserving C^2 diffeomorphism has either zero volume or coincides with the whole manifold.

Markov partitions have played a fundamental role in the development of the theory of hyperbolic dynamical systems. For non-uniformly hyperbolic dynamical systems, Benedicks and Young introduced in [**BY**] some structures of Markov style in certain regions of the phase space with *infinitely many branches* and *variable return times*. This structures enabled them to obtain exponential decay of correlations and deduce the Central Limit Theorem for Hénon maps. Further developments by Young in [**Yo2**] lead to a uniform theory for some non-uniformly hyperbolic diffeomorphisms, including Hénon maps, billiards with convex scatterers and Axiom A attractors. This kind of approach has also been successfully implemented by Young in [**Yo3**] for studying the rates of mixing of non-invertible systems with some non-uniformly expanding behavior. These Markov structures play also a key role in [**ACF**] for proving the continuity of the SRB measure for Hénon maps of the Benedicks-Carleson type. The existence of this Markov structures for partially hyperbolic systems whose central direction is mostly expanding is part of content in [**AP3**].

These notes were prepared for a course lectured in the Smooth Ergodic Theory Workshop at the Morningside Center of Mathematics, Chinese Academy of Sciences, May 2007. Most of this presentation is based on the works [**ABV, ALP, AP1, AP3**], with the fist two sections corresponding to [**ABV, AP1**] and the last one corresponding to [**ALP, AP3**]. The example from [**Bo2**] is also presented in the first section.

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1. Partially hyperbolic sets

Let $f : M \to M$ be a diffeomorphism of a compact connected Riemannian manifold M. We say that f is C^{1+} if f is C^1 and Df is Hölder continuous. We use Leb to denote a normalized volume form extended to the Borel sets of M that we call *Lebesgue measure.* Given a submanifold $\gamma \subset M$ we use Leb_γ to denote the measure on γ induced by the restriction of the Riemannian structure to γ. A set $\Lambda \subset M$ is said to be *invariant* if $f(\Lambda) = \Lambda$, and *forward invariant* if $f(\Lambda) \subset \Lambda$.

Definition 1.1. — Given a forward invariant compact set K, we define

$$\Lambda = \bigcap_{n \geq 0} f^n(K).$$

Suppose that there is a continuous splitting $T_K M = E^{cs} \oplus E^{cu}$ of the tangent bundle restricted to K, and assume that this splitting is Df-invariant over Λ.

1. This is a *dominated splitting* (over Λ) if there is a constant $0 < \lambda < 1$ such that for some choice of a Riemannian metric on M

$$\|Df \mid E^{cs}_x\| \cdot \|Df^{-1} \mid E^{cu}_{f(x)}\| \leq \lambda, \quad \text{for every } x \in \Lambda.$$

   We call E^{cs} the *centre-stable bundle* and E^{cu} the *centre-unstable bundle.*
2. We say that Λ is *partially hyperbolic*, if additionally E^{cs} is *uniformly contracting* or E^{cu} is *uniformly expanding*, meaning that there is $0 < \lambda < 1$ such that, respectively,

$$\|Df \mid E^{cs}_x\| \leq \lambda, \quad \text{for every } x \in \Lambda,$$

   or

$$\|Df^{-1} \mid E^{cu}_{f(x)}\| \leq \lambda, \quad \text{for every } x \in \Lambda.$$

3. We say that f is *non-uniformly expanding along the centre-unstable direction* for a point $x \in K$ if

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Df^{-1} \mid E^{cu}_{f^j(x)}\| < -c. \tag{NUE}$$

Condition NUE means that the derivative has *expanding behavior in the centre-unstable direction in average* over the orbit of x for an infinite number of times. Let us mention that if condition NUE holds for every point in a compact invariant set Λ, then E^{cu} is *uniformly expanding in the centre-unstable direction in* Λ; see [**AAS, Ca**].

Example 1.2. — Take a linear Anosov diffeomorphism f_0 on the d-dimensional torus $M = T^d$ with $d \geq 3$. We write $TM = E^u \oplus E^s$ the corresponding hyperbolic decomposition. Let V be a small closed domain in M such that there exist unit open cubes K^0 and K^1 in $\mathbb{R}^d$ such that $V \subset \pi(K^0)$ and $f_0(V) \subset \pi(K^1)$, where $\pi : \mathbb{R}^d \to T^d$ is the canonical projection. Now, let f be a diffeomorphism on T^d such that

(a) f admits invariant cone fields C^{cu} and C^{cs}, with small width $\alpha > 0$ and containing, respectively, the unstable bundle E^u and the stable bundle E^s of the Anosov diffeomorphism f_0;

(b) there is $\sigma_1 > 1$ so that

$$|\det(Df \mid T_x\mathcal{D}^{cu})| > \sigma_1$$

for any $x \in M$ and any disk $\mathcal{D}^{cu}$ through x tangent to C^{cu}.

(c) there exists $\sigma_2 < 1$ satisfying

$$\|(Df \mid T_x\mathcal{D}^{cu})^{-1}\| < \sigma_2$$

for any $x \in M \setminus V$ and any disk $\mathcal{D}^{cu}$ through x tangent to C^{cu}.

(d) there exists some small $\delta_0 > 0$ satisfying

$$\|(Df \mid T_x\mathcal{D}^{cu})^{-1}\| < (1 + \delta_0)$$

for any $x \in V$ and any disks $\mathcal{D}^{cu}$ tangent to C^{cu}.

For instance, if f_1 is a torus diffeomorphism satisfying (a), (b), (d), and coinciding with f_0 outside V, then any map f in a C^1 neighbourhood of f_1 satisfies all the previous conditions.

We argue that any f satisfying (a)–(d) is non-uniformly expanding along its centre-unstable direction in a strong sense: there is $c > 0$ such that

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Df^{-1} \mid E^{cu}_{f^j(x)}\| < -c. \tag{1}$$

on a full Lebesgue measure set of points $x \in M$.

To explain this, let $B_1, \ldots, B_p, B_{p+1} = V$ be any partition of T^d into small domains, in the sense that there exist open unit cubes K_i^0 and K_i^1 in $\mathbb{R}^d$ such that

$$B_i \subset \pi(K_i^0) \quad \text{and} \quad f(B_i) \subset \pi(K_i^1). \tag{2}$$

Let $\mathcal{F}_0^u$ be the unstable foliation of f_0, and $\mathcal{F}_j = f^j(\mathcal{F}_0^u)$ for every $j \geq 0$. By (a), each $\mathcal{F}_j$ is a foliation of T^d tangent to the centre-unstable cone field C^{cu}. For any subset E of a leaf of $\mathcal{F}_j$ with $j \geq 0$, we denote $\mathrm{Leb}_j(E)$ the Lebesgue measure of E inside that leaf.

Claim 1. — *There is $C_0 > 0$ such that for any small disk D_0 contained in a leaf of $\mathcal{F}_0$*

$$\mathrm{Leb}_0(\{x \in D_0 : f^j(x) \in B_{i_j} \text{ for } 0 \leq j < n\}) \leq C_0\sigma_1^{-n}$$

for every $n \geq 1$ and every $i_0, \ldots, i_{n-1}$ in $\{1, \ldots, p, p+1\}$

Proof. — Let $\tilde{\mathcal{F}}_j$ be the lift to $\mathbb{R}^d$ of $\mathcal{F}_j$, for $j \geq 0$. Using (2) one can easily conclude, by induction on j, that $f^j([\underline{i}])$ is contained in the image $\pi(K^1_{j-1} \cap \tilde{F}_j)$ of the intersection of K^1_{j-1} with some leaf $\tilde{F}_j$ of $\tilde{\mathcal{F}}_j$, for every $0 \leq j \leq n$. So, using (b) and the fact that $(\pi \mid K^1_{n-1})$ is a diffeomorphism and an isometry onto its image,

$$\mathrm{Leb}_0([\underline{i}]) \leq \sigma_1^{-n} \mathrm{Leb}_n(f^n([\underline{i}])) \leq \sigma_1^{-n} \mathrm{Leb}_n(F_n \cap K^1_{n-1}). \tag{3}$$

Recall that we took f_0 linear, so that its unstable foliation $\mathcal{F}_0^u$ lifts to a foliation $\tilde{\mathcal{F}}_0^u$ of $\mathbb{R}^d$ by affine hyperplanes. The leaves of every $\tilde{\mathcal{F}}_n$ are C^1 submanifolds of $\mathbb{R}^d$ transverse to these hyperplanes, with angles uniformly bounded away from zero at every intersection point. Consequently, the intersection of a leaf of $\tilde{\mathcal{F}}_n$ with any unit cube in $\mathbb{R}^d$ has Lebesgue measure (inside the leaf) bounded by some uniform constant C_0. In particular, the last factor in (3) is bounded by C_0. $\square$

Claim 2. — *There exists $\theta > 0$ such that the orbit of Lebesgue almost every point $x \in D_0$ spends a fraction θ of the time in $B_1 \cup \cdots \cup B_p$, that is,*

$$\#\{0 \leq j < n : f^j(x) \in B_1 \cup \cdots \cup B_p\} \geq \theta n$$

for every large n.

Proof. — Let n be fixed. Given a sequence $\underline{i} = (i_0, i_1, \ldots, i_{n-1})$ in $\{1, \ldots, p+1\}$, we denote

$$[\underline{i}] = B_{i_0} \cap f^{-1}(B_{i_1}) \cap \cdots \cap f^{-n+1}(B_{i_{n-1}}).$$

Moreover, we define $g(\underline{i})$ to be the number of values of $0 \leq j \leq n-1$ for which $i_j \leq p$. We begin by noting that, given any $\theta > 0$, the total number of sequences $\underline{i}$ for which $g(\underline{i}) < \theta n$ is bounded by

$$\sum_{k<\theta n} \binom{n}{k} p^k \leq \sum_{k \leq \theta n} \binom{n}{k} p^{\theta n}$$

A standard application of Stirling's formula (see e.g. [**BV**, Section 6.3]) gives that the last expression is bounded by $e^{\gamma n} p^{\theta n}$, where γ depends only on θ and goes to zero when θ goes to zero.

On the other hand, by Claim 1 we have $\mathrm{Leb}([\underline{i}]) \leq C_0 \sigma_1^{-n}$. Then the measure of the union I_n of all the sets $[\underline{i}]$ with $g(\underline{i}) < \theta n$ is less than $C_0 \sigma_1^{-(1-\theta)n} e^{\gamma n} p^{\theta n}$. Since $\sigma_1 > 1$, we may fix θ small so that $e^{\gamma} p^{\theta} < \sigma_1^{1-\theta}$. This means that the Lebesgue measure of I_n goes to zero exponentially fast as $n \to \infty$. Thus, by Borel-Cantelli lemma, Lebesgue almost every point $x \in D_0$ belongs in only finitely many sets I_n. Clearly, any such point x satisfies the conclusion of the lemma. □

Let $\theta > 0$ be the constant given by Claim 2, and fix $\delta > 0$ small enough so that $\sigma_0^{\theta}(1+\delta) \leq e^{-c}$ for some $c > 0$. Let x be any point satisfying the conclusion of the lemma. Then

$$\prod_{j=0}^{n-1} \|Df^{-1} \mid E^{cu}_{f^j(x)}\| \leq \sigma_0^{\theta n}(1+\delta)^{(1-\theta)n} \leq e^{-cn}$$

for every large enough n. This implies that

$$\limsup_{n \to +\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Df^{-1} \mid E^{cu}_{f^j(x)}\| < -c$$

for Leb_0-almost every point $x \in D_0$. Since D_0 was an arbitrary disk inside a leaf of $\mathcal{F}_0^s$, and the latter is an absolutely continuous foliation, we conclude that f is non-uniformly expanding along E^{cu}, Lebesgue almost everywhere in $M = T^d$.

1.1. Sets with positive volume.

— In this section we present some results on the topological structure of partially hyperbolic sets attracting a positive volume set of orbits with non-uniform expansion, whose proofs we leave to the next sections.

Definition 1.3. — We say that an embedded disk $\gamma \subset M$ is an *unstable manifold*, or an *unstable disk*, if $\mathrm{dist}(f^{-n}(x), f^{-n}(y)) \to 0$ exponentially fast as $n \to \infty$, for every $x, y \in \gamma$. Similarly, γ is called a *stable manifold*, or a *stable disk*, if $\mathrm{dist}(f^n(x), f^n(y)) \to 0$ exponentially fast as $n \to \infty$, for every $x, y \in \gamma$.

It is well-known that every point in a hyperbolic set possesses a local stable manifold $W^s_{loc}(x)$ and a local unstable manifold $W^u_{loc}(x)$ which are disks tangent to E^s_x and E^u_x at x respectively.

Theorem 1.4. — *Let $f : M \to M$ be a C^{1+} diffeomorphism and let $K \subset M$ be a forward invariant compact set with a continuous splitting $T_K M = E^{cs} \oplus E^{cu}$ dominated over $\Lambda = \bigcap_{n\geq 0} f^n(K)$. If NUE holds for a positive Lebesgue set of points $x \in K$, then Λ contains some local unstable manifold.*

The next result is a direct consequence of Theorem 1.4, whenever E^{cu} is uniformly expanding. If, on the other hand, E^{cs} is uniformly contracting, then we just have to apply Theorem 1.4 to f^{-1}.

Corollary 1.5. — *Let $f : M \to M$ be a C^{1+} diffeomorphism and let $\Lambda \subset M$ be a compact invariant set with a dominated splitting and* $\mathrm{Leb}(\Lambda) > 0$.

1. *If E^{cs} is uniformly contracting, then Λ contains some local stable manifold.*
2. *If E^{cu} is uniformly expanding, then Λ contains some local unstable manifold.*

The same conclusions hold for partially hyperbolic sets intersecting a local stable manifold or a local unstable manifold in a positive Lebesgue measure subset, as Corollary 1.7 below shows.

Theorem 1.6. — *Let $f : M \to M$ be a C^{1+} diffeomorphism and let $K \subset M$ be a forward invariant compact set with a continuous splitting $T_K M = E^{cs} \oplus E^{cu}$ dominated over $\Lambda = \bigcap_{n\geq 0} f^n(K)$. Assume that there is a local unstable manifold γ such that NUE holds for every x in a positive* Leb_γ *subset of $\gamma \cap K$. Then Λ contains some local unstable manifold.*

The next result is an immediate consequence of Theorem 1.6, in the case that E^{cu} is uniformly expanding, and a consequence of the same theorem applied to f^{-1} when E^{cs} is uniformly contracting. Actually, we shall prove a stronger version of this result in Theorem 1.17.

Corollary 1.7. — *Let $f : M \to M$ be a C^{1+} diffeomorphism and let $\Lambda \subset M$ with* $\mathrm{Leb}(\Lambda) > 0$ *be a compact invariant set with a dominated splitting.*

1. *If E^{cs} is uniformly contracting and there exists a local stable manifold γ such that* $\mathrm{Leb}_\gamma(\gamma \cap \Lambda) > 0$, *then Λ contains some local stable manifold.*
2. *If E^{cu} is uniformly expanding and there exists a local unstable manifold γ such that* $\mathrm{Leb}_\gamma(\gamma \cap \Lambda) > 0$, *then Λ contains some local unstable manifold.*

The previous results give in particular that there cannot be partially hyperbolic horseshoes with positive volume, provided the dynamics is C^{1+}. If the diffeomorphism is just C^1 then the previous conclusions do not hold. The following example was presented by Bowen in [**Bo1**] and makes this point clear.

Example 1.8. — We shall construct a C^1 diffeomorphism which has a compact invariant hyperbolic Cantor set (horseshoe) with positive Lebesgue measure. First we consider some notation for Cantor sets. Let I be a closed interval and $(\alpha_n)_n$ a squence of positive numbers with $\sum_{n\geq 0}\alpha_n \leq \text{length}(I)$. Let $\underline{a}=a_1a_2\dots a_n$ denote a sequence of 0's and l's of length $n=n(\underline{a})$; we allow the empty sequence $\underline{a}=\emptyset$ with $n(\emptyset)=0$. Define

$$I_\emptyset = I=[a,b], \quad I^*_\emptyset=\left[\frac{a+b}{2}-\frac{\alpha_0}{2},\frac{a+b}{2}+\frac{\alpha_0}{2}\right]$$

and $I^*_{\underline{a}}\subset I_{\underline{a}}$ recursively as follows: let $I_{\underline{a}0}$ and $I_{\underline{a}1}$ be the left and right intervals remaining when the interior of $I^*_{\underline{a}}$ is removed from $I_{\underline{a}}$; let $I^*_{\underline{a}k}$ (k=0, 1) be the closed interval of length $\alpha_{n(\underline{a}k)}2^{-n(\underline{a}k)}$ and having the same center as $I_{\underline{a}k}$. The Cantor set K_I is given as

$$K_I=\bigcap_{m\geq 0}\bigcup_{n(\underline{a})=m} I_{\underline{a}}.$$

This is the standard construction of the Cantor set except that we permit some flexibility in the lengths of intervals removed. The measure of K_I is

$$m(K_I)=\text{length}(I)-\sum_{n\geq 0}\alpha_n$$

Assume now we have another interval J and $\beta_n>0$ with $\sum_{n\geq 0}\beta_n\leq \text{length}(I)$. One can then construct $J^*_{\underline{a}}\subset J_{\underline{a}}$ and K_J as above. Suppose that $\beta_n\alpha_n^{-1}\to\gamma\geq 0$ as $n\to\infty$. Pick a sequence $\delta_n\to 0$ and for each $\underline{a}$ let $g: I^*_{\underline{a}}\to J^*_{\underline{a}}$ be a C^1 orientation preserving diffeomorphism such that:

(i) $g'(x)=\gamma$ for x and endpoint of $I^*_{\underline{a}}$;

(ii) $g'(I^*_{\underline{a}})$ is contained in the interval spanned by $\gamma\pm\delta_n$ and $\dfrac{\beta_n}{\alpha_n}\pm\delta_n$.

Then g extends from $\cup_{\underline{a}}I^*_{\underline{a}}$ by continuity to a homeomorphism $g: I\to J$ which is C^1 with $g'(x)=\gamma$ at each point $x\in K_I$.

