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SRB measures for partially hyperbolic systems whose central direction is mostly expanding*

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Abstract. We construct Sinai-Ruelle-Bowen (SRB) measures supported on partially hyperbolic sets of diffeomorphisms – the tangent bundle splits into two invariant subbundles, one of which is uniformly contracting – under the assumption that the complementary subbundle is *non-uniformly* expanding. If the rate of expansion (Lyapunov exponents) is bounded away from zero, then there are only finitely many SRB measures. Our techniques extend to other situations, including certain maps with singularities or critical points, as well as diffeomorphisms having only a dominated splitting (and no uniformly hyperbolic subbundle).

1. Introduction

The following approach has been most effective in studying the dynamics of complicated systems: one tries to describe the average time spent by typical orbits in different regions of the phase space. According to the ergodic theorem of Birkhoff, such times are well defined for almost all point, with respect to any invariant probability measure. However, the notion of typical orbit is usually meant in the sense of volume (Lebesgue measure), which is not always captured by invariant measures. Indeed, it is a fundamental open problem to understand under which conditions the behaviour of typical points is well defined, from this statistical point of view.

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This problem can be precisely formulated by means of the following notion, introduced by Sinai, Ruelle, and Bowen. Let us consider discrete time systems, namely maps $f : M \to M$ on a manifold M. Given an f-invariant Borelian probability μ in M, we call *basin* of μ the set $B(\mu)$ of the points $x \in M$ such that the averages of Dirac measures along the orbit of x converge to μ in the weak* sense:

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi \, d\mu \quad \text{for any continuous } \varphi : M \to \mathbb{R}. \quad (1)$$

Then we say that μ is a *physical* or *Sinai-Ruelle-Bowen (SRB) measure* for f if the basin $B(\mu)$ has positive Lebesgue measure in M. Then, one would like to know whether, for most systems, the basins of all SRB measures cover a full Lebesgue measure subset of the whole manifold.

This question has an affirmative answer in the context of uniformly hyperbolic systems, after [Sin72,BR75,Bow75,Rue76]. A detailed picture is also available for maps of the circle or the interval, see [Jak81,Lyu]. However, in higher dimensions the problem is mostly open, outside the uniformly hyperbolic setting, despite substantial progress in the study of certain classes of maps and flows with some properties of non-uniform hyperbolicity, including the Lorenz-like attractors, see e.g. [Pes92,Sat92,Tuc99], and the Hénon-like attractors, see [BC91,BY92,BV].

In this work we deal with diffeomorphisms admitting *partially hyperbolic* invariant sets: the tangent bundle over the set has an invariant dominated splitting into two subbundles, one of which is uniformly hyperbolic (contracting or expanding). Precise definitions are given in the next subsection. This property yields a fair amount of geometric information, e.g. invariant foliations, so that it is natural to try to recover for these systems as much as possible of the geometric and ergodic properties of the hyperbolic ones.

Indeed, some knowledge of such properties is already available, through works of several authors. For the foundations concerning invariant foliations see [BP74,HPS77]. Gibbs u-states were constructed by [PS82], and used by [You90,Car93,BV99] to construct SRB measures for some types of partially hyperbolic systems. SRB measures and decay of correlations were also studied by [Alv97,Cas98,Dol]. Moreover, [Shu71,Mañ78,GPS94,Kan94,BD96,Via97,SW] constructed examples where partial hyperbolicity is used to get robustness (stability) of topological or ergodic properties, which is also a main topic in [BP74]. Conversely, [Mañ82,DPU99,BDP], showed that partial hyperbolicity (or, in high dimensions, existence of a dominated splitting) is in fact a necessary condition for robust topological transitivity.

The interest in this class of systems was further stressed by these recent results, that suggest that partial hyperbolicity (or, at least, existence of a dominated splitting) should be a crucial ingredient in a global theory of Dynamics. A program towards such a theory has been proposed a few years ago by [Pal99], the cornerstone of which is a conjecture containing the following statement: every system can be approximated by one having finitely many SRB measures, whose basins cover a full Lebesgue measure subset of the phase space.

In the present paper we obtain results of existence and finiteness of SRB measures for a large class of partially hyperbolic maps. Previous constructions depended on the existence of a uniformly expanding (strong-unstable) invariant subbundle. In fact, except for the situation of [Via97,Alv97], the SRB measures coincide with the Gibbs u-states (invariant measures absolutely continuous in the strong-unstable direction) constructed by [PS82]. As shown in [BV99], this happens whenever the central subbundle is (at least) non-uniformly contracting.