We will now construct a horseshoe with positive measure. Choose $\beta_n>0$ with

$$\sum_{n\geq 0}\beta_n<2 \quad\text{and}\quad \frac{\beta_{n+1}}{\beta_n}\to 1 \quad \left(\text{e.g. } \beta_n=\frac{1}{(n+100)^2}\right).$$

Let

$$J=[-1,1],\quad I=\left[\frac{\beta_0}{2},1\right] \quad\text{and}\quad \alpha_n=\frac{\beta_{n+1}}{2}.$$

Then

$$\sum_{n\geq 0}\alpha_n<\text{length}(I) \quad\text{and}\quad \gamma=\lim\frac{\beta_n}{\alpha_n}=\lim\frac{2\beta_n}{\beta_{n+1}}=2.$$

So we get a C^1 diffeomorphism $g\colon I\to J$ as above. One defines a diffeomorphism f of the square $S=J\times J$ into $\mathbb{R}^2$ by

(i) $f(x,y)=(g(x),g^{-1}(y)$ for $(x,y)\in I\times J$;
(ii) $f(x,y)=(g(-x),-g^{-1}(y)$ for $(x,y)\in(-I)\times J$;
(iii) $f(T)\cap(J\times J)=\emptyset$, where $T=\left(-\dfrac{\beta_0}{2},\dfrac{\beta_0}{2}\right)\times J$.

Then

$$\Lambda=\bigcap_{n=-\infty}^{+\infty}f^n(S)=K_J\times K_J$$

has Lebesgue measure

$$m(\Lambda) = m(K_J)^2 = \left(2 - \sum_{n\geq 0} \beta_n\right)^2 > 0.$$

The mapping f extends to a diffeomorphism of S^2 exactly as in [**Sm2**].

1.2. Hölder control of tangent direction. — In this section we present some results from [**ABV**] concerning the Hölder control of the tangent direction of certain submanifolds.

Let K be a forward invariant compact set for which there is a continuous splitting $T_K M = E^{cs} \oplus E^{cu}$ of the tangent bundle restricted to K which is Df-invariant over

$$\Lambda = \bigcap_{n\geq 0} f^n(K).$$

We fix continuous extensions of the two bundles E^{cs} and E^{cu} to some compact neighborhood U of Λ, that we still denote by E^{cs} and E^{cu}. Replacing K by a forward iterate of it, if necessary, we may assume that $K \subset U$.

Definition 1.9. — Given $0 < a < 1$, we define the *centre-unstable cone field* $(C_a^{cu}(x))_{x\in U}$ *of width* a by

$$C_a^{cu}(x) = \big\{v_1 + v_2 \in E_x^{cs} \oplus E_x^{cu} \text{ such that } \|v_1\| \leq a\|v_2\|\big\}. \tag{4}$$

We define the *centre-stable cone field* $(C_a^{cs}(x))_{x\in U}$ *of width* a in a similar way, just reversing the roles of the subbundles in (4).

We fix $a > 0$ and U small enough so that, up to slightly increasing $\lambda < 1$, the domination condition remains valid for any pair of vectors in the two cone fields, i.e.

$$\|Df(x)v^{cs}\| \cdot \|Df^{-1}(f(x))v^{cu}\| \leq \lambda \|v^{cs}\| \, \|v^{cu}\|,$$

for every $v^{cs} \in C_a^{cs}(x)$, $v^{cu} \in C_a^{cu}(f(x))$, and any $x \in U \cap f^{-1}(U)$. Note that the centre-unstable cone field is forward invariant:

$$Df(x)C_a^{cu}(x) \subset C_a^{cu}(f(x)), \quad \text{whenever } x, f(x) \in U.$$

Indeed, the domination property together with the invariance of E^{cu} over Λ imply that

$$Df(x)C_a^{cu}(x) \subset C_{\lambda a}^{cu}(f(x)), \quad \text{for every } x \in \Lambda. \tag{5}$$

This extends to any $x \in U \cap f^{-1}(U)$ just by continuity, slightly increasing $\lambda < 1$, if necessary.

Definition 1.10. — We say that an embedded C^1 submanifold $N \subset U$ is *tangent to the centre-unstable cone field* if the tangent subspace to N at each point $x \in N$ is contained in the corresponding cone $C_a^{cu}(x)$. Then $f(N)$ is also tangent to the centre-unstable cone field, if it is contained in U, by the domination property.

We choose $\delta_0 > 0$ small enough so that the inverse of the exponential map $\exp_x$ is defined on the δ_0 neighbourhood of every point x in U. From now on we identify this neighbourhood of x with the corresponding neighbourhood U_x of the origin in T_xN, through the local chart defined by $\exp_x^{-1}$. Reducing δ_0, if necessary, we may suppose that E_x^{cs} is contained in the centre-stable cone $C_a^{cs}(y)$ of every $y \in U_x$. In particular, the intersection of $C_a^{cu}(y)$ with

E_x^{cs} reduces to the zero vector. Then, the tangent space to N at y is parallel to the graph of a unique linear map $A_x(y) : T_xN \to E_x^{cs}$.

Definition 1.11. — Given constants $C > 0$ and $0 < \zeta \le 1$, we say that *the tangent bundle to N is (C, ζ)-Hölder* if for every $y \in N \cap U_x$ and $x \in U$

$$\|A_x(y)\| \le C d_x(y)^\zeta, \tag{6}$$

where $d_x(y)$ denotes the distance from x to y along $N \cap U_x$, defined as the length of the shortest curve connecting x to y inside $N \cap U_x$.

Recall that we have chosen the neighbourhood U and the cone width a sufficiently small so that the domination property remains valid for vectors in the cones $C_a^{cs}(z)$, $C_a^{cu}(z)$, and for any point z in U. Then, there exist $\lambda_1 \in (\lambda, 1)$ and $\zeta \in (0, 1]$ such that

$$\|Df(z)v^{cs}\| \cdot \|Df^{-1}(f(z))v^{cu}\|^{1+\zeta} \le \lambda_1 < 1 \tag{7}$$

for every norm 1 vectors $v^{cs} \in C_a^{cs}(z)$ and $v^{cu} \in C_a^{cu}(z)$, at any $z \in U$. Then, up to reducing $\delta_0 > 0$ and slightly increasing $\lambda_1 < 1$, condition (7) remains true if we replace z by any $y \in U_x$, with $x \in U$ (taking $\|\cdot\|$ to mean the Riemannian metric in the corresponding local chart).

We fix ζ and λ_1 as above, choosing ζ with the additional property that f is of class $C^{1+\zeta}$. Given a C^1 submanifold $N \subset U$, we define

$$\kappa(N) = \inf\{C > 0 : \text{the tangent bundle of } N \text{ is } (C, \zeta)\text{-Hölder}\}. \tag{8}$$

Proposition 1.12. — *There exist $\lambda_0 < 1$ and $C_0 > 0$ so that if $N \subset U \cap f^{-1}(U)$ is any C^1 submanifold tangent to the centre-unstable cone field, then*

$$\kappa(f(N)) \le \lambda_0\, \kappa(N) + C_0\,.$$

Proof. — We only need to consider the case when $\kappa(N)$ is finite, that is, the tangent bundle of N is (C, ζ)-Hölder for some $C > 0$. Let $x \in N$ be fixed. We use $(u, s) \in T_xN \oplus E_x^{cs}$ and $(u_1, s_1) \in T_{f(x)}f(N) \oplus E_{f(x)}^{cs}$, respectively, to represent the local coordinates in U_x and $U_{f(x)}$ introduced above. We write the expression of our map in these local coordinates as $f(u, s) = (u_1(u, s), s_1(u, s))$. Observe that if $x \in K$ then the partial derivatives of u_1 and s_1 at the origin $0 \in T_xN$ are

$$\partial_u u_1(0) = Df|T_xN, \quad \partial_s u_1(0) = 0, \quad \partial_u s_1(0) = 0, \quad \partial_s s_1(0) = Df|E_x^{cs}\,.$$

This is because $E_x^{cs} = E_x^{cs}$ is mapped to $E_{f(x)}^{cs} = E_{f(x)}^{cs}$ under $Df(x)$ and, similarly, T_xN is mapped to $T_{f(x)}N$. Then, given any small $\varepsilon_0 > 0$ we have that

$$\|\partial_u u_1(y) - Df|T_xN\|, \quad \|\partial_s u_1(y)\|, \quad \|\partial_u s_1(y)\|, \quad \|\partial_s s_1(y) - Df|E_x^{cs}\|, \tag{9}$$

are all less than ε_0 for every $x \in U$ and $y \in U_x$, as long as δ_0 and U are small. Taking the cone width a also small, we get

$$\|Df|T_yN - Df|E_x^{cu}\| \le \varepsilon_0 \quad \text{and} \quad \|Df^{-1}|T_{f(y)}f(N) - Df^{-1}|E_{f(x)}^{cu}\| \le \varepsilon_0\,, \tag{10}$$

for every $x \in U$ and $y \in U_x$. Since f is C^2, there is also some constant $K_2 > 0$ such that

$$\|\partial_s u_1(y)\| \le K_2 d_x(y)^\zeta \quad \text{and} \quad \|\partial_u s_1(y)\| \le K_2 d_x(y)^\zeta. \tag{11}$$

For y_1 in $U_{f(x)}$, let $A_{f(x)}(y_1)$ be the linear map from $T_{f(x)}f(N)$ to $E^{cs}_{f(x)}$ whose graph is parallel to $T_{y_1}f(N)$. We are going to prove that, fixing ε_0 sufficiently small, then $A_{f(x)}(y_1)$ satisfies (6) for any $C > \lambda_0\kappa(N) + C_0$, with convenient λ_0 and C_0. Let us begin by noting that $\|A_{f(x)}(y_1)\|$ is bounded by some uniform constant $K_1 > 0$, since $f(N)$ is tangent to the centre-unstable cone field. We will choose the constant $C_0 \geq K_1/(\delta_0/\|Df^{-1}\|)^\zeta$, so that (6) is immediate when $d_{f(x)}(y_1) \geq \delta_0/\|Df^{-1}\|$:

$$\|A_{f(x)}(y_1)\| \leq K_1 \leq C_0(\delta_0/\|Df^{-1}\|)^\zeta \leq C_0 d_{f(x)}(y_1)^\zeta.$$

Here $\|Df^{-1}\|$ is the supremum of all $\|Df^{-1}(z)\|$ with $z \in U_w$, $w \in U$, where the norms are taken with respect to the Riemannian metrics in the local charts. This permits us to restrict to the case when $d_{f(x)}(y_1) < \delta_0/\|Df^{-1}\|$ in all that follows. Let Γ_1 be any curve on $f(N)\cap U_{f(x)}$ joining $f(x)$ to y_1 and whose length approximates $d_{f(x)}(y_1)$. Then $\Gamma = f^{-1}(\Gamma_1)$ is a curve in $N \cap U_x$ joining x to $y = f^{-1}(y_1)$, with length less than δ_0. In fact, cf. (10),

$$d_x(y) \leq \operatorname{length}(\Gamma) \leq (\|Df^{-1}|E^{cu}_{f(x)}\| + \varepsilon_0)\operatorname{length}(\Gamma_1).$$

This shows that $d_x(y) \leq (\|Df^{-1}|E^{cu}_{f(x)}\| + \varepsilon_0)d_{f(x)}(y_1)$.

Now we observe that

$$A_{f(x)}(y_1) = \big[\partial_u s_1(y) + \partial_s s_1(y)\cdot A_x(y)\big]\cdot\big[\partial_u u_1(y) + \partial_s u_1(y)\cdot A_x(y)\big]^{-1}.$$

On the one hand, by (9) and (11),

$$\begin{aligned}\|\partial_u s_1(y) + \partial_s s_1(y)\cdot A_x(y)\| &\leq K_2 d_x(y)^\zeta + \big(\|Df|E^{cs}_x\| + \varepsilon_0\big)\kappa(N)d_x(y)^\zeta\\ &\leq \big(K_2 + \big(\|Df|E^{cs}_x\| + \varepsilon_0\big)\kappa(N)\big)d_x(y)^\zeta.\end{aligned}$$

On the other hand, $\|\partial_s u_1(y)\cdot A_x(y)\| \leq \varepsilon_0 K_1$, which can be made much smaller than $1/\|(\partial_u u_1(y)^{-1}\|$. As a consequence, recall (10) and (11),

$$\|\big[\partial_u u_1(y) + \partial_s s_1(y)\cdot A_x(y)\big]^{-1}\| \leq \|Df^{-1}|E^{cu}_{f(x)}\| + \varepsilon_1\,,$$

where ε_1 can be made arbitrarily small by reducing ε_0. Putting these bounds together, we conclude that $\|A_{f(x)}(y_1)\|\,d_{f(x)}(y_1)^{-\zeta}$ is less than

$$\frac{(\|Df|E^{cs}_x\| + \varepsilon_0)(\|Df^{-1}|E^{cu}_{f(x)}\| + \varepsilon_1)}{(\|Df^{-1}|E^{cu}_{f(x)}\| + \varepsilon_0)^{-\zeta}}\kappa(N) + \frac{K_2(\|Df^{-1}|E^{cu}_{f(x)}\| + \varepsilon_1)}{(\|Df^{-1}|E^{cu}_{f(x)}\| + \varepsilon_0)^{-\zeta}}.$$

Hence, choosing δ_0, U, a sufficiently small, we can make ε_0, ε_1 sufficiently close to zero so that the factor multiplying $\kappa(N)$ is less than some $\lambda_0 \in (\lambda_1, 1)$; recall (7). Moreover, the second term in the expression above is bounded by some constant that depends only on f. We take C_0 larger than this constant. □

Corollary 1.13. — *There exists $C_1, \zeta > 0$ such that, given any C^1 submanifold $N \subset U$ tangent to the centre-unstable cone field, there is $n_0 \geq 1$ such that:*

1. *$\kappa(f^n(N)) \leq C_1$ for every $n \geq n_0$ such that $f^k(N) \subset U$ for all $0 \leq k \leq n$;*
2. *if $\kappa(N) \leq C_1$, then $\kappa(f^n(N)) \leq C_1$ for $n \geq 1$ such that $f^k(N) \subset U$ for all $0 \leq k \leq n$;*
3. *if N and n are as in 2, then the functions*

$$J_k : f^k(N) \ni x \longmapsto \log|\det\big(Df \mid T_x f^k(N)\big)|, \quad 0 \leq k \leq n,$$

are (L, ζ)-Hölder continuous with $L > 0$ depending only on C_1 and f.

Proof. — It suffices to take any $C_1 \geq C_0/(1-\lambda_0)$. □

1.3. Hyperbolic times and bounded distortion. — Let $K \subset M$ be a forward invariant compact set and let $\Lambda \subset K \subset U$ be as in Section 1.2. The following notion will allow us to derive *uniform behaviour* (expansion, distortion) from the non-uniform expansion.

Definition 1.14. — Given $\sigma < 1$, we say that n is a *σ-hyperbolic time* for $x \in K$ if

$$\prod_{j=n-k+1}^{n} \|Df^{-1} \mid E^{cu}_{f^j(x)}\| \leq \sigma^k, \quad \text{for all } 1 \leq k \leq n.$$

In particular, if n is a σ-hyperbolic time for x, then $Df^{-k} \mid E^{cu}_{f^n(x)}$ is a contraction for every $1 \leq k \leq n$:

$$\|Df^{-k} \mid E^{cu}_{f^n(x)}\| \leq \prod_{j=n-k+1}^{n} \|Df^{-1} \mid E^{cu}_{f^j(x)}\| \leq \sigma^k. \tag{12}$$

If $a > 0$ is taken sufficiently small in the definition of our cone fields, and we choose $\delta_1 > 0$ also small so that the δ_1-neighborhood of K should be contained in U, then by continuity

$$\|Df^{-1}(f(y))v\| \leq \frac{1}{\sqrt{\sigma}} \|Df^{-1}|E^{cu}_{f(x)}\| \, \|v\|, \tag{13}$$

whenever $x \in K$, $\mathrm{dist}(f(x), f(y)) \leq \delta_1$ and $v \in C^{cu}_a(f(y))$.

Given any disk $\Delta \subset M$, we use $\mathrm{dist}_\Delta(x,y)$ to denote the distance between $x, y \in \Delta$ measured along Δ. The distance from a point $x \in \Delta$ to the boundary of Δ is $\mathrm{dist}_\Delta(x, \partial\Delta) = \inf_{y \in \partial\Delta} \mathrm{dist}_\Delta(x,y)$.

Lemma 1.15. — *Take any C^1 disk $\Delta \subset U$ of radius δ, with $0 < \delta < \delta_1$, tangent to the centre-unstable cone field. There is $n_0 \geq 1$ such that for $x \in \Delta \cap K$ with $\mathrm{dist}_\Delta(x, \partial\Delta) \geq \delta/2$ and $n \geq n_0$ a σ-hyperbolic time for x, then there is a neighborhood V_n of x in Δ such that:*

1. *f^n maps V_n diffeomorphically onto a disk of radius δ_1 around $f^n(x)$ tangent to the centre-unstable cone field;*
2. *for every $1 \leq k \leq n$ and $y, z \in V_n$,*
$$\mathrm{dist}_{f^{n-k}(V_n)}(f^{n-k}(y), f^{n-k}(z)) \leq \sigma^{k/2}\, \mathrm{dist}_{f^n(V_n)}(f^n(y), f^n(z));$$
3. *for every $1 \leq k \leq n$ and $y \in V_n$,*
$$\prod_{j=n-k+1}^{n} \|Df^{-1} \mid E^{cu}_{f^j(y)}\| \leq \sigma^{k/2}.$$

Proof. — First we show that $f^n(\Delta)$ contains some disk of radius δ_1 around $f^n(x)$, as long as

$$n > 2\frac{\log(\delta/(2\delta_1))}{\log(\sigma)}. \tag{14}$$

Define Δ_1 as the connected component of $f(\Delta) \cap U$ containing $f(x)$. For $k \geq 1$, we inductively define $\Delta_{k+1} \subset f^{k+1}(\Delta)$ as the connected component of $f(\Delta_k) \cap U$ containing

$f^{k+1}(x)$. We shall prove that Δ_n contains some disk of radius δ_1 around $f^n(x)$ for n as in (14). Observe that since $\Delta_j \subset U$, the invariance (5) gives that for every $j \geq 1$

$$T_w\Delta_j \subset C^{cu}_{\lambda^j a}(w), \quad \text{for every } w \in \Delta_j. \tag{15}$$

Let η_0 be a curve of minimal length in Δ_n connecting $f^n(x)$ to $f^n(y) \in \Delta_n$ for which $\operatorname{dist}_{\Delta_n}(f^n(x), f^n(y)) < \delta_1$. For $0 \leq k \leq n$, writing $\eta_k = f^{-k}(\eta_0)$ we have $\eta_k \subset \Delta_{n-k}$. We prove by induction that $\operatorname{length}(\eta_k) < \sigma^{k/2}\delta_1$, for $0 \leq k \leq n$. Let $1 \leq k \leq n$ and assume that

$$\operatorname{length}(\eta_j) < \sigma^{j/2}\delta_1, \quad \text{for } 0 \leq j \leq k-1.$$

Denote by $\dot\eta_0(w)$ the tangent vector to the curve η_0 at the point w. Using the fact that $\eta_k \subset \Delta_{n-k}$ and (15) we have

$$Df^{-j}(w)\dot\eta_0(w) \in C^{cu}_{\lambda^{n-j}a}(f^{-j}(w)) \subset C^{cu}_a(f^{-j}(w)).$$

Then, by the choice of δ_1 in (13) and the definition of σ-hyperbolic time,

$$\|Df^{-k}(w)\dot\eta_0(w)\| \leq \sigma^{-k/2}\, \|\dot\eta_0(w)\| \prod_{j=n-k+1}^{n} \|Df^{-1}|E^{cu}_{f^j(x)}\| \leq \sigma^{k/2}\|\dot\eta_0(w)\|.$$

Hence,

$$\operatorname{length}(\eta_k) \leq \sigma^{k/2}\operatorname{length}(\eta_0) < \sigma^{k/2}\delta_1\,.$$

This completes our induction.

In particular we have $\operatorname{length}(\eta_n) < \sigma^{n/2}\delta_1$. The k preimage of the ball of radius δ_1 in Δ_n centered at $f^n(x)$ is contained in U for each $1 \leq k \leq n$. If η_n is a curve in Δ connecting x to $y \in \partial\Delta$, then we must have

$$n < 2\frac{\log(\delta/(2\delta_1))}{\log(\sigma)}.$$

Moreover, $f^n(\Delta)$ contains some disk of radius δ_1 around $f^n(x)$ for n as in (14).

Let now D_1 be the disk of radius δ_1 around $f^n(x)$ in $f^n(\Delta)$ and let $V_n = f^{-n}(D_1)$, for n as in (14). Take any $y, z \in V_n$ and let η_0 be a curve of minimal length in D_1 connecting $f^n(y)$ to $f^n(z)$. Defining $\eta_k = f^{-n+k}(\eta_0)$, for $1 \leq k \leq n$, and arguing as before we inductively prove that for $1 \leq k \leq n$

$$\operatorname{length}(\eta_k) \leq \sigma^{k/2}\operatorname{length}(\eta_0) = \sigma^{k/2}\operatorname{dist}_{f^n(V_n)}(f^n(y), f^n(z)),$$

which implies that for $1 \leq k \leq n$

$$\operatorname{dist}_{f^{n-k}(V_n)}(f^{n-k}(y), f^{n-k}(z)) \leq \sigma^{k/2}\operatorname{dist}_{f^n(V_n)}(f^n(y), f^n(z)).$$

This completes the proof of the first two items of the lemma.

Given $y \in V_n$ we have $\operatorname{dist}(f^j(x), f^j(y)) \leq \delta_1$ for every $1 \leq j \leq n$, which together with (13) gives

$$\prod_{j=n-k+1}^{n} \|Df^{-1} \mid E^{cu}_{f^j(y)}\| \leq \sigma^{-k/2} \prod_{j=n-k+1}^{n} \|Df^{-1} \mid E^{cu}_{f^j(x)}\| \leq \sigma^{k/2}.$$

Recall that $f^j(x) \in K$ for every j, and n is a σ-hyperbolic time for x. □

We shall sometimes refer to the sets V_n as *hyperbolic pre-balls* and to their images $f^n(V_n)$ as *hyperbolic balls*. Notice that the latter are indeed balls of radius δ_1.

***Corollary 1.16* (Bounded Distortion).** — *There exists $C_2 > 1$ such that given Δ as in Lemma 1.15 with $\kappa(\Delta) \le C_1$, and given any hyperbolic pre-ball $V_n \subset \Delta$ with $n \ge n_0$, then for all $y, z \in V_n$*

$$\frac{1}{C_2} \le \frac{|\det Df^n \mid T_y\Delta|}{|\det Df^n \mid T_z\Delta|} \le C_2.$$

Proof. — For $0 \le i < n$ and $y \in \Delta$, we denote $J_i(y) = |\det Df \mid T_{f^i(y)} f^i(\Delta)|$. Then,

$$\log \frac{|\det Df^n \mid T_y\Delta|}{|\det Df^n \mid T_z\Delta|} = \sum_{i=0}^{n-1} \big(\log J_i(y) - \log J_i(z) \big).$$

By Corollary 1.13, $\log J_i$ is (L,ζ)-Hölder continuous, for some uniform constant $L > 0$. Moreover, by Lemma 1.15, the sum of all $\mathrm{dist}_{f^j(\Delta)}(f^j(y), f^j(z))^\zeta$ over $0 \le j \le n$ is bounded by $2\delta_1/(1-\sigma^{\zeta/2})$. Then it suffices to take $C_2 = \exp(2\delta_1 L/(1-\sigma^{\zeta/2}))$. □

1.4. A local unstable disk inside the attractor. — Now we are able to prove Theorem 1.4 and Theorem 1.6. Those results will be obtained as corollaries of the next slightly more general result. Take $K \subset M$ a forward invariant compact set and let $\Lambda \subset K \subset U$ be as before.

Theorem 1.17. — *Let $f : M \to M$ be a C^{1+} diffeomorphism and let $K \subset M$ be a forward invariant compact set with a continuous splitting $T_K M = E^{cs} \oplus E^{cu}$ dominated over $\Lambda = \bigcap_{n\ge 0} f^n(K)$. Assume that there is a disk Δ tangent to the centre-unstable cone field intersecting K in a positive* Leb_Δ *set of points where NUE holds. Then Λ contains some local unstable disk.*

1.4.1. Consequences of Theorem 1.17. — Let us show that Theorem 1.17 implies Theorem 1.4. Assume that NUE holds for Lebesgue almost every $x \in K$ with $\mathrm{Leb}(K) > 0$. Choosing a Leb density point of K, we laminate a neighborhood of that point into disks tangent to the centre-unstable cone field contained in U. Since the relative Lebesgue measure of the intersections of these disks with K cannot be all equal to zero, we obtain some disk Δ as in the assumption of Theorem 1.17.

For showing that Theorem 1.17 implies Theorem 1.6, we just have to observe that local unstable manifolds are tangent to the centre-unstable subspaces.

1.4.2. Proof of Theorem 1.17. — Let $H \subset K$ be the set of points where NUE holds. It is easy to check that $f(H) \subset H$. Let Δ be a disk of radius $\delta > 0$ tangent to the centre-unstable cone field intersecting H in a positive Leb_Δ. Since NUE remains valid under positive iteration, by Corollary 1.13 we may assume that $\kappa(\Delta) < C_1$. It is no restriction to assume that H intersects the sub-disk of Δ of radius $\delta/2$, for some $0 < \delta < \delta_1$, in a positive Leb_Δ subset, and we do so.

***Lemma 1.18* (Pliss).** — *Given $A > c_2 > c_1 > 0$ let $\theta = (c_2 - c_1)/(A - c_1)$. Given real numbers $a_1, \dots, a_N$ satisfying $a_j \le A$ for every $1 \le j \le N$ and*

$$\sum_{j=1}^{N} a_j \ge c_2 N,$$

there are $l > \theta N$ and $1 < n_1 < \cdots < n_l \leq N$ so that

$$\sum_{j=n+1}^{n_i} a_j \geq c_1(n_i - n)$$

for every $0 \leq n < n_i$ and $i = 1, \ldots, l$.

Proof. — Define for each $1 \leq n \leq N$,

$$S_n = \sum_{j=1}^{n}(a_j - c_1), \quad \text{and also} \quad S_0 = 0.$$

Then define $1 < n_1 < \cdots < n_l \leq N$ to be the maximal sequence such that $S_{n_i} \geq S_n$ for every $0 \leq n < n_i$ and $i = 1, \ldots, l$. Note that l cannot be zero, since $S_N > S_0$. Moreover, the definition means that

$$\sum_{j=n+1}^{n_i} a_j \geq c_1(n_i - n), \quad \text{for} \quad 0 \leq n < n_i \quad \text{and} \quad i = 1, \ldots, l.$$

So, we only have to check that $l > \theta_0 N$. Observe that, by definition,

$$S_{n_i - 1} < S_{n_{i-1}} \quad \text{and so} \quad S_{n_i} < S_{n_{i-1}} + (A - c_1)$$

for every $1 < i \leq l$. Moreover,

$$S_{n_1} \leq (A - c_1) \quad \text{and} \quad S_{n_l} \geq S_N \geq N(c_2 - c_1).$$

This gives,

$$N(c_2 - c_1) \leq S_{n_l} = \sum_{i=2}^{l}\left(S_{n_i} - S_{n_{i-1}}\right) + S_{n_1} < l(A - c_1),$$

which completes the proof. $\square$

Corollary 1.19. — *There is $0 < \sigma < 1$ (depending only on f and λ) such that every $x \in H$ has infinitely many σ-hyperbolic times.*

Proof. — Given $x \in H$, by NUE we have infinitely many positive integers N for which

$$\sum_{j=1}^{N} \log \|Df^{-1}|E^{cu}_{f^j(x)}\| \leq -cN.$$

Take $c_1 = c/2$, $c_2 = c$, $A = \sup\big|\log \|Df^{-1}|E^{cu}\|\big|$, and $a_j = -\log \|Df^{-1}|E^{cu}_{f^j(x)}\|$ in the previous lemma. $\square$

Note that under assumption NUE we are unable to prove the existence of positive frequency of hyperbolic times at infinity. This will be possible in Section 3 where we shall use a stronger form of non-uniform expansion replacing lim inf by lim sup in the definition of NUE. The existence of infinitely many hyperbolic times is enough for the present situation.

Lemma 1.20. — *Let O be an open set in Δ such that $\mathrm{Leb}_\Delta(O \cap H) > 0$. Given any small $\rho > 0$ there is a hyperbolic time n, a hyperbolic pre-ball $V \subset O$ and $W \subset V$ such that $\Delta_n = f^n(W)$ is a disk of radius $\delta_1/4$ tangent to the centre-unstable cone field and*

$$\frac{\mathrm{Leb}_{\Delta_n}(f^n(H))}{\mathrm{Leb}_{\Delta_n}(\Delta_n)} \geq 1 - \rho.$$

Proof. — Take a small number $\epsilon > 0$. By regularity of Leb_Δ measure, there is a compact set C contained in $O \cap H$ and A an open neighborhood of $O \cap H$ in Δ such that

$$\mathrm{Leb}_\Delta(A \setminus C) < \epsilon \mathrm{Leb}_\Delta(C).$$

It follows from Corollary 1.19 and Lemma 1.15 that we can choose for each $x \in C$ a σ-hyperbolic time $n(x)$ and a hyperbolic pre-ball V_x such that $V_x \subset A$. Recall that V_x is the neighborhood of x which is mapped diffeomorphically by $f^{n(x)}$ onto the ball $B_{\delta_1}(f^{n(x)}(x))$ of radius δ_1 around $f^{n(x)}(x)$, tangent to the centre-unstable cone field. Let $W_x \subset V_x$ be the pre-image of the ball $B_{\delta_1/4}(f^{n(x)}(x))$ of radius $\delta_1/4$ under this diffeomorphism. By compactness there are $x_1, ..., x_m \in C$ such that $C \subset W_{x_1} \cup ... \cup W_{x_m}$. Assume that

$$\{n(x_1), ..., n(x_m)\} = \{n_1, ..., n_s\}, \quad \text{with } n_1 < n_2 < ... < n_s. \tag{16}$$

Let $I_1 \subset \mathbb{N}$ be a maximal set of $\{1, ..., m\}$ such that if $i \in I_1$ then $n(x_i) = n_1$ and $W_{x_i} \cap W_{x_j} = \emptyset$ for all $j \in I_1$ with $j \neq i$. Inductively we define I_k for $2 \leq k \leq s$ as follows: Supposing that I_{k-1} has already been defined, let $I_k \subset \mathbb{N}$ be a maximal set of $\{1, \ldots, m\}$ such that if $i \in I_k$, then $n(x_i) = n_k$ and $W_{x_i} \cap W_{x_j} = \emptyset$ for all $j \in I_k$ with $j \neq i$, and also $W_{x_i} \cap W_{x_j} = \emptyset$ for all $j \in I_1 \cup ... \cup I_{k-1}$.

Let $I = I_1 \cup \cdots \cup I_s$. By maximality, each W_{x_j}, for $1 \leq j \leq m$, intersects some W_{x_i} with $i \in I$ and $n(x_j) \geq n(x_i)$. Thus, given any $1 \leq j \leq m$, taking $i \in I$ such that $W_{x_j} \cap W_{x_i} \neq \emptyset$ and $n(x_j) \geq n(x_i)$, we get

$$f^{n(x_i)}(W_{x_j}) \cap B_{\delta_1/4}(f^{n(x_i)}(x_i)) \neq \emptyset.$$

Lemma 1.15 assures that

$$\mathrm{diam}(f^{n(x_i)}(W_{x_j})) \leq \frac{\delta_1}{2} \sigma^{(n(x_j) - n(x_i))/2} \leq \frac{\delta_1}{2},$$

and so

$$f^{n(x_i)}(W_{x_j}) \subset B_{\delta_1}(f^{n(x_i)}(x_i)).$$

This implies that $W_{x_j} \subset V_{x_i}$. Hence $\{V_{x_i}\}_{i \in I}$ is a covering of C. It follows from Corollary 1.16 that there is a uniform constant $\gamma > 0$ such that

$$\frac{\mathrm{Leb}_\Delta(W_{x_i})}{\mathrm{Leb}_\Delta(V_{x_i})} \geq \gamma, \quad \text{for every } i \in I.$$

Hence

$$\begin{aligned}
\mathrm{Leb}_\Delta\left(\cup_{i \in I} W_{x_i}\right) &= \sum_{i \in I} \mathrm{Leb}_\Delta(W_{x_i}) \\
&\geq \sum_{i \in I} \gamma \, \mathrm{Leb}_\Delta(V_{x_i}) \\
&\geq \gamma \, \mathrm{Leb}_\Delta\left(\cup_{i \in I} V_{x_i}\right) \\
&\geq \gamma \, \mathrm{Leb}_\Delta(C).
\end{aligned}$$

Let

$$\tau = \min\left\{\frac{\mathrm{Leb}_\Delta(W_{x_i} \setminus C)}{\mathrm{Leb}_\Delta(W_{x_i})} : i \in I\right\}.$$

We essentially want to prove that τ can be made arbitrarily small. This will achieved by taking $\epsilon > 0$ small. We have

$$\begin{aligned}
\varepsilon\,\mathrm{Leb}_\Delta(C) &> \mathrm{Leb}_\Delta(A \setminus C) \\
&\geq \mathrm{Leb}_\Delta\left(\cup_{i\in I} W_{x_i} \setminus C\right) \\
&= \sum_{i\in I} \mathrm{Leb}_\Delta(W_{x_i} \setminus C) \\
&\geq \tau\,\mathrm{Leb}_\Delta\left(\cup_{i\in I} W_{x_i}\right) \\
&\geq \tau\gamma\,\mathrm{Leb}_\Delta(C).
\end{aligned}$$

This implies that $\tau < \varepsilon/\gamma$. Since $\varepsilon > 0$ can be taken arbitrarily small, we may choose W_{x_i} with the relative Lebesgue measure of C in W_{x_i} arbitrarily close to 1. Then, by bounded distortion, the relative Lebesgue measure of $f^{n(x_i)}(H) \supset f^{n(x_i)}(C)$ in $f^{n(x_i)}(W_{x_i})$ can also be made arbitrarily close to 1. Recalling that $f^{n(x_i)}(W_{x_i})$ is a disk of radius $\delta_1/4$ around $f^{n(x_i)}(x_i)$ tangent to centre-unstable cone field, we just have to take $V = V_{x_i}$, $W = W_{x_i}$ and $n = n(x_i)$. □

Remark 1.21. — Observe that we did not use the forward invariance of H in the proof of Lemma 1.20. We have just used the fact that points in H have infinitely many hyperbolic times.

Corollary 1.22. — *There are sets $W_1 \supset W_2 \supset \cdots$ and integers $1 \leq n_1 \leq n_2 \leq \cdots$ such that:*

1. *W_k is contained in some hyperbolic pre-ball with hyperbolic time n_k;*
2. *$\Delta_k = f^{n_k}(W_k)$ is a disk of radius $\delta_1/4$, centered at some point x_k, tangent to the centre-unstable cone field;*
3. *$f^{n_k}(W_{k+1})$ is contained in the disk of radius $\delta_1/8$ centered at x_k;*
4. $\displaystyle\lim_{k\to\infty} \frac{\mathrm{Leb}_{\Delta_k}(f^{n_k}(H))}{\mathrm{Leb}_{\Delta_k}(\Delta_k)} = 1.$

Proof. — Take a constant $0 < \rho < 1$ such that for any disk D of radius $\delta_1/4$ centered at a point x tangent to the centre-unstable cone field the following holds:

> *If* $\mathrm{Leb}_D(A) \geq (1-\rho)\mathrm{Leb}_D(D)$ *for some $A \subset D$, then we must have* $\mathrm{Leb}_{D^*}(A) > 0$ *for the disk $D^* \subset D$ of radius $\delta_1/8$ centered at the same point x.*

Note that it is possible to make a choice of ρ in these conditions only depending on the radius of the disk and the dimension of the disk. Surely, once we have chosen some ρ satisfying the required property, then any smaller number still has that property.