One main novelty here is that we do not assume existence of a strong-unstable direction. Using only a condition of non-uniform expansion in an invariant (centre-unstable) subbundle, we are able to construct invariant measures absolutely continuous in this centre-unstable direction. Actually, this does not even require the full strength of partial hyperbolicity, it suffices to have a dominated splitting. In some cases, these *Gibbs cu-states* are SRB measures: in particular, this is always the case if the complementary subbundle is uniformly contracting on the support of the measure.

1.1. Partially hyperbolic diffeomorphisms

Let $f : M \to M$ be a C^1 diffeomorphism on a manifold M. Here we say that a compact set $K \subset M$ is *partially hyperbolic* for f if it is positively invariant, i.e. $f(K) \subset K$, and there exists a continuous Df-invariant splitting $T_K M = E^{ss} \oplus E^{cu}$ of the tangent bundle restricted to K and a constant $\lambda < 1$ satisfying (for some choice of a Riemannian metric on M)

1. E^{ss} is uniformly contracting: $\|Df \mid E^{ss}_x\| \leq \lambda$ for all $x \in K$;
2. E^{cu} is dominated by E^{ss}: $\|Df \mid E^{ss}_x\| \cdot \|Df^{-1} \mid E^{cu}_{f(x)}\| \leq \lambda$ for all $x \in K$.

We call E^{ss} strong-stable subbundle and E^{cu} centre-unstable subbundle.

Theorem A. *Let $f : M \to M$ be a C^2 diffeomorphism having a partially hyperbolic set K. Assume that f is non-uniformly expanding along the centre-unstable direction, meaning that*

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \log \left\| Df^{-1} \mid E^{cu}_{f^j(x)} \right\| < 0 \tag{2}$$

for all x in a positive Lebesgue measure set $H \subset K$. Then f has some ergodic SRB measure with support contained in $\cap_{j=0}^{\infty} f^j(K)$. In fact, Lebesgue almost every point in H belongs in the basin of some such SRB measure.

As we shall explain below, for the proof it is enough to have condition (2) on a positive Lebesgue measure subset of some disk transverse to the strong-stable direction.

The SRB measures produced by this theorem have dim E^{cu} positive Lyapunov exponents and dim E^{ss} negative Lyapunov exponents. Moreover, they have absolutely continuous conditional measures along corresponding Pesin unstable manifolds (which are tangent to the centre-unstable subbundle E^{cu} at almost every point).

We ignore whether the conclusion of Theorem A remains true if the non-uniform expansion condition is replaced by

$$\limsup_{n\to+\infty} \frac{1}{n} \log \left\| Df^{-n} \mid E^{cu}_{f^n(x)} \right\| < 0 \tag{3}$$

(i.e. positivity of all the Lyapunov exponents in the centre-unstable direction). Clearly, the two formulations (2) and (3) are equivalent when the non-uniformly expanding direction is 1-dimensional, that is, when E^{cu} can be split as $E^c \oplus E^{uu}$ with E^c having dimension 1 and E^{uu} being uniformly expanding. Let us also note that for a set of points with full probability (full measure with respect to any invariant probability measure) condition (3) implies condition (2) for some f^k, $k \geq 1$. Of course, the theorem is not affected when f is replaced by any positive iterate of it.

As a consequence of the proof of the theorem we also get

Corollary B. *Under the assumptions of Theorem A, if the limit in (2) is bounded away from zero, then the set H is contained in the union of the basins of finitely many SRB measures, up to a zero Lebesgue measure subset.*

These results have the following curious consequence: in the setting of Theorem A the set of values of the limit in (2) over a full Lebesgue measure subset of H is discrete, with zero as the unique possible accumulation value.

The assumption of partial hyperbolicity in Theorem A can be somewhat relaxed, as we explain in Sect. 6. Assuming only the existence of a dominated splitting $E^{cs} \oplus E^{cu}$ with E^{cu} non-uniformly expanding, we prove that the diffeomorphism admits invariant probability measures with absolutely continuous conditional measures along the centre-unstable direction. This may be thought of as a non-uniform version of the results of [PS82] on existence of Gibbs u-states. We also show that in some cases where E^{cs} is non-uniformly contracting these invariant measures are SRB measures for the corresponding system.

1.2. Maps with singular sets

Similar methods allow us to construct SRB measures for certain maps with singularities and/or critical points. Apart from a condition of non-uniform expansion on a positive Lebesgue measure subset H, similar to (2), we also need the points of H not to spend most of the time too close to the singular set. This is properly expressed in (6) below. Before that, let us explain what we mean by singular set of a map.

Let M be a compact manifold, $\mathcal{S} \subset M$ a compact subset, and $f : M \setminus \mathcal{S} \to M$ a C^2 map on $M \setminus \mathcal{S}$. We assume that f behaves like a power of the distance to $\mathcal{S}$ close to the *singular set* $\mathcal{S}$, in the following sense: there exist constants $B > 1$ and $\beta > 0$ such that

(S1) $\displaystyle \frac{1}{B}\operatorname{dist}(x, \mathcal{S})^{\beta} \le \frac{\|Df(x)v\|}{\|v\|} \le B\operatorname{dist}(x, \mathcal{S})^{-\beta};$

(S2) $\displaystyle |\log\|Df(x)^{-1}\| - \log\|Df(y)^{-1}\|| \le B\frac{\operatorname{dist}(x, y)}{\operatorname{dist}(x, \mathcal{S})^{\beta}};$

(S3) $\displaystyle |\log|\det Df(x)| - \log|\det Df(y)|| \le B\frac{\operatorname{dist}(x, y)}{\operatorname{dist}(x, \mathcal{S})^{\beta}};$

for every $v \in T_xM$ and $x, y \in M \setminus \mathcal{S}$ with $\operatorname{dist}(x, y) < \operatorname{dist}(x, \mathcal{S})/2$.

Given $\delta > 0$ and $x \in M \setminus \mathcal{S}$ we define the δ-*truncated distance* from x to $\mathcal{S}$

$$\operatorname{dist}_{\delta}(x, \mathcal{S}) = \begin{cases} 1 & \text{if } \operatorname{dist}(x, \mathcal{S}) \ge \delta \\ \operatorname{dist}(x, \mathcal{S}) & \text{otherwise.} \end{cases} \tag{4}$$

Note that this is not really a distance function: $\operatorname{dist}(x, y) + \operatorname{dist}_{\delta}(y, \mathcal{S})$ may be smaller than $\operatorname{dist}_{\delta}(x, \mathcal{S})$. We also denote $\mathcal{S}_{\infty} = \cup_{n=0}^{\infty} f^n(\mathcal{S})$.

Theorem C. *Assume that f satisfies (S1), (S2), (S3) and is non-uniformly expanding, in the sense that*

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log\|Df(f^j(x))^{-1}\| < 0 \tag{5}$$

for all x in a positive Lebesgue measure set $H \subset M \setminus \mathcal{S}_{\infty}$. Assume moreover that, given any $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in H$

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} -\log\operatorname{dist}_{\delta}(f^j(x), \mathcal{S}) \le \varepsilon. \tag{6}$$

Then Lebesgue almost every point in H belongs in the basin of some ergodic absolutely continuous invariant measure.

As a by-product, corresponding to the case when the singular set is empty,

Corollary D. *Let $f : M \to M$ be a C^2 covering map (local diffeomorphism) on a compact manifold M, which is non-uniformly expanding: (5) holds for all x in a set $H \subset M$ with positive Lebesgue measure. Then Lebesgue almost every point $x \in H$ belongs in the basin of some ergodic absolutely continuous invariant measure.*

In the settings of these last two results, we also have an analogue of Corollary B: if the limit in (5) is bounded away from zero then H is covered by the basins of finitely many ergodic absolutely continuous invariant measures. See also the comments at the end of Sect. 5 concerning similar results for partially hyperbolic maps with singularities.

For unimodal maps of the interval with negative Schwarzian derivative, results in a similar spirit had been obtained by [Led81] and [Kel90]. In particular, [Kel90] contains a strong version of Corollary D for such maps.

This paper is organized as follows. To construct the SRB measures in Theorem A we fix any C^2 disk transverse to the strong-stable direction and intersecting H on a positive Lebesgue measure subset, and we consider the sequence of averages of forward iterates of Lebesgue measure restricted to such a disk. We prove, in Sect. 3, that a definite fraction of each average corresponds to a measure that is absolutely continuous with respect to Lebesgue measure along the iterates of the disk, with uniformly bounded densities. This uses distortion bounds that we obtain in Sect. 2, together with the key notion of *hyperbolic times*, first introduced in [Alv97]. The construction of the SRB measures is completed in Sect. 4, where we show that the absolute continuity property passes to the limit. In Sect. 4 we also obtain enough information about the basins of these measures to prove the finiteness statement in Corollary B.