We shall use Lemma 1.20 repeatedly in order to define the sequence of sets $(W_k)_k$ and hyperbolic times $(n_k)_k$ inductively. Let us start with $O = \Delta$ and $0 < \rho < 1$ with the property mentioned above. By Lemma 1.20 there are $n_1 \geq 1$ and $W_1 \subset V_1 \subset O$, where V_1 is a hyperbolic pre-ball with hyperbolic time n_1, such that $\Delta_1 = f^{n_1}(W_1)$ is a disk of

radius $\delta_1/4$ centered at some point x_1, tangent to the centre-unstable cone field, such that
$$\frac{\mathrm{Leb}_{\Delta_1}(f^{n_1}(H))}{\mathrm{Leb}_{\Delta_1}(\Delta_1)} \geq 1-\rho.$$
Considering $\Delta_1^* \subset \Delta_1$ the disk of radius $\delta_1/8$ centered at x_1, then by the choice of ρ we have $\mathrm{Leb}_{\Delta_1^*}(H) > 0$. Let $O_1 \subset W_1$ be the part of W_1 which is sent by f^{n_1} diffeomorphically onto Δ_1^*. We have $\mathrm{Leb}_\Delta(O_1 \cap H) > 0$.

Next we apply Lemma 1.20 to $O = O_1$ and $\rho/2$ in the place of ρ. Then we find a hyperbolic time n_2 and $W_2 \subset O_1$ such that $\Delta_2 = f^{n_2}(W_2)$ satisfies
$$\frac{\mathrm{Leb}_{\Delta_2}(f^{n_2}(H)))}{\mathrm{Leb}_{\Delta_2}(\Delta_2)} \geq 1-\frac{\rho}{2}.$$
Observe that $W_2 \subset O_1 \subset W_1$. Then we take $O_2 \subset W_2$ as that part of W_2 which is sent by f^{n_2} diffeomorphically onto the disk Δ_2^* of radius $\delta_1/8$ and proceed inductively. □

The next result gives the conclusion of Theorem 1.17.

Lemma 1.23. — *The sequence $(\Delta_k)_k$ has a subsequence converging to a local unstable disk Δ_∞ of radius $\delta_1/4$ inside Λ.*

Proof. — Let $(\Delta_k)_k$ be the sequence of disks given by Corollary 1.22 and $(x_k)_k$ be the sequence of points at which these disks are centered. Up to taking subsequences, we may assume that the centers of the disks converge to some point x. Using Ascoli-Arzela, a subsequence of the disks converge to some disk Δ_∞ centered at x, which must necessarily be contained in Λ.

Note that each Δ_k is contained in the n_k-iterate of Δ, which is a disk tangent to the centre-unstable cone field. The domination property implies that the angle between Δ_k and E^{cu} goes uniformly to 0 as $n \to \infty$. In particular, Δ_∞ is tangent to E^{cu} at every point in $\Delta_\infty \subset \Lambda$. By Lemma 1.15, given any $n \geq 1$, then f^{-n} is a $\sigma^{n/2}$-contraction on Δ_k for every large k. Passing to the limit, we get that f^{-n} is a $\sigma^{n/2}$-contraction in the E^{cu} direction over Δ_∞ for every $n \geq 1$. The fact that the Df-invariant splitting $T_\Lambda M = E^{cs} \oplus E^{cu}$ is dominated implies that any expansion Df may exhibit along the complementary direction E^{cs} is weaker than the expansion in the E^{cu} direction. Then there exists a unique unstable manifold $W^u_{loc}(x)$ tangent to E^{cu} and which is contracted by the negative iterates of f; see [**Pe**]. Since Δ_∞ is contracted by every f^{-n}, and all its negative iterates are tangent to centre-unstable cone field, then Δ_∞ is contained in $W^u_{loc}(x)$. □

1.5. Limit sets. — Using the previous results we are able to give a description of the ω-limit of Lebesgue almost every point in a partially hyperbolic set whose center-unstable direction displays non-uniform expansion in a subset with positive volume. Recall that the *ω-limit* of a point in M is the set of accumulation points of its orbit.

Theorem 1.24. — *Let $f : M \to M$ be a C^{1+} diffeomorphism and let $K \subset M$ with $\mathrm{Leb}(K) > 0$ be a forward invariant compact set with a continuous splitting $T_K M = E^{cs} \oplus E^{cu}$ which is dominated over $\Lambda = \bigcap_{n\geq 0} f^n(K)$. Assume that E^{cs} is uniformly contracting and NUE holds for Lebesgue almost every $x \in K$. Then there are hyperbolic periodic points $p_1, \ldots, p_k \in \Lambda$ such that:*

1. $\overline{W^u(p_i)} \subset \Lambda$ *for each* $1 \le i \le k$;
2. *for* Leb *almost every* $x \in K$ *there is* $1 \le i \le k$ *for which* $\omega(x) \subset \overline{W^u(p_i)}$.

Moreover, if E^{cu} *has dimension one, then for each* $1 \le i \le k$

3. $\overline{W^u(p_i)}$ *attracts an open neighborhood of itself.*

The last conclusion also holds whenever E^{cu} is uniformly expanding. Actually, more can be said in the case of uniformly hyperbolic sets with positive volume, as we shall see in the next subsection.

Open problem. — *Can we obtain the conclusion of the third item for higher dimensional centre-unstable direction? Is there any counter-example?*

In the remaining of this section we prove Theorem 1.24. By Corollary 1.22 there exist a sequence of sets $W_1 \supset W_2 \supset \cdots$ contained in Δ and a sequence of positive integers $n_1 \le n_2 \le \cdots$ such that:

1. W_k is contained in some hyperbolic pre-ball with hyperbolic time n_k;
2. $\Delta_k = f^{n_k}(W_k)$ is a disk of radius $\delta_1/4$, centered at some point x_k, tangent to the centre-unstable cone field;
3. $f^{n_k}(W_{k+1})$ is contained in the disk Δ_k^* of radius $\delta_1/8$ centered at x_k.

Taking a subsequence, if necessary, we have by Lemma 1.23 that the sequence of disks $(\Delta_k)_k$ accumulates on a local unstable disk Δ_∞ of radius $\delta_1/4$ which is contained in Λ. Our aim now is to prove that Λ contains the unstable manifold of some periodic point.

We choose $\delta > 0$ small so that *i)* $W^s_\delta(z)$ is defined for every $z \in \Lambda$, *ii)* the 2δ-neighborhood of Λ is contained in U, and *iii)*

$$\|Df^{-1}(f(y))v\| \le \sigma^{-1/4}\|Df^{-1}|E^{cu}_{f(x)}\|\,\|v\|, \tag{17}$$

whenever $x \in U$, $\operatorname{dist}(f(x), f(y)) \le 2\delta$, and $v \in C^{cu}_a(f(y))$.

Lemma 1.25. — *Given* $K_1 \subset K$ *with* $\operatorname{Leb}(K_1) > 0$, *there exist a hyperbolic periodic point* $p \in \Lambda$ *and* $\delta_2 > 0$ *(not depending on* p*) such that:*

1. $\overline{W^u(p)} \subset \Lambda$;
2. *the size of* $W^u_{loc}(p)$ *is at least* δ_2;
3. $\operatorname{Leb}_{W^u_{loc}(p)}$ *almost every point in* $W^u_{loc}(p)$ *belongs to* H;
4. *there is* $x \in K_1$ *with* $\omega(x) \subset \overline{W^u(p)}$.

Proof. — Let x denote the center of the accumulation disk Δ_∞. Let us consider the cylinder

$$\mathcal{C}_\delta = \bigcup_{y \in \Delta_\infty} W^s_\delta(y),$$

and the projection along local stable manifolds

$$\pi\colon \mathcal{C}_\delta \longrightarrow \Delta_\infty.$$

Slightly diminishing the radius of the disk Δ_∞, if necessary, we may assume that there is a positive integer k_0 such that for every $k \ge k_0$

$$\pi(\Delta_k \cap \mathcal{C}_\delta) = \Delta_\infty \quad \text{and} \quad \Delta_k^* \subset \mathcal{C}_\delta. \tag{18}$$

For each $k \geq k_0$ let

$$\pi_k\colon \Delta_\infty \longrightarrow \Delta_k$$

be the projection along the local stable manifolds. Notice that these projections are continuous and $\pi \circ \pi_k = \mathrm{id}_{\Delta_\infty}$. Take a positive integer $k_1 > k_0$ sufficiently large so that

$$\pi(\Delta_{k_1} \cap \mathcal{C}_{\delta/2}) = \Delta_\infty \quad \text{and} \quad \lambda^{n_{k_1}-n_{k_0}} \leq \frac{1}{4}. \tag{19}$$

We have

$$\Delta_{k_1} = f^{n_{k_1}}(W_{k_1}) \subset f^{n_{k_1}-n_{k_0}}\left(f^{n_{k_0}}(W_{k_0+1})\right) \subset f^{n_{k_1}-n_{k_0}}(\Delta^*_{k_0}),$$

which together with (18) and (19) implies that there is some disk $\Delta_0 \subset \Delta_\infty$ such that

$$\pi \circ f^{n_{k_1}-n_{k_0}} \circ \pi_{k_0}(\Delta_0) = \Delta_\infty.$$

Thus there must be some $z \in \Delta_0 \subset \Delta_\infty$ which is a fixed point for the continuous map $\pi \circ f^{n_{k_1}-n_{k_0}} \circ \pi_{k_0}$. This means that there are $z_{k_0}, z_{k_1} \in W^s_\delta(z)$ with $z_{k_0} \in \Delta_{k_0}$ and $z_{k_1} \in \Delta_{k_1}$ such that $f^{n_{k_1}-n_{k_0}}(z_{k_0}) = z_{k_1}$. Letting $\gamma = W^s_\delta(z)$, we have $\mathrm{dist}_\gamma(w, z_{k_1}) \leq 2\delta$ for every $w \in \gamma$. This implies that

$$\mathrm{dist}_\gamma(f^{n_{k_1}-n_{k_0}}(w), z_{k_1}) = \mathrm{dist}_\gamma(f^{n_{k_1}-n_{k_0}}(w), f^{n_{k_1}-n_{k_0}}(z_{k_0})) \leq 2\delta\lambda^{n_{k_1}-n_{k_0}},$$

which together with (19) gives

$$\mathrm{dist}_\gamma(f^{n_{k_1}-n_{k_0}}(w), z) \leq \mathrm{dist}_\gamma(f^{n_{k_1}-n_{k_0}}(w), z_{k_1}) + \mathrm{dist}_\gamma(z_{k_1}, z) \leq \delta.$$

We conclude that $f^{n_{k_1}-n_{k_0}}(W^s_\delta(z)) \subset W^s_\delta(z)$. Since $W^s_\delta(z)$ is a topological disk, this implies that $W^s_\delta(z)$ must necessarily contain some periodic point p of period $m = n_{k_1} - n_{k_0}$. As $z \in \Delta_\infty$ and $p \in W^s_\delta(z)$ it follows that $p \in \Lambda$, by closeness of Λ.

Let us now prove that p is a hyperbolic point. As $p \in W^s_\delta(z)$, it is enough to show that $\|Df^{-m} \mid E^{cu}_{f^m(p)}\| < 1$, where $m = n_{k_1} - n_{k_0}$. Let $q = W^s_\delta(z) \cap f^{n_{k_0}}(W_{k_1})$. Observe that since $p \in \Lambda \cap W^s_\delta(z)$, then q belongs to the 2δ-neighborhood of Λ, which is contained in U. Since W_{k_1} is contained in some hyperbolic pre-ball with hyperbolic time n_{k_1}, it follows from Lemma 1.15 and the choice of δ in (17) that

$$\begin{aligned}
\|Df^{-m} \mid E^{cu}_{f^m(p)}\| &\leq \prod_{j=1}^{m} \|Df^{-1} \mid E^{cu}_{f^j(p)}\| &(20)\\
&\leq \sigma^{-m/4} \prod_{j=1}^{m} \|Df^{-1} \mid E^{cu}_{f^j(q)}\| \\
&\leq \sigma^{m/4}. &(21)
\end{aligned}$$

Thus we have proved the hyperbolicity of p.

Now since p is a hyperbolic periodic point, there is $W^u_{\mathrm{loc}}(p)$ a local unstable manifold through p tangent to the center unstable bundle. As Δ_∞ cuts transversely the local stable manifold through p, then using the inclination lemma we deduce that the positive iterates of Δ_∞ accumulate on the unstable manifold through p. Since these iterates are all contained in the closed set Λ, we must have $W^u(p) \subset \Lambda$, which then implies that $\overline{W^u(p)} \subset \Lambda$. Thus we have proved the first part of the result.

By (20) and (21) we deduce that every multiple of m is a $\sigma^{1/4}$-hyperbolic time for p. Then we choose $\delta_2 > 0$ such that an inequality as in (13) holds with δ_2 in the place of δ_1

and $\sigma^{1/8}$ in the place of $\sigma^{1/2}$. Using Lemma 1.15 with $W^u_{\mathrm{loc}}(p)$ in the place of Δ and taking a sufficiently large $\sigma^{1/4}$-hyperbolic time for p we deduce that there is a hyperbolic pre-ball inside $W^u_{\mathrm{loc}}(p)$. This imples that its image by the hypebolic time, which is a disk of radius δ_2 around p, is contained in the local unstable manifold of p. This gives the second part of the result.

Observe that as long as we take the local unstable manifold through p small enough, every point in $W^u_{\mathrm{loc}}(p)$ belongs to the local stable manifold of some point in Δ_∞. By construction, Δ_∞ is accumulated by the disks $\Delta_k = f^{n_k}(W_k)$ which, by Corollary 1.22, satisfy

$$\lim_{k\to\infty} \frac{\mathrm{Leb}_{\Delta_k}(f^{n_k}(H))}{\mathrm{Leb}_{\Delta_k}(\Delta_k)} = 1. \tag{22}$$

Since H is forward invariant, we have

$$\lim_{k\to\infty} \frac{\mathrm{Leb}_{\Delta_k}(H)}{\mathrm{Leb}_{\Delta_k}(\Delta_k)} = 1.$$

Let now $\varphi\colon \Lambda \to \mathbb{R}$ be the continuous function given by

$$\varphi(x) = \log \|Df^{-1} \mid E^{cu}_x\|.$$

Since time averages of φ are constant for points on local stable manifolds and the local stable foliation is absolutely continuous, we deduce that

$$\frac{\mathrm{Leb}_{\Delta_\infty}(H)}{\mathrm{Leb}_{\Delta_\infty}(\Delta_\infty)} = 1.$$

The same conclusion holds for the local unstable manifold of p in the place of Δ_∞ by the same reason.

Let us now prove the last item. Since H has full Lebesgue measure in K and $K_1 \subset K$ has positive Lebesgue measure, we may start our construction with the set $H_1 = H \cap K_1$ in the place of H intersecting the disk Δ in a positive Leb_Δ measure set of points. Although we have not invariance of H_1, by Corollary 1.22 we still have that the iterates of $H_1 \subset \Lambda_1$ accumulate on the whole Δ_∞; recall Remark 1.21. Since the stable manifolds through points in $W^u_{\mathrm{loc}}(p)$ intersect Δ_∞, there must be points in H_1 whose orbits accumulate on $W^u_{\mathrm{loc}}(p)$. $\square$

Let p_1 be a hyperbolic periodic point as in Lemma 1.25. Let B_1 be the basin of $\overline{W^u(p_1)}$, i.e. the set of points whose ω-limit is contained in $\overline{W^u(p_1)}$. If $\mathrm{Leb}(K \setminus B_1) = 0$, then we have proved the theorem. Otherwise, let $K_1 = K \setminus B_1$. Using again Lemma 1.25 we obtain a point $p_2 \in \Lambda$ such that the basin B_2 of $\overline{W^u(p_2)}$ attracts the orbit of some point in K_1. By definition of K_1 we must have $\overline{W^u(p_1)} \neq \overline{W^u(p_2)}$.

We proceed inductively, thus obtaining periodic points $p_1, \dots, p_n \in \Lambda$ with $\overline{W^u(p_i)} \neq \overline{W^u(p_j)}$ for every $i \neq j$. This process must stop after a finite number of steps. Indeed, if there were infinitely many points as above, by compactness, choosing p_{i_1}, p_{i_2} sufficiently close we would get $\overline{W^u(p_{i_1})} = \overline{W^u(p_{i_2})}$ by the Inclination Lemma. We have proved the first two items of Theorem 1.24.

1.5.1. Dimension one centre-unstable direction. — Assume now that E^{cu} has dimension one. We want to show that each $\overline{W^u(p_i)}$ attracts an open set containing $\overline{W^u(p_i)}$. Given $1 \le i \le k$, by Lemma 1.25 we can find at least one point in each connected component of $W^u(p_i) \setminus \{p_i\}$ belonging to H. Since these points have infinitely many hyperbolic times, then each connected component of $W^u(p_i) \setminus \{p_i\}$ necessarily has infinite arc length; recall Lemma 1.15. This implies that each point $x \in W^u(p_i)$ has an unstable arc $\gamma^u(x) \subset W^u(p_i)$ of a fixed length passing through it. Let

$$B(x) = \bigcup_{y \in \gamma^u(x)} W^s_\delta(y).$$

By domination, the angles of $\gamma^u(x)$ and the local stable manifolds $W^s_\delta(y)$ with $y \in \gamma^u(x)$ are uniformly bounded away from zero. Thus, $B(x)$ must contain some ball of uniform radius (not depending on x), and so the set $\bigcup_{x \in W^u(p_i)} B(x)$ is a neighborhood of $\overline{W^u(p_i)}$. Since, for each $x \in W^u(p_i)$, the points in $B(x)$ have their ω-limit set contained in $\overline{W^u(p_i)}$, we are done.

2. Hyperbolic sets with positive volume

A compact invariant set Λ is called *hyperbolic* if there is an invariant splitting $T_\Lambda M = E^s \oplus E^u$ of the tangent bundle restricted to Λ, and a constant $0 < \lambda < 1$ such that for some choice of a Riemannian metric on M we have

$$\|Df \mid E^s_x\| < \lambda \quad \text{and} \quad \|Df^{-1} \mid E^u_x\| < \lambda,$$

for every $x \in \Lambda$. Observe that in this case the splitting is obviously a dominated splitting.

In the next subsections we will derive several consequences for hyperbolic sets from the results obtained before in the partially hyperbolic context.

2.1. Transitive sets. — We are able to prove that transitive hyperbolic sets with positive volume necessarily coincide with the whole manifold, i.e. the diffeomorphism is Anosov. The main reason why we cannot generalize the next result to the context of partially hyperbolic sets is that the length of local stable/unstable manifolds may shrink to zero when iterated back/forth, respectively.