In Sect. 5 we explain how condition (6) allows us to bypass the difficulty caused by the presence of singularities, and obtain Theorem C and Corollary D, as well as the results on partially hyperbolic maps with singularities that we mentioned before. Sect. 6 contains related results for systems with a dominated splitting $E^{cs} \oplus E^{cu}$ where neither of the factors needs have any uniform subbundle, that we also mentioned above. In the Appendix we give a few simple criteria allowing to check the assumptions of our results in specific situations.

2. Curvature and distortion

A first main step in the proof of Theorem A is to prove a bounded distortion property for iterates of f over disks whose tangent space is contained in a centre-unstable cone at each point. This section is devoted to the precise statement and proof of this property.

Remark 2.1. Throughout this section, as well as in Sect. 3 and Subsect. 4.1, we do not need the full strength of partial hyperbolicity. In Subsect. 2.1 we only use existence of a dominated splitting $E^{cs} \oplus E^{cu}$. Subsections 2.2 and 2.3, Sect. 3, and Subsect. 4.1 depend also on the condition of non-uniform expansion along the centre-unstable direction (2). Existence of a strong-stable subbundle E^{ss} is used for the first time in Subsect. 4.2.

We fix continuous extensions of the two subbundles E^{cs} and E^{cu} to some neighbourhood V_0 of K, that we denote $\tilde{E}^{cs}$ and $\tilde{E}^{cu}$. It should be noted that we do not require these extensions to be invariant under Df. Then, given $0 < a < 1$, we define the *centre-unstable cone field* $C_a^{cu} = \big(C_a^{cu}(x)\big)_{x\in V_0}$ *of width* a by

$$C_a^{cu}(x) = \big\{v_1 + v_2 \in \tilde{E}_x^{cs} \oplus \tilde{E}_x^{cu} \text{ such that } \|v_1\| \le a\|v_2\|\big\}. \tag{7}$$

We define the *centre-stable cone field* $C_a^{cs} = \big(C_a^{cs}(x)\big)_{x\in V_0}$ *of width* a in a similar way, just reversing the roles of the subbundles in (7).

We fix $a > 0$ and V_0 small enough so that, up to slightly increasing $\lambda < 1$, the domination condition 2. in Subsect. 1.1 remains valid for any pair of vectors in the two cone fields:

$$\|Df(x)v^{cs}\| \cdot \|Df^{-1}(f(x))v^{cu}\| \le \lambda\|v^{cs}\|\,\|v^{cu}\|$$

for every $v^{cs} \in C_a^{cs}(x)$, $v^{cu} \in C_a^{cu}(f(x))$, and any point $x \in V_0 \cap f^{-1}(V_0)$. Note that the centre-unstable cone field is positively invariant: $Df(x)C_a^{cu}(x) \subset C_a^{cu}(f(x))$ whenever x and $f(x)$ are in V_0. Indeed, the domination property together with the invariance of $E^{cu} = (\tilde{E}^{cu} \mid K)$ imply

$$Df(x)C_a^{cu}(x) \subset C_{\lambda a}^{cu}(f(x)) \subset C_a^{cu}(f(x)),$$

for every $x \in K$, and this extends to any $x \in V_0 \cap f^{-1}(V_0)$ by continuity.

Wherever we presume E^{cs} to be uniformly contracting (as already mentioned, this will happen not happen before Subsect. 4.2), we denote it by E^{ss} instead, and represent by $\tilde{E}^{ss}$ its extension to V_0. Moreover, in that case the cone field C_a^{cs} is denoted C_a^{ss}, and called *strong-stable*.

2.1. Hölder control of the tangent direction

We say that an embedded C^1 submanifold $N \subset V_0$ is *tangent to the centre-unstable cone field* C_a^{cu} if the tangent subspace to N at each point $x \in N$ is contained in the corresponding cone $C_a^{cu}(x)$. Then $f(N)$ is also tangent to the centre-unstable cone field, if it is contained in V_0.

The tangent bundle of N is said to be Hölder continuous if $x \mapsto T_xN$ defines a Hölder continuous section from N to the corresponding Grassman bundle of M. In this subsection we show that the tangent bundle of the iterates of a C^2 submanifold are Hölder continuous (as long as they do not leave V_0), with uniform Hölder constants.

The basic idea is contained in the following observation, which the reader may easily check. Let E_1, E_2 be two Euclidean spaces and L be a linear isomorphism on $E_1 \oplus E_2$ leaving both factors invariant. Assume that we have the domination property

$$\|L \mid E_1\| \cdot \|L^{-1} \mid E_2\| < 1.$$

Then there exist $C > 0$ and $0 < \zeta \le 1$ such that if $\Gamma \subset E_1 \oplus E_2$ is the graph of a $C^{1+\zeta}$ map $\phi : E_1 \to E_2$, with Hölder constant C, then the same is true for $L(\Gamma)$. In fact, it suffices to take any ζ such that $\|L \mid E_1\| \cdot \|L^{-1} \mid E_2\|^{1+\zeta} < 1$.