Theorem 2.1. — *Let $f : M \to M$ be a C^{1+} diffeomorphism and let $\Lambda \subset M$ be a transitive hyperbolic set.*

1. *If Λ has positive volume, then $\Lambda = M$.*
2. *If Λ attracts a set with positive volume, then Λ attracts a neighborhood of itself.*

If Λ has positive volume, it follows from Corollary 1.5 that Λ must contain some local unstable disk and some local stable disk. The first item of Theorem 2.1 is a consequence of the following easy lemma.

Lemma 2.2. — *If Λ is a transitive hyperbolic set containing the local unstable manifold of some point, then Λ contains the local unstable manifolds of all its points.*

Proof. — Take $\delta > 0$ small such that $W^s_\delta(x)$ and $W^u_\delta(y)$ intersects at most in one point, for every $x, y \in \Lambda$, and assume that $W^u_\delta(x_0) \subset \Lambda$ for some $x_0 \in \Lambda$. Let $z \in \Lambda$ be a point with dense orbit in Λ. It is no restriction to assume that $W^s_\delta(z)$ intersects $W^u_\delta(x_0)$, and we do so. Let $x_1 = W^s_\delta(z) \cap W^u_\delta(x_0)$. We also have $W^u_\delta(x_1) \subset \Lambda$. Given any point $y \in \Lambda$, we take a sequence of integers $0 = n_1 < n_2 < \cdots$ such that $f^{n_k}(z) \to y$, when $k \to \infty$. Since $x_1 \in W^s(z)$ we also have $x_k := f^{n_k}(x_1) \to y$, when $k \to \infty$. The local unstable manifolds through the points $x_1, x_2, ...$ are necessarily contained in Λ and accumulate on a disk $D(y)$ contained in Λ and containing y. Since the local unstable disks are tangent to the unstable spaces, the continuity of these spaces implies that $T_w D(y) = E^u_w$ for every $w \in D(y)$. By uniqueness of the unstable foliation, we must have $D(y)$ coinciding with the local unstable manifold through y. □

Using the previous lemma applied to f^{-1}, we have that Λ must also contain the stable manifolds through its points. Then we easily deduce that every point in Λ belongs in the interior of Λ, and so Λ is an open set. Since Λ is assumed to be closed, we conclude that $\Lambda = M$, thus having proved the first part of Theorem 2.1.

Lemma 2.3. — *Let Λ be a hyperbolic set attracting a set with positive volume. Then there is a hyperbolic periodic point in Λ whose local unstable manifold is contained in Λ.*

Proof. — We fix continuous extensions (not necessarily continuous) of the two bundles E^{cs} and E^{cu} to some neighborhood U of Λ. Let A be the set of points which are attracted to Λ under positive iteration. Since A has positive volume, there must be some compact set $C \subset A$ with positive volume, and some $N \in \mathbb{N}$ such that $f^n(C) \subset U$ for every $n \geq N$. Letting

$$K = \bigcup_{n \geq N} f^n(C) \cup \Lambda$$

we have that K is compact forward invariant set with positive volume for which

$$\Lambda = \bigcap_{n \geq 1} f^n(K).$$

The conclusion of the lemma then follows from Theorem 1.24. □

The second part of Theorem 2.1 is now a consequence of Lemma 2.2 and Lemma 2.3. Actually, it follows from the lemmas that $\bigcup_{x \in \Lambda} W^s_\delta(x)$ is a neighborhood of Λ whose points are attracted to Λ under positive iteration.

The following example appeared in [**ABD**] and provides a hyperbolic set with nonempty interior that does not coincide with the whole manifold, thus showing that the transitivity assumption is necessary; see [**Fi**] for a detailed construction.

Example 2.4. — Consider an Axiom A diffeomorphism on a surface M having a Plykin attractor A and a Plykin repeller R for which $W^s(A)$ and $W^u(R)$ have nonempty interior; see [**Ply**]. Consider a disk D contained in $W^s(A) \cap W^u(R)$ where the stable foliation of $W^s(A)$ is transverse to the unstable foliation of $W^u(R)$. Now $\Lambda = \overline{\bigcup_{n \in \mathbb{Z}} f^n(D)}$ is a hyperbolic set with nonempty interior which does not coincide with the whole manifold.

2.2. Conservative case. — We say that a diffeomorphism $f\colon M \to M$ is *conservative* if it preserves Lebesgue measure, meaning that $\mathrm{Leb}(f^{-1}(A)) = \mathrm{Leb}(A)$ for every Borel set $A \subset M$. The next result shows that for conservative diffeomorphisms we do not transitivity for similar conclusions of Theorem 2.1. The conclusion is even stronger in the case of the second item.

Theorem 2.5. — *Let $f : M \to M$ be a C^{1+} conservative diffeomorphism and $\Lambda \subset M$ a hyperbolic set.*

1. *If Λ has positive volume, then $\Lambda = M$.*
2. *If Λ attracts a set with positive volume, then $\Lambda = M$.*

Note that the first item is actually a consequence of the second one. We prove the second item following the argument in the final part of [**BV**, Appendix B]. By Lemma 2.3, there is some periodic point $p \in \Lambda$ such that $\overline{W^u(p)} \subset \Lambda$. Let $\Lambda_0 = \overline{W^u(p)}$ and define

$$W^s_\epsilon(\Lambda_0) = \bigcup_{x\in\Lambda_0} W^s_\epsilon(x).$$

Lemma 2.6. — *$W^s_\epsilon(\Lambda_0)$ is a neighborhood of Λ_0.*

Proof. — Let $z \in \Lambda_0$. There is a sequence $(z_k)_k$ of points in $W^u(p)$ accumulating on z. The ϵ-unstable manifolds of all z_k are contained in Λ_0 and they accumulate on the ϵ-unstable manifold of z. Then $\bigcup_{x\in W^u_\epsilon(z)} W^s_\epsilon(x)$ is a neighborhood of z contained in $W^s_\epsilon(\Lambda_0)$. □

Lemma 2.7. — *$f(W^s_\epsilon(\Lambda_0)) = W^s_\epsilon(\Lambda_0)$.*

Proof. — For any $\delta \in (\lambda\epsilon, \epsilon)$ we have

$$f(W^s_\delta(\Lambda_0)) \subset f(\overline{W^s_\delta(\Lambda_0)}) \subset f(W^s_\epsilon(\Lambda_0)) \subset W^s_{\lambda\epsilon}(\Lambda_0) \subset W^s_\delta(\Lambda_0).$$

Since f preserves volume

$$\mathrm{Leb}(W^s_\delta(\Lambda_0) \setminus f(\overline{W^s_\delta(\Lambda_0)})) \le \mathrm{Leb}(W^s_\delta(\Lambda_0) \setminus f(W^s_\delta(\Lambda_0))) = 0.$$

It follows that

$$W^s_\delta(\Lambda_0) \setminus f(\overline{W^s_\delta(\Lambda_0)}) = \emptyset,$$

because Leb is positive on open sets. Hence,

$$W^s_\delta(\Lambda_0) \setminus f(W^s_\epsilon(\Lambda_0)) = \emptyset.$$

Taking the union over all $\delta < \epsilon$, we get

$$W^s_\epsilon(\Lambda_0) \setminus f(W^s_\epsilon(\Lambda_0)) = \emptyset$$

□

It follows that

$$W^s_\epsilon(\Lambda_0) = \bigcap_{n\ge 0} f^n(W^s_\epsilon(\Lambda_0)) = \Lambda_0.$$

Since $W^s_\epsilon(\Lambda_0)$ is open and Λ_0 we must have $\Lambda_0 = M$.

2.3. Spectral decomposition. — The next result gives a description of the ω-limit set of Lebesgue almost every point in a hyperbolic set with positive volume. A similar decomposition holds for α-limits just by taking f^{-1} in the place of f.

Theorem 2.8. — *Let $f : M \to M$ be a C^{1+} diffeomorphism and let $\Lambda \subset M$ be a hyperbolic set with positive volume. There are hyperbolic sets $\Omega_1, \dots, \Omega_q \subset \Lambda$ such that:*

1. *for* Leb *almost every $x \in \Lambda$ there is $1 \le i \le q$ such that $\omega(x) \subset \Omega_i$;*
2. *Ω_j attracts a neighborhood of itself in M, for each $1 \le j \le q$;*
3. *$f|\Omega_k$ is transitive;*
4. $\mathrm{Per}(f)$ *is dense in Ω_j, for each $1 \le j \le q$.*

Moreover, for each $1 \le k \le q$ there is a decomposition of Ω_k into disjoint hyperbolic sets $\Omega_k = \Omega_{k,1} \cup \dots \cup \Omega_{k,n_k}$ such that:

(5) *$f(\Omega_{k,i}) = \Omega_{k,i+1}$, for $1 \le i < n_k$, and $f(\Omega_{k,n_k}) = \Omega_{k,1}$;*
(6) *$f^{n_k} \colon \Omega_{k,i} \to \Omega_{k,i}$ is topologically mixing for every $1 \le i \le n_k$.*

Let $\Sigma = \overline{W^u(p)} \subset \Lambda$, where p is a hyperbolic periodic point given by Lemma 1.25. We claim that Σ contains the local unstable manifolds of all its points. Indeed, if $x \in \Sigma$, then there is a sequence $(x_n)_n$ of points in $W^u(p)$ converging to x. The continuous variation of the local unstable manifolds gives that the local unstable manifolds of the points x_n, which are contained in Σ, accumulate on the local unstable manifold of x. By closeness, the local unstable manifold of x must be contained in Σ. Thus, defining

$$A = \bigcup_{x \in \Sigma} W^s_\delta(x)$$

we have that A is a neighborhood of Σ whose points have their ω-limit set contained in Σ.

Since Σ is a hyperbolic set with a local product structure attracting an open neighborhood of itself, then by [**KH**, Theorem 18.3.1] there are hyperbolic invariant sets $\Omega_1, \dots, \Omega_s \subset \Sigma \subset \Lambda$ verifying (3)-(6) of Theorem 2.8. Moreover, their union is the set of non-wandering points of f in Σ,

$$NW(f|\Sigma) = \Omega_1 \cup \dots \cup \Omega_s.$$

Since $L(f|\Sigma) \subset NW(f|\Sigma)$, this implies that $\omega(x) \subset \Omega_1 \cup \dots \cup \Omega_s$ for every $x \in A$. Recall that every point in A belongs to the stable manifold of some point in Σ. Now since $\Omega_1, \dots, \Omega_s$ are disjoint compact invariant sets, given $x \in A$, we must even have $\omega(x) \subset \Omega_i$ for some $1 \le i \le s$. Reordering these sets if necessary, let $\Omega_1, \dots, \Omega_q$, for some $q \le s$, be those which attract a set with positive Lebesgue measure. By Theorem 2.1 and transitivity, each $\Omega_1, \dots, \Omega_q$ attracts a neighborhood of itself.

3. Markov structures for partially hyperbolic sets

Assume that Λ is forward invariant compact set with a partially hyperbolic splitting $T_\Lambda M = E^{ss} \oplus E^{cu}$. We assume that the bundle E^{ss} is uniformly contracting. In this section we need some form of non-uniform expansion in the centre-unstable direction which is stronger than the one we have considered in NUE.

Fixing some small $c > 0$ we say that f is *strongly non-uniformly expanding in the central-unstable direction* for $x \in \Lambda$ if

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{j=1}^{n} \log \|Df^{-1} \mid E^{cu}_{f^j(x)}\| < -c. \tag{23}$$

Comparing this to NUE, we are simply replacing lim inf by lim sup. A consequence of this stronger non-uniform expansion is the existence of positive frequency of hyperbolic times. For the points x satisfying (23) we may define the *expansion time*

$$\mathcal{E}(x) = \min\left\{ N \geq 1\colon \frac{1}{n}\sum_{i=0}^{n-1} \log \|Df^{-1} \mid E^{cu}_{f^i(x)}\| < -c, \quad \text{for all } n \geq N \right\}. \tag{24}$$

Arguing as in Corollary 1.19, we obtain the following result as an immediate consequence of Lemma 1.18.

Corollary 3.1. — *There are $\theta > 0$ and $\sigma > 0$ such that for every $x \in \Lambda$ with $\mathcal{E}(x) \leq n$ there exist σ-hyperbolic times $1 \leq n_1 < \cdots < n_l \leq n$ for x with $l \geq \theta n$.*

Given $n \geq 1$ we define

$$H_n = \{x \in \Lambda\colon\; n \text{ is a } \sigma\text{-hyperbolic time for } x\ \}.$$

The next result will play an important role in the metric estimates of Section 3.5.

Corollary 3.2. — *Let D be a centre-unstable disk for which* (23) *holds* Leb_D *almost everywhere. Given $n \geq 1$ and $A \subset D \setminus \{\mathcal{E} > n\}$ with* $\mathrm{Leb}_D(A) > 0$ *we have*

$$\frac{1}{n}\sum_{j=1}^{n} \frac{\mathrm{Leb}_D(A \cap H_j)}{\mathrm{Leb}_D(A)} \geq \theta.$$

Proof. — Take $n \geq 1$ and $A \subset D \setminus \{\mathcal{E} > n\}$ with positive Leb_D measure. Let ξ_n be the measure in $\{1, \ldots, n\}$ defined by $\xi_n(J) = \#J/n$, for each subset J. Then, using Fubini Theorem

$$\begin{aligned}
\frac{1}{n}\sum_{j=1}^{n} \mathrm{Leb}_D(A \cap H_j) &= \int \left(\int_A \chi(x,i)\, d\,\mathrm{Leb}_D(x) \right) d\xi_n(i) \\
&= \int_A \left(\int \chi(x,i)\, d\xi_n(i) \right) d\,\mathrm{Leb}_D(x),
\end{aligned}$$

where $\chi(x,i) = 1$ if $x \in H_i$ and $\chi(x,i) = 0$ otherwise. Corollary 3.1 implies that the integral with respect to $d\xi_n$ is larger than $\theta > 0$. Hence, the last expression above is bounded from below by $\theta\,\mathrm{Leb}_D(A)$. □

3.1. Markov structures. — Benedicks and Young introduced in [**BY**] a Markov structure with infinitely many branches and variable return times for studying the rates of mixing of the Hénon attractor. Some modifications on the definition of these structures were introduced by Young [**Yo3**] and Alves, Pinheiro [**AP2**] in order to make them applicable to more situations.

Definition 3.3. — Let $\mathrm{Emb}^1(D^u, M)$ be the space of C^1 embeddings from D^u into M. We say that $\Gamma^u = \{\gamma^u\}$ is a *continuous family of C^1 unstable manifolds* if there is a compact set K^s, a unit disk D^u of some $\mathbb{R}^n$, and a map $\Phi^u\colon K^s \times D^u \to M$ such that

i) $\gamma^u = \Phi^u(\{x\} \times D^u)$ is an unstable manifold;
ii) Φ^u maps $K^s \times D^u$ homeomorphically onto its image;
iii) $x \mapsto \Phi^u|(\{x\} \times D^u)$ defines a continuous map from K^s into $\mathrm{Emb}^1(D^u, M)$.

Continuous families of C^1 stable manifolds are defined similarly.

Definition 3.4. — We say that $\Omega \subset M$ has a *hyperbolic product structure* if there exist a continuous family of unstable manifolds $\Gamma^u = \{\gamma^u\}$ and a continuous family of stable manifolds $\Gamma^s = \{\gamma^s\}$ such that

i) $\Omega = (\cup\gamma^u) \cap (\cup\gamma^s)$;
ii) $\dim\gamma^u + \dim\gamma^s = \dim M$;
iii) each γ^s meets each γ^u in exactly one point;
iv) stable and unstable manifolds are transversal with angles bounded away from 0.

Let $\Omega \subset M$ have a hyperbolic product structure, whose defining families are Γ^s and Γ^u. A subset $\Omega_0 \subset \Omega$ is called an *s-subset* if Ω_0 also has a hyperbolic product structure and its defining families Γ_0^s and Γ_0^u can be chosen with $\Gamma_0^s \subset \Gamma^s$ and $\Gamma_0^u = \Gamma^u$; *u-subsets* are defined analogously. Given $x \in \Omega$, let $\gamma^*(x)$ denote the element of Γ^* containing x, for $* = s, u$. For each $n \geq 1$ let $(f^n)^u$ denote the restriction of the map f^n to γ^u-disks, and let $\det D(f^n)^u$ be the Jacobian of $D(f^n)^u$. We require that the hyperbolic product structure Ω satisfies several properties:

(P$_1$) Markov: there are pairwise disjoint s-subsets $\Omega_1, \Omega_2, \dots \subset \Omega$ such that
(a) $\mathrm{Leb}_\gamma\big((\Omega \setminus \cup\Omega_i) \cap \gamma\big) = 0$ on each $\gamma \in \Gamma^u$;
(b) for each $i \in \mathbb{N}$ there is $R_i \in \mathbb{N}$ such that $f^{R_i}(\Omega_i)$ is u-subset, and for all $x \in \Omega_i$
$$f^{R_i}(\gamma^s(x)) \subset \gamma^s(f^{R_i}(x)) \quad \text{and} \quad f^{R_i}(\gamma^u(x)) \supset \gamma^u(f^{R_i}(x)).$$

In the statements of the remaining properties about the hyperbolic structure we assume that $C > 0$ and $0 < \beta < 1$ are constants which only depend on f and Ω.

(P$_2$) Contraction on stable leaves: $\mathrm{dist}(f^n(y), f^n(x)) \leq C\beta^n$, $\forall y \in \gamma^s(x)\, \forall n \geq 1$.

In the statements of the last two properties we use a return time function $R\colon \Omega \to \mathbb{N}$ and a return map $f^R\colon \Omega \to \Omega$, defined as
$$R|_{\Omega_i} = R_i \quad \text{and} \quad f^R|_{\Omega_i} = f^{R_i}|_{\Omega_i}, \quad \text{for each } i \in \mathbb{N}.$$
Given $x, y \in \Omega$ we define the *separation time* $s(x, y)$ as the minimum integer $n \geq 0$ such that $(f^R)^n(x)$ and $(f^R)^n(y)$ lie in distinct Ω_i's.

(P$_3$) Regularity of stable foliation: given $\gamma, \gamma' \in \Gamma^u$, we define $\Theta\colon \gamma' \cap \Omega \to \gamma \cap \Omega$ by taking $\Theta(x)$ equal to $\gamma^s(x) \cap \gamma$. Then
(a) Θ is absolutely continuous and
$$\frac{d(\Theta_* \mathrm{Leb}_{\gamma'})}{d\,\mathrm{Leb}_\gamma}(x) = \prod_{i=0}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(\Theta^{-1}(x)))};$$

(b) letting $u(x)$ denote the density in item (a), we have

$$\log \frac{u(x)}{u(y)} \le C\beta^{s(x,y)}, \quad \text{for } x, y \in \gamma' \cap \Omega.$$

(P_4) Bounded distortion: for $\gamma \in \Gamma^u$ and $x, y \in \Omega \cap \gamma$

$$\log \frac{\det D(f^R)^u(x)}{\det D(f^R)^u(y)} \le C\beta^{s(f^R(x), f^R(y))}.$$

In the next result we show that Markov structures can be constructed for partially hyperbolic transitive sets, provided we have some control on the decay of the tail set of the function introduced in (24). The remaining part of these notes is devoted to the proof of this theorem.

Theorem 3.5. — *Let $f : M \to M$ be a C^{1+} diffeomorphism and $\Lambda \subset M$ be a compact invariant partially hyperbolic transitive set. Assume that there is a local unstable disk $D \subset \Lambda$ and $\tau > 0$ such that*

$$\mathrm{Leb}_D\{\mathcal{E} > n\} \le \mathcal{O}(n^{-\tau}).$$

Then there is $\Omega \subset M$ with a hyperbolic product structure for which (P_1)-(P_4) hold, and

$$\mathrm{Leb}_\gamma\{R > n\} \le \mathcal{O}(n^{-\tau}), \quad \textit{for some } \gamma \in \Gamma^u.$$

The proof of this result will be given in the next sections.

Open problem. — *Can we obtain similar result for exponential decay?*

3.2. Returning disks. — Let U be a small neighborhood of Λ as in Section 1.2 and take δ_1 as in (13). Here we assume that $K = \Lambda$. Let D be as Theorem 3.5. Diminishing $\delta_1 > 0$, if necessary, we may assume that D has radius δ_1. Take $0 < \delta_s < \delta_1/2$ such that local stable disks centered at points in Λ have at least radius δ_s. In particular, these local stable manifolds are contained in U.

Lemma 3.6. — *There are $N_0 \ge 1$ and $q \in \Lambda$ such that:*

1. *$W^s_{\delta_s/2}(q)$ intersects D in a point p with $\mathrm{dist}_D(p, \partial D) > \delta_1/2$;*
2. *for each centre-unstable disk γ^u_1 of radius δ_1 centered at a point in Λ there are $0 \le j \le N_0$ and a disk $\gamma^u_2 \subset \gamma^u_1$ of radius $\delta_1/2$ centered at a point $z \in W^s_{\delta_s/4}(f^{-j}(q))$.*

Proof. — We start by observing that there is a constant $A > 0$ (depending only on the minimum angle between the two spaces of the dominated splitting over Λ) with the following property: *given $x \in \Lambda$, $\rho > 0$ and $y \in \Lambda$ with $\mathrm{dist}(x, y) < \rho$ having a local unstable disk of radius δ_1 centered at y, then $W^s_{\delta_s}(x)$ intersects $W^u_{\delta_1}(y)$ in a point z with*

$$\mathrm{dist}_{W^s_{\delta_s}(x)}(z, x) < A\rho \quad \textit{and} \quad \mathrm{dist}_{W^u_{\delta_1}(y)}(z, y) < \delta_1/2.$$

In particular, such a point z has a neighborhood of radius $\delta_1/2$ inside $W^u_{\delta_1}(y)$.

Since we are assuming $f|\Lambda$ transitive, we may fix $q \in \Lambda$, a small number $0 < \rho < \delta_s/(4A)$ and $N_0 \in \mathbb{N}$ such that both *i*) $W^s_{\delta_s/2}(q)$ intersects D in a point p with $\mathrm{dist}_D(p, \partial D) > \delta_1/2$, and *ii*) $\{f^{-N_0}(q), \dots, f^{-1}(q), q\}$ is ρ-dense in Λ. By ρ-dense we mean that any other point in Λ has one of those pre-images at a distance less than ρ.

Hence, given any centre-unstable disk γ_1^u of radius δ_1 centered at a point $y \in \Lambda$, there is $0 \leq j \leq N_0$ such that $\mathrm{dist}(f^{-j}(q), y) < \rho$. Then, by the choice of A and ρ, we have that $W^s_{\delta_s}(f^{-j}(q))$ intersects γ_1^u in a point z with $\mathrm{dist}_{W^s_{\delta_s}(f^{-j}(q))}(z, f^{-j}(q)) < A\rho < \delta_s/4$ and $\mathrm{dist}_{\gamma_1^u}(z, y) < \delta_1/2$. □

Lemma 3.7. — *There is $\delta_2 > 0$ such that if γ^u is a centre-unstable disk of radius $\delta_1/2$ centered at a point $z \in W^s_{\delta_s}(w)$ with $w \in \Lambda$, then $f^j(\gamma^u)$ contains a centre-unstable disk of radius δ_2 centered at $f^j(z)$, for each $1 \leq j \leq N_0$.*

Proof. — Let us first prove the result for $j = 1$. Let $f(y)$ be a point in $\partial f(\gamma^u)$ minimizing the distance from $f(z)$ to $\partial f(\gamma^u)$, and let η_1 be a curve of minimal length in $f(\gamma^u)$ connecting $f(z)$ to $f(y)$. Define $\eta_0 = f^{-1}(\eta_1)$. Denote by $\dot\eta_1(x)$ the tangent vector to the curve η_1 at the point x. Then,

$$\|Df^{-1}(w)\dot\eta_1(x)\| \leq C\,\|\dot\eta_1(x)\|,$$

where

$$C = \max_{x\in M}\left\{\|Df^{-1}(x)\|\right\} \geq 1.$$

Hence,

$$\mathrm{length}(\eta_0) \leq C\,\mathrm{length}(\eta_1).$$

Noting that η_0 is a curve connecting z to $y \in \partial\gamma^u$, this implies that $\mathrm{length}(\eta_0) \geq \delta_1/2$. Hence

$$\mathrm{length}(\eta_1) \geq C^{-1}\,\mathrm{length}(\eta_0) \geq C^{-1}\delta_1/2.$$

Thus $f(\gamma^u)$ contains the disk γ_1^u of radius $C^{-1}\delta_1/2$ around $f(z)$. Moreover,

$$\mathrm{dist}(f(z), f(w)) \leq \lambda\delta_s < \delta_s,$$

and so, by the choice of δ_s, we have that γ_1^u is also a centre-unstable disk. Making now γ_1^u play the role of γ^u and $f^2(z)$ play the role of $f(z)$ we prove that:

(a) $f(\gamma_1^u)$ contains a centre-unstable disk of radius $C^{-2}\delta_1/2^2$ centered at $f^2(z)$;
(b) $\mathrm{dist}(f^2(z), f^2(w)) \leq \lambda^2\delta_s < \delta_s$.

Item (a) gives in particular that $f^2(\gamma^u)$ contains a centre-unstable disk of radius $C^{-2}\delta_1/2^2$ centered at $f^2(z)$. Arguing inductively we are able to prove that $f^j(\gamma^u)$ contains a disk of radius $C^{-j}\delta_1/2^j \geq C^{-N_0}\delta_1/2^{N_0}$ around $f^j(z)$, for each $1 \leq j \leq N_0$. Hence, we just have to take $\delta_2 = C^{-N_0}\delta_1/2^{N_0}$. □

3.3. Partition on a reference leaf. — The construction we are going to explain below requires that we use several constants. For the sake of clearness we start by setting out the way these constants are related. First we take $\delta_1 > 0$ as in (13), and $0 < \delta_2 < \delta_1$ as in Lemma 3.7. Then we take $\delta_0 > 0$ and $\varepsilon > 0$ so that

$$\delta_0 \ll \delta_2 \quad\text{and}\quad \varepsilon \ll \delta_0.$$

We are going to describe the construction of the (m_D mod 0) partition $\mathcal{P}$ of the centre-unstable disk of radius δ_0 centered at p contained in D. Let us consider the following neighborhoods of p in D

$$\Delta_0^0 = B^u(p,\delta_0),\quad \Delta_0^1 = B^u(p,2\delta_0),\quad \Delta_0^2 = B^u(p,\sqrt{\delta_0})\quad\text{and}\quad \Delta_0^3 = B^u(p,2\sqrt{\delta_0}),$$

and the cylinders whose respective bases are the sets above

$$\mathcal{C}^i = \bigcup_{x\in\Delta_0^i} W^s_{\delta_s}(x), \quad \text{for } i=0,1,2,3.$$

Letting π denote the projection from $\mathcal{C}^3$ to Δ_0^3 along local stable leaves, we have

$$\pi(\mathcal{C}^i) = \Delta_0^i, \quad \text{for } i=0,1,2,3.$$

We say that a centre-unstable disk γ^u *u-crosses* $\mathcal{C}^i$ if $\pi(\gamma^u)=\Delta_0^i$.

Remark 3.8. — To simplify the exposition we shall pretend that each centre-unstable disk γ^u u-crossing $\mathcal{C}^i$ is still a disk centered at a point of $W^s_{\delta_s}(p)$ and with the same radius of Δ_0^i. Actually, we have such a γ^u contained and containing disks with radiuses proportional to that radius, with the proportion constant depending only on the height of the cylinder and the angles of the two fibre bundles of the dominated splitting.

Let $\partial^u\mathcal{C}_1^3$ denote the *top* and *bottom* components of $\partial\mathcal{C}^3$, i.e. the set of points $z\in\partial\mathcal{C}^3$ such that $z\in\partial W^s_{\delta_s}(x)$ for some $x\in\Delta_0^3$. By the domination property, we may take $\delta_0>0$ small so that no centre-unstable disk contained in $\mathcal{C}^3$ and intersecting $W^s_{\delta_s/2}(p)$ can reach $\partial^u\mathcal{C}^3$.

For $0<\sigma<1$ given by Corollary 1.19, let

$$I_k = \left\{x\in\Delta_0^1 \,:\, \delta_0(1+\sigma^{k/2}) < \mathrm{dist}_D(x,p) < \delta_0(1+\sigma^{(k-1)/2})\right\}, \quad k\geq 1,$$

be a partition (Leb_D mod 0) into countably many rings of $\Delta_0^1\setminus\Delta_0^0$.

Now we are able to describe the inductive construction of the partition $\mathcal{P}$ of Δ_0. The construction requires that we introduce inductively several objects. In particular, we will consider sequences of sets (Δ_n), (A_n), and (B_n). For each $n\geq 0$, the set Δ_n is the part of Δ_0 that has not be partitioned. The set Δ_n is the disjoint union of A_n and B_n, where A_n is essentially the part of Δ_n where new elements of partition may be constructed in the next step, and B_n is some protection that we put around the sets previously constructed in order to avoid overlaps. For technical reasons, a small neighborhood A_n^ε of each A_n will also be considered.

First step of induction. — We fix R_0 some large integer, and we ignore any dynamics occurring up to time R_0. Let $k\geq R_0+1$ be the first time that $\Delta_0\cap H_k\neq\emptyset$. For $j<k$ we define formally the objects

$$A_j = A_j^\varepsilon = \Delta_j = \Delta_0, \quad \text{and} \quad B_j=\emptyset.$$

Let $(\omega^3_{k,j})_j$ be all the center-unstable disks in A^ε_{k-1} contained in hyperbolic pre-balls V_m, with $k-N_0\leq m\leq k$, which are mapped by f^k onto a centre-unstable disk u-crossing $\mathcal{C}^3$ and intersecting $W^s_{\delta_s/4}(p)$. Then we let

$$\omega^i_{k,j} = \omega^3_{k,j}\cap f^{-k}(\mathcal{C}^i), \quad i=0,1,2$$

and set $R(x)=k$ for $x\in\omega^0_{k,j}$. We take

$$\Delta_k = \Delta_{k-1}\setminus\{R=k\},$$

and define a function $t_k : \Delta_k \to \mathbb{N}$ by

$$t_k(x) = \begin{cases} s, & \text{if } x \in \omega^1_{k,j} \text{ and } \pi(f^k(x)) \in I_s \text{ for some } j; \\ 0, & \text{otherwise.} \end{cases}$$

Finally let

$$A_k = \{x \in \Delta_k : t_k(x) = 0\}, \quad B_k = \{x \in \Delta_k : t_k(x) > 0\}$$

and

$$A^\varepsilon_k = \{x \in \Delta_k : \operatorname{dist}_{f^{k+1}(D)}(f^{k+1}(x), f^{k+1}(A_k)) < \varepsilon\}.$$

General step of induction. — The general step of the construction follows the ideas above with minor modifications. Assume that the sets Δ_i, A_i, A^ε_i B_i, $\{R = i\}$ and functions $t_i : \Delta_i \to \mathbb{N}$ are defined for each $i \le n-1$. Let $(\omega^3_{n,j})_j$ be all the centre-unstable disks in A^ε_{n-1} contained in hyperbolic pre-balls V_m, with $n - N_0 \le m \le n$, which are mapped by f^n onto a centre-unstable disk u-crossing $\mathcal{C}^3$ and intersecting $W^s_{\delta_s/4}(p)$. Take

$$\omega^i_{n,j} = \omega^3_{n,j} \cap f^{-n}(\mathcal{C}^i_1), \quad i = 0, 1, 2 \tag{25}$$

set $R(x) = n$ for $x \in \omega^0_{n,j}$, and let

$$\Delta_n = \Delta_{n-1} \setminus \{R = n\}.$$

The definition of the function $t_n : \Delta_n \to \mathbb{N}$ is slightly different in the general case:

$$t_n(x) = \begin{cases} x, & \text{if } x \in \omega^1_{n,j} \setminus \omega^0_{n,j} \text{ and } f^n(x), \in I_s \text{ for some } j, \\ 0, & \text{if } x \in A_{n-1} \setminus \cup_j \omega^1_{n,j}, \\ t_{n-1}(x) - 1, & \text{if } x \in B_{n-1} \setminus \cup_j \omega^1_{n,j}. \end{cases}$$

Finally, we let

$$A_n = \{x \in \Delta_n : t_n(x) = 0\}, \quad B_n = \{x \in \Delta_n : t_n(x) > 0\}$$

and

$$A^\varepsilon_n = \{x \in \Delta_n : \operatorname{dist}_{f^{n+1}(D)}(f^{n+1}(x), f^{n+1}(A_n)) < \varepsilon\}.$$

At this point we have described the construction of the sets A_n, A^ε_n, B_n and $\{R = n\}$.

Since the components of $\{R = n\}$ are taken in A^ε_{n-1}, it could happen that these new components intersect B_{n-1}. The next lemma shows that this is not the case (not even for the new parts of B_n!) as long as $\epsilon > 0$ is taken small enough. For notational simplicity we will drop the index j in the elements defined at (25).

Lemma 3.9. — *If $\epsilon > 0$ is sufficiently small, then $\omega^1_n \cap \{t_{n-1} \ge 1\} = \emptyset$ for all n.*

Proof. — Take $k \ge 1$ and let ω^0_{n-k} be a component of $\{R = n-k\}$. Let Q_k be the part of ω^1_{n-k} that is mapped by $\pi \circ f^{n-k}$ onto I_k, and assume that Q_k intersects some ω^3_n. Recall that, by construction, Q_k is precisely that part of ω^1_{n-k} on which $t_{n-1} = 1$, and ω^3_n is contained in a hyperbolic pre-ball V_m with $n - N_0 \le m \le n$.

Let q_1 and q_2 be any two points in distinct components (inner and outer, respectively) of the boundary of Q_k. If we assume that $q_1, q_2 \in \omega^3_n$, then $q_1, q_2 \in V_m$, and so by Lemma 1.15 we have

$$\operatorname{dist}_{f^{n-k}(D)}(f^{n-k}(q_1), f^{n-k}(q_2)) \le C_0 \sigma^{k/2} \operatorname{dist}_{f^n(D)}(f^n(q_1), f^n(q_2)) \tag{26}$$

for some C_0 depending on N_0. We also have for some $C_1>0$ depending on the angle of the stable and centre-unstable spaces over K_∞

$$\begin{aligned}\mathrm{dist}_{f^{n-k}(D)}(f^{n-k}(q_1),f^{n-k}(q_2)) &\geq C_1\delta_0(1+\sigma^{(k-1)/2})-\delta_0(1+\sigma^{k/2})\\ &= C_1\delta_0\sigma^{k/2}(\sigma^{-1/2}-1),\end{aligned}$$

which combined with (26) gives

$$\mathrm{dist}_{f^n(D)}(f^n(q_1),f^n(q_2))\geq\frac{C_1}{C_0}\delta_0(\sigma^{-1/2}-1).$$

On the other hand, since $\omega_n^3\subset A_{n-1}^\varepsilon$ by construction of ω_n^3, taking

$$\varepsilon<\frac{C_1}{C_0}\delta_0(\sigma^{-1/2}-1) \tag{27}$$

we have $\omega_n^3\cap\{t_{n-1}>1\}=\emptyset$. This implies $\omega_n^1\cap\{t_{n-1}\geq 1\}=\emptyset$. □

3.4. Product structure. — Consider the center-unstable disk $\Delta_0\subset D$ and the partition $\mathcal{P}$ of Δ_0 (Leb_D mod 0) constructed in Section 3.3. We shall use the elements of $\mathcal{P}$ to define the s-subsets that give rise to the hyperbolic structure. Given an arbitrary element $\omega\in\mathcal{P}$, we have by construction some $R(\omega)\in\mathbb{N}$ such that $f^{R(\omega)}(\omega)$ is a centre-unstable disk u-crossing $\mathcal{C}^0$. We define $\mathcal{C}_\omega$ as the cylinder made by the stable leaves passing through the points in ω, i.e.

$$\mathcal{C}_\omega=\bigcup_{x\in\omega}W^s_{\delta_s}(x).$$

The sets $\mathcal{C}_\omega$, with $\omega\in\mathcal{P}$, are by definition the pairwise disjoint s-subsets $\Lambda_1,\Lambda_2,\dots$ which define the Markovian structure.

Now we define inductively some sets of centre-unstable manifolds u-crossing $\mathcal{C}^0$ that will give rise to the family Γ^u. The first one is

$$\Gamma_0=\{\Delta_0\}.$$

Having defined Γ_j, for some $j\geq 0$, we define

$$\Gamma_{j+1}=\left\{f^{R(\omega)}(\mathcal{C}_\omega\cap\gamma):\ \omega\in\mathcal{P}\text{ and }\gamma\in\Gamma_j\right\}.$$

Observe that each element of Γ_j is equal to an iterate of a subset of Δ_0. In particular, the elements of each Γ_j are unstable manifolds. Moreover, since by construction $f^{R(\omega)}(\omega)$ intersects $W^s_{\delta_s/4}(p)$, then according to the choice of δ_0 and the invariance of the stable foliation, we have that each element of Γ_j must u-cross $\mathcal{C}^0$.

Since the union of the leaves of the sets Γ_j, with $j\geq 0$, is not necessarily compact, we still need to take accumulation points of that union. Let

$$\Delta_\infty=\overline{\bigcup_{j\geq 0}\bigcup_{\gamma_j\in\Gamma_j}\gamma_j}.$$

Given $x\in\Delta_\infty$, there are $(j_k)_k\to\infty$, disks $\gamma_{j_k}\in\Gamma_{j_k}$ and points $x_k\in\gamma_{j_k}$ converging to x as $k\to\infty$. Using the domination property and Ascoli-Arzela theorem we conclude that the disks γ_{j_k} converge to a disk γ_∞ containing x. Since the disk γ_∞ is accumulated by disks u-crossing $\mathcal{C}^0$ then it also must u-cross $\mathcal{C}^0$. We define Γ_∞ as the set of all these accumulation disks. Finally, we take

$$\Gamma^u = \bigcup_{j\geq 0} \Gamma_j \cup \Gamma_\infty.$$

Lemma 3.10. — *There is $C > 0$ such that, given $\omega \in \mathcal{P}$ and $\gamma \in \Gamma^u$, we have for all $1 \leq k \leq R(\omega)$ and all $x, y \in \mathcal{C}_\omega \cap \gamma$*

$$\mathrm{dist}_{f^{R(\omega)-k}(\mathcal{C}_\omega\cap\gamma)}(f^{R(\omega)-k}(x), f^{R(\omega)-k}(y)) \leq C\sigma^{k/2}\,\mathrm{dist}_{f^{R(\omega)}(\mathcal{C}_\omega\cap\gamma)}(f^{R(\omega)}(x), f^{R(\omega)}(y)).$$

Proof. — Recall that, by construction, for each $\omega \in \mathcal{P}$ there is a hyperbolic pre-ball $V_{n(\omega)}(x)$ containing ω associated to some point $x \in D$ with σ-hyperbolic time $n(\omega)$ satisfying $R(\omega) - N_0 \leq n(\omega) \leq R(\omega)$. Taking $\delta_s, \delta_0 < \delta_1/2$, it follows from (13) that $n(\omega)$ is a $\sqrt{\sigma}$-hyperbolic time for every point in $\mathcal{C}_\omega \cap \gamma$. Then, recall (12), this implies that for all $1 \leq k \leq n(\omega)$ and all $x, y \in \mathcal{C}_\omega \cap \gamma$ we have

$$\mathrm{dist}_{f^{n(\omega)-k}(\mathcal{C}_\omega\cap\gamma)}(f^{n(\omega)-k}(x), f^{n(\omega)-k}(y)) \leq \sigma^{k/2}\,\mathrm{dist}_{f^{n(\omega)}(\mathcal{C}_\omega\cap\gamma)}(f^{n(\omega)}(x), f^{n(\omega)}(y)).$$

Since the difference between $R(\omega)$ and $n(\omega)$ is at most N_0, the result follows with C depending only on N_0 and the the derivative of f. □

3.5. Metric estimates. — In this section we obtain some metric estimates that enable us to obtain the conclusion on the decay of $\mathrm{Leb}_D\{R > n\}$ of Theorem 3.5. We start by proving some estimates arising from the construction preformed in Section 3.3.

Lemma 3.11. — *There is $a_0 > 0$ such that $\mathrm{Leb}_D(B_{n-1} \cap A_n) \geq a_0 \,\mathrm{Leb}_D(B_{n-1})$, for all $n \geq 1$.*

Proof. — It is enough to see this for each component of B_{n-1}. Let C be a component of B_{n-1} and Q be its outer ring, corresponding to $t_{n-1} = 1$. Observe that by Lemma 3.9 we have $Q = C \cap A_n$. Moreover, there must be some $k < n$ and a component ω_k^0 of $\{R = k\}$ such that $\pi \circ f^k$ maps C diffeomorphically onto $\bigcup_{i=k}^\infty I_i$ and Q onto I_k, both with uniform bounded distortion given by Corollary 1.16. Thus, it is sufficient to compare the Lebesgue measures of $\bigcup_{i=k}^\infty I_i$ and I_k. We have

$$\frac{\mathrm{Leb}_D(I_k)}{\mathrm{Leb}_D(\cup_{i=k}^\infty I_i)} \approx \frac{[\delta_0(1+\sigma^{(k-1)/2})]^u - [\delta_0(1+\sigma^{k/2})]^u}{[\delta_0(1+\sigma^{(k-1)/2})]^u - \delta_0^u} \approx 1 - \sigma^{1/2},$$

where u is the dimension of E^{cu}. □

Lemma 3.12. — *There exist $b_0, c_0 > 0$, with $b_0, c_0 \to 0$ as $\delta_0 \to 0$, such that, for all $n \geq 1$,*

1. *$\mathrm{Leb}_D(A_{n-1} \cap B_n) \leq b_0 \,\mathrm{Leb}_D(A_{n-1})$;*
2. *$\mathrm{Leb}_D(A_{n-1} \cap \{R = n\}) \leq c_0 \,\mathrm{Leb}_D(A_{n-1})$.*

Proof. — It is enough to prove this for each neighborhood of a component ω_n^0 of $\{R = n\}$. Observe that by construction we have $\omega_n^3 \subset A_{n-1}^\varepsilon$, which means that $\omega_n^2 \subset A_{n-1}$, because we are taking $\varepsilon < \delta_0$. Using the uniform bounded distortion of Corollary 1.16 we obtain

$$\frac{\mathrm{Leb}_D(\omega_n^1 \setminus \omega_n^0)}{\mathrm{Leb}_D(\omega_n^2 \setminus \omega_n^1)} \approx \frac{\mathrm{Leb}_D(\Delta_0^1 \setminus \Delta_0^0)}{\mathrm{Leb}_D(\Delta_0^2 \setminus \Delta_0^1)} \approx \frac{\delta_0^d}{\delta_0^{d/2}} \ll 1,$$

which gives the first estimate. Moreover,

$$\frac{\mathrm{Leb}_D(\omega_n^0)}{\mathrm{Leb}_D(\omega_n^2\setminus\omega_n^1)}\approx\frac{\mathrm{Leb}_D(\Delta_0^0)}{\mathrm{Leb}_D(\Delta_0^2\setminus\Delta_0^1)}\approx\frac{\delta_0^d}{\delta_0^{d/2}}\ll 1,$$

and this gives the second one. □

We take $\delta_s>0$ sufficiently small so that

$$\big|\log\|Df^{-1}\mid E^{cu}_x\|-\log\|Df^{-1}\mid E^{cu}_y\|\big|<\frac{c}{4}. \tag{28}$$

for each $x\in\Lambda$ and $y\in W^s_{\delta_s}(x)$.

Lemma 3.13. — *Given $0<\varepsilon<\delta_1$, there exists $N_\varepsilon>0$ such that for every $n\geq n_0$ and every $x\in D\cap H_n$, the ball of radius ε centered at $f^n(z)$ inside the hyperbolic ball $f^n(V_n(x))$ contains a hyperbolic pre-ball $V_k(z)$ with $k\leq N_\varepsilon$.*

Proof. — Given $n\geq n_0$ and $x\in D\cap H_n$, let B^n_x denote the hyperbolic ball associated to x with hyperbolic time n. Recall that $B^n_x=f^n(V_n(x))$ is a centre-unstable ball of radius δ_1 around $f^n(x)$. We define the cylinder

$$C^n_x=\bigcup_{y\in B^n_x}W^s_{\delta_s}(y).$$

Since NUE holds for Leb_D almost every point and it remains valid by forward iteration, it follows that $\mathrm{Leb}_{B^n_x}$ almost every point in B^n_x also satisfies NUE. Given $z\in C^n_x$ we define

$$\tilde{\mathcal{E}}(z)=\min\left\{N\geq 1\colon \frac{1}{n}\sum_{i=0}^{n-1}\log\|Df^{-1}\mid E^{cu}_{f^i(w)}\|<-\frac{3c}{4},\quad\forall n\geq N\right\}.$$

The fact that NUE holds for $\mathrm{Leb}_{B^n_x}$ almost every point in B^n_x together with (28) imply that $\tilde{\mathcal{E}}(z)$ is well defined for Leb almost every point $z\in C^n_x$. Hence $\tilde{\mathcal{E}}(z)$ is well defined for Leb almost every point z belonging to the set

$$C=\bigcup_{\substack{n\geq n_0\\ x\in D\cap H_n}}C^n_x.$$

Using Lemma 1.15 we may choose n_ε large enough so that any hyperbolic pre-ball of σ-hyperbolic time $n\geq n_\varepsilon$ will have diameter not exceeding $\varepsilon/2$. Let now $B^n_x(\varepsilon/2)$ denote the ball of radius $\varepsilon/2$ around $f^n(x)$ inside B^n_x, and take

$$v_\varepsilon=\min_{\substack{n\geq n_0\\ x\in D\cap H_n}}\left\{\mathrm{Leb}\left(\bigcup_{y\in B^n_x(\varepsilon/2)}W^s_{\delta_s}(y)\right)\right\}.$$

Since the sizes and tangent directions of hyperbolic balls and stable disks are uniformly controlled, this minimum v_ε must be strictly positive. Hence, as

$$\mathrm{Leb}\left\{z\in C\colon\tilde{\mathcal{E}}(z)>n\right\}\to 0,\qquad\text{when } n\to\infty,$$

it is possible to choose $N_\varepsilon\in\mathbb{N}$ large enough so that

$$\mathrm{Leb}\left\{z\in C\colon\tilde{\mathcal{E}}(z)>N_\varepsilon\right\}\leq v_\varepsilon. \tag{29}$$

We take N_ε also satisfying $\theta N_\varepsilon > n_\varepsilon$. By the choice of v_ε, given $n \geq n_0$ and $x \in D \cap H_n$ there must be some $z \in W^s_{\delta_s}(y)$ with $y \in B^n_x(\varepsilon/2)$ such that $\tilde{\mathcal{E}}(z) \leq N_\varepsilon$. Using (28) we easily deduce that $\mathcal{E}(z) \leq N_\varepsilon$; recall (24). So, by Proposition 1.19 there exists some σ-hyperbolic time h for y with $\theta N_\varepsilon < h \leq N_\varepsilon$. Since we have taken $\theta N_\varepsilon > n_\varepsilon$, attending to the choice of n_ε we are done. □

Proposition 3.14. — *There exist $c_1 > 0$ and a positive integer $N = N(\varepsilon)$ such that*

$$\mathrm{Leb}_D\left(\cup_{i=0}^N \{R = n+i\}\right) \geq c_1 \,\mathrm{Leb}_D(A_{n-1} \cap H_n)$$

for every $n \geq 1$.

Proof. — Let $K_0 = \max_{x\in\Lambda} \|Df^{-1}\|$ and take $r = 5\delta_0 K_0^{N_0}$, where N_0 is given by Lemma 3.6. Recall that by Lemma 1.15, for each $z \in f^n(A_{n-1}\cap H_n)$ there is $x \in H_n$ and a σ-hyperbolic pre-ball $V_n(x) \subset D$ which is sent diffeomorphically onto the centre-unstable ball of radius δ_1 around z. Let $\{z_j\}$ be a maximal set in $f^n(A_{n-1} \cap H_n)$ with the property that the sets $B(z_j, r)$ are pairwise disjoint, where each $B(z_j, r)$ is the ball of radius r centered at z_j inside the hyperbolic ball around z_j. By maximality we have

$$\bigcup\nolimits_j B(z_j, 2r) \supset f^n(A_{n-1} \cap H_n). \tag{30}$$

For each j let $x_j \in H_n$ be the point such that $f^n(x_j) = z_j$.

Claim 1. *There is $0 \leq k \leq N_\epsilon + N_0$ such that t_{n+k} is not identically zero in $f^{-n}(B(z, \epsilon))$.* Assume, by contradiction, that $t_{n+k}|f^{-n}(B(z, \epsilon)) = 0$ for all $0 \leq k \leq N_\epsilon + N_0$. This implies that $f^{-n}(B(z, \epsilon)) \subset A_{n+k}$ for all $0 \leq k \leq N_\epsilon + N_0$. Using Lemma 3.13 we may find a hyperbolic pre-ball $V_m \subset B(z, \epsilon)$ with σ-hyperbolic time $m \leq N_\epsilon$. Now, since $f^m(V_m)$ is a centre-unstable disk of radius δ_1, it follows from Lemma 3.6 and Lemma 3.7 that there are $V \subset f^m(V_m)$ and $m' \leq N_0$ such that u-crossing $\mathcal{C}^3$ and intersecting $W^s_{\delta_s/4}(p)$. Thus, taking $k = m + m'$ we have that $0 \leq k \leq N_\epsilon + N_0$ and $f^{-n}(V_m)$ contains an element of $\{R = n + k\}$ inside $f^{-n}(B(z, \epsilon))$. This contradicts the fact that $t_{n+k}|f^{-n}(B(z, \epsilon)) = 0$ for all $0 \leq k \leq N_\epsilon + N_0$.

Claim 2. *$f^{-n}(B(z, \delta_1/4))$ contains a component of $\{R = n + k\}$ with $0 \leq k \leq N_\epsilon + N_0$.* Let k be the smallest integer $0 \leq k \leq N_\epsilon + N_0$ for which t_{n+k} is not identically zero in $f^{-n}(B(z, \epsilon))$. Since $f^{-n}(B(z, \epsilon)) \subset A^\epsilon_{n-1} \subset \{t_{n-1} \leq 1\}$, there must be some component ω^0_{n+k} of $\{R = n + k\}$ for which $f^{-n}(B(z, \epsilon)) \cap \omega^1_{n+k} \neq \emptyset$. Recall that, by definition, f^{n+k} sends ω^1_{n+k} diffeomorphically onto a centre-unstable disk (of radius $2\delta_0$) u-crossing $\mathcal{C}^1$ and intersecting $W^s_{\delta_s/4}(p)$. Thus, the diameter of $f^n(\omega^1_{n+k})$ is at most $4\delta_0 K_0^{N_0}$. Since $B(z, \epsilon)$ intersects $f^n(\omega^1_{n+k})$ and $\epsilon < \delta_0 < \delta_0 K_0^{N_0}$, we have $f^{-n}(B(z, \delta_1/4)) \supset \omega^0_{n+k}$, as long as we take $\delta_0 > 0$ small so that $5\delta_0 K_0^{N_0} < \delta_1/4$. Hence, we have shown that $f^{-n}(B(z, \delta_1/4))$ contains some component of $\{R = n + k\}$ with $0 \leq k \leq N_\epsilon + N_0$, and so we have proved the claim.

Since n is a hyperbolic time for x_j, we have by the distortion control given by Corollary 1.16 that there is some constant C only depending on C_2 and δ_1 for which

$$\frac{\mathrm{Leb}_D(f^{-n}(B(z_j, 2r)))}{\mathrm{Leb}_D(f^{-n}(B(z_j, r)))} \leq C \frac{\mathrm{Leb}_{f^n(D)}(B(z_j, 2r))}{\mathrm{Leb}_{f^n(D)}(B(z_j, r))} \tag{31}$$

and

$$\frac{\mathrm{Leb}_D(f^{-n}(B(z_j,r)))}{\mathrm{Leb}_D(\omega^0_{n+k})} \le C\frac{\mathrm{Leb}_{f^n(D)}(B(z_j,r))}{\mathrm{Leb}_{f^n(D)}(f^n(\omega^0_{n+k}))}. \tag{32}$$

Recalling that from time n to $n+k$ we have at most N_0 iterates, from (31) and (31) we easily deduce that there is some positive constant, that we still denote by C, for which

$$\mathrm{Leb}_D(f^{-n}(B(z_j,2r))) \le C\,\mathrm{Leb}_D(f^{-n}(B(z_j,r)))$$

and

$$\mathrm{Leb}_D(f^{-n}(B(z_j,r))) \le C\,\mathrm{Leb}_D(\omega^0_{n+k_j}).$$

Finally, let us compare the Lebesgue measure of the sets $\bigcup_{i=0}^N \big\{R=n+i\big\}$ and $A_{n-1}\cap H_n$. By (30) we have

$$\mathrm{Leb}_D\big(A_{n-1}\cap H_n\big) \le \sum_j \mathrm{Leb}_D(f^{-n}(B(z_j,2r))) \le C\sum_j \mathrm{Leb}_D(f^{-n}(B(z_j,r))).$$

On the other hand, by the disjointness of the balls $B(z_j,r)$ we have

$$\sum_j \mathrm{Leb}_D(f^{-n}(B(z_j,r))) \le C\sum_j \mathrm{Leb}_D(\omega^0_{n+k}) \le C\,\mathrm{Leb}_D\left(\bigcup_{i=0}^N\{R=n+i\}\right).$$

We just have to take $c_1=C^{-2}$. $\square$

For completing the proof of Theorem 3.5, it is enough to show that

$$\mathrm{Leb}_D\{\mathcal{E}>n\} \le \mathcal{O}(n^{-\gamma}) \quad\Rightarrow\quad \mathrm{Leb}_D\{R>n\} \le \mathcal{O}(n^{-\gamma}).$$

Recall that for $n\ge 1$ we have defined H_n as the set of points for which n is a σ-hyperbolic time. In Corollary 3.1 we have obtained the following estimate:

(m_1) There is $\theta>0$ such that for $n\ge 1$ and $A\subset M\setminus\{\mathcal{E}>n\}$ with $\mathrm{Leb}_D(A)>0$

$$\frac{1}{n}\sum_{j=1}^n \frac{\mathrm{Leb}_D(A\cap H_j)}{\mathrm{Leb}_D(A)} \ge \theta.$$

In the construction of the Markov structure have taken a disk Δ of radius $\delta_0>0$ and defined inductively the subsets A_n, B_n, $\{R=n\}$ and Δ_n related in the following way:

$$\Delta_n = \Delta\setminus\{R\le n\} = A_n\dot\cup B_n.$$

Moreover, we have proved in Lemma 3.11, Lemma 3.12 and Proposition 3.14 that the following metric relations hold:

(m_2) There is $a_0>0$ (bounded away from 0 with δ_0) such that for all $n\ge 1$

$$\mathrm{Leb}_D(B_{n-1}\cap A_n) \ge a_0\mathrm{Leb}_D(B_{n-1}).$$

(m_3) There are $b_0, c_0>0$ with $b_0,c_0\to 0$ as $\delta_0\to 0$, such that for all $n\ge 1$

$$\frac{\mathrm{Leb}_D(A_{n-1}\cap B_n)}{\mathrm{Leb}_D(A_{n-1})} \le b_0 \quad\text{and}\quad \frac{\mathrm{Leb}_D(A_{n-1}\cap\{R=n\})}{\mathrm{Leb}_D(A_{n-1})} \le c_0.$$

(m_4) There is $c_1>0$ and an integer $N\ge 0$ such that for all $n\ge 1$

$$\mathrm{Leb}_D\big(\cup_{i=0}^N\{R=n+i\}\big) \ge c_1\,\mathrm{Leb}_D(A_{n-1}\cap H_n).$$

Estimates (m_1)-(m_4) are enough to obtain the conclusion on the decay of $\mathrm{Leb}_D\{R > n\}$ of Theorem 3.5.

In the inductive process of construction of the sets A_n, B_n, $\{R = n\}$ and Δ_n we have fixed some large integer R_0, being this the first step at which the construction began. Recall that $A_n = \Delta_n = \Delta_0$ and $B_n = \{R = n\} = \emptyset$ for $n \le R_0$. For technical reasons we will assume that

$$R_0 > \max\left\{2(N+1), \frac{12}{\theta}\right\}. \tag{33}$$

Note that since N and θ do not depend on R_0 this is always possible.

This is the abstract setting under which we will be completing the proof of Theorem 3.5. From now on we will only make use of the metric relations (m_1)-(m_4) and will not be concerned with any other properties about these sets.

Lemma 3.15. — *There is $a_1 > 0$, with $a_1 \to 0$ when $\delta_0 \to 0$, such that for all $n \ge 1$*

$$\mathrm{Leb}_D(B_n) \le a_1 \,\mathrm{Leb}_D(A_n).$$

Proof. — We have by (m_3)

$$\mathrm{Leb}_D(A_n \cap A_{n-1}) \ge \eta \,\mathrm{Leb}_D(A_{n-1}), \tag{34}$$

where $\eta = 1 - b_0 - c_0$. Then we take

$$\widehat{a} = \frac{b_0 + c_0}{a_0} \quad \text{and} \quad a_1 = \frac{(1+a_0)b_0 + c_0}{a_0 \eta}. \tag{35}$$

The fact that $a_1 \to 0$ when $\delta_0 \to 0$ is a consequence of $b_0, c_0 \to 0$ when $\delta_0 \to 0$, and a_0 being bounded away from 0. Observe that $0 < \eta < 1$ and $\widehat{a} < a_1$.

Now the proof follows by induction. The result is obviously true for n up to R_0. Assume that it holds for $n-1 \ge R_0$. We consider separately the cases $\mathrm{Leb}_D(B_{n-1}) > \widehat{a}\,\mathrm{Leb}_D(A_{n-1})$ and $\mathrm{Leb}_D(B_{n-1}) \le \widehat{a}\,\mathrm{Leb}_D(A_{n-1})$.

Assume first that $\mathrm{Leb}_D(B_{n-1}) > \widehat{a}\,\mathrm{Leb}_D(A_{n-1})$. We may write

$$\mathrm{Leb}_D(B_{n-1}) = \mathrm{Leb}_D(B_{n-1} \cap A_n) + \mathrm{Leb}_D(B_{n-1} \cap B_n),$$

which by (m_2) gives

$$\mathrm{Leb}_D(B_{n-1} \cap B_n) \le (1 - a_0)\,\mathrm{Leb}_D(B_{n-1}). \tag{36}$$

Since $\mathrm{Leb}_D(B_n) = \mathrm{Leb}_D(B_n \cap B_{n-1}) + \mathrm{Leb}_D(B_n \cap A_{n-1})$, it follows from (36) and (m_3) that $\mathrm{Leb}_D(B_n) \le (1-a_0)\,\mathrm{Leb}_D(B_{n-1}) + b_0\,\mathrm{Leb}_D(A_{n-1})$. According to the case we are considering this leads to

$$\mathrm{Leb}_D(B_n) \le (1 - a_0)\,\mathrm{Leb}_D(B_{n-1}) + b_0\widehat{a}\;\mathrm{Leb}_D(B_{n-1}) \le \mathrm{Leb}_D(B_{n-1}). \tag{37}$$

On the other hand, we have $\mathrm{Leb}_D(A_n) = \mathrm{Leb}_D(A_n \cap A_{n-1}) + \mathrm{Leb}_D(A_n \cap B_{n-1})$, which together with ($m_2$) and (34) gives $m(A_n) \ge \eta m(A_{n-1}) + a_0 m(B_{n-1})$. Again by the case we are considering we have

$$\mathrm{Leb}_D(A_n) \ge \eta\,\mathrm{Leb}_D(A_{n-1}) + a_0\widehat{a}\;\mathrm{Leb}_D(A_{n-1}) \ge \mathrm{Leb}_D(A_{n-1}). \tag{38}$$

Inequalities (37) and (38), together with the inductive hypothesis, yield the result in this first case.

Assume now that $\mathrm{Leb}_D(B_{n-1}) \le \widehat{a}\,\mathrm{Leb}_D(A_{n-1})$. Since $\mathrm{Leb}_D(B_n) = \mathrm{Leb}_D(B_n \cap B_{n-1}) + \mathrm{Leb}_D(B_n \cap A_{n-1})$, it follows from ($\mathrm{m}_3$) that $\mathrm{Leb}_D(B_n) \le \mathrm{Leb}_D(B_{n-1}) + b_0\,\mathrm{Leb}_D(A_{n-1})$. On the other hand, (34) implies that $m(A_n) \ge \eta m(A_{n-1})$. Hence

$$\frac{\mathrm{Leb}_D(B_n)}{\mathrm{Leb}_D(A_n)} \le \frac{\mathrm{Leb}_D(B_{n-1}) + b_0\,\mathrm{Leb}_D(A_{n-1})}{\eta\,\mathrm{Leb}_D(A_{n-1})} \le \frac{\widehat{a} + b_0}{\eta} = a_1,$$

which also proves the result in this case. □

Corollary 3.16. — *There exists $c_2 > 0$ such that for every $n \ge 1$*

$$\mathrm{Leb}_D(\Delta_n) \le c_2\,\mathrm{Leb}_D(\Delta_{n+1}).$$

Proof. — Using (m_3) we obtain

$$\mathrm{Leb}_D(\Delta_{n+1}) \ge \mathrm{Leb}_D(A_{n+1}) \ge (1 - b_0 - c_0)\,\mathrm{Leb}_D(A_n).$$

On the other hand, by Lemma 3.15,

$$\mathrm{Leb}_D(\Delta_n) = \mathrm{Leb}_D(A_n) + \mathrm{Leb}_D(B_n) \le (1 + a_1^{-1})\,\mathrm{Leb}_D(A_n).$$

It is enough to take $c_2 = (1 + a_1^{-1})/(1 - b_0 - c_0)$. □

At this point we are able to definitely specify the choice of δ_0. First of all, let us recall that the number θ in (m_1) does not depend on δ_0. Assume that $\mathrm{Leb}_D(\Gamma_n) \le Cn^{-\gamma}$, for some $C, \gamma > 0$, and pick $\alpha > 0$ such that

$$\alpha < \left(\frac{\theta}{12}\right)^{\gamma+1}. \tag{39}$$

Then we choose $\delta_0 > 0$ small enough so that

$$a_1 < 2\alpha. \tag{40}$$

This is possible because $a_1 \to 0$ as $\delta_0 \to 0$ by Lemma 3.15.

Since $m(\Delta_n) = m(A_n) + m(B_n)$, we easily deduce from (m_4) and Lemma 3.15 that if we take

$$b_1 = \frac{c_1}{1 + a_1}, \tag{41}$$

then

$$\mathrm{Leb}_D\left(\cup_{i=0}^N \{R = n + i\}\right) \ge b_1 \frac{\mathrm{Leb}_D(A_{n-1} \cap H_n)}{\mathrm{Leb}_D(A_{n-1})}\,\mathrm{Leb}_D(\Delta_{n-1}).$$

This immediately implies that

$$\mathrm{Leb}_D\left(\Delta_{n+N}\right) \le \left(1 - b_1 \frac{\mathrm{Leb}_D(A_{n-1} \cap H_n)}{\mathrm{Leb}_D(A_{n-1})}\right) \mathrm{Leb}_D(\Delta_{n-1}). \tag{42}$$

At this point we obtained some recurrence relation for the Lebesgue measure of the sets Δ_n. Since $(\Delta_n)_n$ forms a decreasing sequence of sets we finally have

$$\mathrm{Leb}_D\left(\Delta_{n+N}\right) \le \exp\left(-\frac{b_1}{N+1} \sum_{j=R_0}^{n} \frac{\mathrm{Leb}_D(A_{j-1} \cap H_j)}{\mathrm{Leb}_D(A_{j-1})}\right) \mathrm{Leb}_D(\Delta_0). \tag{43}$$

We will complete the proof of Theorem 3.5 by considering several different cases, according to the behavior of the proportions $\mathrm{Leb}_D(A_{j-1} \cap H_j)/\mathrm{Leb}_D(A_{j-1})$. We define for each $n \geq 1$

$$E_n = \left\{ j \leq n \colon \frac{\mathrm{Leb}_D(A_{j-1} \cap H_j)}{\mathrm{Leb}_D(A_{j-1})} < \alpha \right\},$$

and

$$F = \left\{ n \in \mathbb{N} \colon \frac{\# E_n}{n} > 1 - \frac{\theta}{12} \right\}.$$

Proposition 3.17. — *Take any $n \in F$ with $n \geq R_0$. If $\mathrm{Leb}_D(A_n) \geq 2\,\mathrm{Leb}_D(\Gamma_n)$, then there is some $0 < k = k(n) < n$ for which $\mathrm{Leb}_D(A_n) < (k/n)^\gamma\,\mathrm{Leb}_D(A_k)$.*

Proof. — We have for $j \leq n$

$$\frac{\mathrm{Leb}_D(A_n \cap H_j)}{\mathrm{Leb}_D(A_n)} \geq \frac{\mathrm{Leb}_D(A_n \setminus \Gamma_n)}{\mathrm{Leb}_D(A_n)} \cdot \frac{\mathrm{Leb}_D((A_n \setminus \Gamma_n) \cap H_j)}{\mathrm{Leb}_D(A_n \setminus \Gamma_n)} \geq \frac{1}{2} \cdot \frac{\mathrm{Leb}_D((A_n \setminus \Gamma_n) \cap H_j)}{\mathrm{Leb}_D(A_n \setminus \Gamma_n)},$$

which together with the conclusion of Corollary 3.2 for the set $A_n \setminus \Gamma_n$ gives

$$\frac{1}{n} \sum_{j=1}^{n} \frac{\mathrm{Leb}_D(A_n \cap H_j)}{\mathrm{Leb}_D(A_n)} \geq \frac{\theta}{2}. \tag{44}$$

Let

$$G_n = \left\{ j \in E_n \colon \frac{\mathrm{Leb}_D(A_{j-1})}{\mathrm{Leb}_D(A_n)} > \frac{\theta}{12\alpha} \right\}.$$

Since $n \in F$, we have

$$\begin{aligned}
\frac{1}{n} \sum_{j=1}^{n} \frac{\mathrm{Leb}_D(A_n \cap H_j)}{\mathrm{Leb}_D(A_n)} &\leq \frac{\theta}{12} + \frac{1}{n} \sum_{j \in E_n} \frac{\mathrm{Leb}_D(A_n \cap H_j)}{\mathrm{Leb}_D(A_n)} \\
&\leq \frac{\theta}{12} + \frac{1}{n} \sum_{j \in E_n \setminus G_n} \frac{\mathrm{Leb}_D(A_n \cap H_j)}{\mathrm{Leb}_D(A_n)} + \frac{\# G_n}{n}.
\end{aligned}$$

Now, for $j \in E_n \setminus G_n$,

$$\begin{aligned}
\frac{\mathrm{Leb}_D(A_n \cap H_j)}{\mathrm{Leb}_D(A_n)} &= \frac{\mathrm{Leb}_D(A_n \cap H_j)}{\mathrm{Leb}_D(A_{j-1})} \cdot \frac{\mathrm{Leb}_D(A_{j-1})}{\mathrm{Leb}_D(A_n)} \\
&\leq \left(\frac{\mathrm{Leb}_D(A_n \cap A_{j-1} \cap H_j)}{\mathrm{Leb}_D(A_{j-1})} + \frac{\mathrm{Leb}_D((A_n \setminus A_{j-1}) \cap H_j)}{\mathrm{Leb}_D(A_{j-1})} \right) \frac{\mathrm{Leb}_D(A_{j-1})}{\mathrm{Leb}_D(A_n)} \\
&\leq \left(\frac{\mathrm{Leb}_D(A_{j-1} \cap H_j)}{\mathrm{Leb}_D(A_{j-1})} + a_0 \right) \frac{\theta}{12\alpha}.
\end{aligned}$$

For this last inequality we used the fact that $(A_n \setminus A_{j-1}) \subset B_{j-1}$ and $j \notin G_n$. Hence

$$\begin{aligned}
\frac{1}{n} \sum_{j=1}^{n} \frac{\mathrm{Leb}_D(A_n \cap H_j)}{\mathrm{Leb}_D(A_n)} &\leq \frac{\theta}{12} + \frac{1}{n} \sum_{j \in E_n \setminus G_n} \frac{\mathrm{Leb}_D(A_{j-1} \cap H_j)}{\mathrm{Leb}_D(A_{j-1})} \frac{\theta}{12\alpha} + a_0 \frac{\theta}{12\alpha} + \frac{\# G_n}{n} \\
&< \frac{\theta}{12} + \alpha \frac{\theta}{12\alpha} + a_0 \frac{\theta}{12\alpha} + \frac{\# G_n}{n}.
\end{aligned}$$

By the choice of a_0 we have that the third term in the last sum above is smaller than $\theta/6$. So, using (44) we obtain

$$\frac{\# G_n}{n} > \frac{\theta}{6}. \tag{45}$$

Now, defining

$$k = \max(G_n) - 1,$$

we have

$$\mathrm{Leb}_D(A_n) < \frac{12\alpha}{\theta}\,\mathrm{Leb}_D(A_k).$$

It follows from (45) that $k+1 > \theta n/6$, and so $k/n > \theta/12$, because $n \ge R_0 > 12/\theta$. Since we have chosen $\alpha < (\theta/12)^{\gamma+1}$, it follows that

$$\left(\frac{k}{n}\right)^{\gamma} > \frac{12}{\theta}\left(\frac{\theta}{12}\right)^{\gamma+1} > \frac{12\alpha}{\theta}.$$

This completes the proof of the result. □

Let us now complete the proof of Theorem 3.5. From Lemma 3.15 we get

$$\mathrm{Leb}_D(\Delta_n) \le (1+a_1)\,\mathrm{Leb}_D(A_n). \tag{46}$$

Hence, it is enough to derive the tail estimate of Theorem 3.5 for $\mathrm{Leb}_D(A_n)$ in the place of $\mathrm{Leb}_D\{R > n\} = \mathrm{Leb}_D(\Delta_n)$. Given any large integer n, we consider the following two cases:

1. If $n \in \mathbb{N}\setminus F$, then by (43) and Corollary 3.16 we have
$$\mathrm{Leb}_D(\Delta_n) \le c_2^N \exp\left(-\frac{b_1\theta\alpha}{12(N+1)}(n-R_0)\right)\mathrm{Leb}_D(\Delta_0).$$
2. If $n \in F$, then we distinguish the next two subcases:
   (a) If $\mathrm{Leb}_D(A_n) < 2\,\mathrm{Leb}_D(\Gamma_n)$, then nothing has to be done.
   (b) If $\mathrm{Leb}_D(A_n) \ge 2\,\mathrm{Leb}_D(\Gamma_n)$, then we apply Proposition 3.17 and get some $k_1 < n$ for which
   $$\mathrm{Leb}_D(A_n) < \left(\frac{k_1}{n}\right)^{\gamma}\mathrm{Leb}_D(A_{k_1}).$$

The only case we are left to consider is 2(b). In such case, either k_1 is in situation 1 or 2(a), or by Proposition 3.17 we can find $k_2 < k_1$ for which

$$\mathrm{Leb}_D(A_{k_1}) < \left(\frac{k_2}{k_1}\right)^{\gamma}\mathrm{Leb}_D(A_{k_2}).$$

Arguing inductively we are able to show that there is a sequence of integers $0 < k_s < \cdots < k_1 < n$ for which one of the following situations eventually holds:

(A) $\quad \mathrm{Leb}_D(A_n) < \left(\frac{k_s}{n}\right)^{\gamma} c_2^N \exp\left(-\frac{b_1\theta\alpha}{12(N+1)}(k_s-R_0)\right)\mathrm{Leb}_D(\Delta_0).$

(B) $\quad \mathrm{Leb}_D(A_n) < \left(\frac{k_s}{n}\right)^{\gamma}\mathrm{Leb}_D(\Gamma_{k_s}).$

(C) $\quad \mathrm{Leb}_D(A_n) < \left(\frac{R_0}{n}\right)^{\gamma}\mathrm{Leb}_D(\Delta_0).$

In all these three situations we arrive at the desired conclusion of Theorem 3.5. Situation (C) corresponds to falling in case 2(b) above successively until $k_s \le R_0$.

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