In order to apply similar arguments in our situation, it is useful to express the notion of Hölder variation of the tangent bundle in local coordinates, as follows.

We choose $\delta_0 > 0$ small enough so that the inverse of the exponential map $\exp_x$ is defined on the δ_0 neighbourhood of every point x in V_0. From now on we identify this neighbourhood of x with the corresponding neighbourhood U_x of the origin in $T_x N$, through the local chart defined by $\exp_x^{-1}$. Accordingly, we identify x with $0 \in T_x N$. Reducing δ_0, if necessary, we may suppose that $\tilde{E}_x^{cs}$ is contained in the centre-stable cone $C_a^{cs}(y)$ of every $y \in U_x$. In particular, the intersection of $C_a^{cu}(y)$ with $\tilde{E}_x^{cs}$ reduces to the zero vector. Then, the tangent space to N at y is parallel to the graph of a unique linear map $A_x(y) : T_x N \to \tilde{E}_x^{cs}$. Given constants $C > 0$ and $0 < \zeta \le 1$, we say that *the tangent bundle to N is (C, ζ)-Hölder* if

$$\|A_x(y)\| \le C d_x(y)^\zeta \quad \text{for every } y \in N \cap U_x \text{ and } x \in V_0 . \tag{8}$$

Here, $d_x(y)$ denotes the distance from x to y along $N \cap U_x$, defined as the length of the shortest curve connecting x to y inside $N \cap U_x$.

Recall that we have chosen the neighbourhood V_0 and the cone width a sufficiently small so that the domination property remains valid for vectors in the cones $C_a^{cs}(z)$, $C_a^{cu}(z)$, and for any point z in V_0. Then, there exist $\lambda_1 \in (\lambda, 1)$ and $\zeta \in (0, 1]$ such that

$$\|Df(z)v^{cs}\| \cdot \|Df^{-1}(f(z))v^{cu}\|^{1+\zeta} \le \lambda_1 < 1 \tag{9}$$

for every norm 1 vectors $v^{cs} \in C_a^{cs}(z)$ and $v^{cu} \in C_a^{cu}(z)$, at any $z \in V_0$. Then, up to reducing $\delta_0 > 0$ and slightly increasing $\lambda_1 < 1$, (9) remains true if we replace z by any $y \in U_x$, $x \in V_0$ (taking $\|\cdot\|$ to mean the Riemannian metric in the corresponding local chart).

We fix ζ and λ_1 as above in all that follows. Then, given a C^1 submanifold $N \subset V_0$, we denote

$$\kappa(N) = \inf\{C > 0 : \text{the tangent bundle of } N \text{ is } (C, \zeta)\text{-Hölder}\}. \tag{10}$$

Proposition 2.2. *There exist $\lambda_0 < 1$ and $C_0 > 0$ so that if $N \subset V_0 \cap f^{-1}(V_0)$ is any C^1 submanifold tangent to the centre-unstable cone field then*

$$\kappa(f(N)) \le \lambda_0 \, \kappa(N) + C_0 .$$

Proof. Of course, we only need to consider the case when $\kappa(N)$ is finite, that is, the tangent bundle of N is (C, ζ)-Hölder for some $C > 0$. Let $x \in N$ be fixed. We use $(u, s) \in T_x N \oplus \tilde{E}_x^{cs}$ and $(u_1, s_1) \in T_{f(x)} f(N) \oplus \tilde{E}_{f(x)}^{cs}$, respectively, to represent the local coordinates in U_x and $U_{f(x)}$ introduced above. We write the expression of our map in these local coordinates as

$f(u, s) = (u_1(u, s), s_1(u, s))$. Observe that if $x \in K$ then the partial derivatives of u_1 and s_1 at the origin $0 \in T_x N$ are

$$\partial_u u_1(0) = Df|T_x N, \quad \partial_s u_1(0) = 0, \quad \partial_u s_1(0) = 0, \quad \partial_s s_1(0) = Df|\tilde{E}^{cs}_x .$$

This is because $E^{cs}_x = \tilde{E}^{cs}_x$ is mapped to $E^{cs}_{f(x)} = \tilde{E}^{cs}_{f(x)}$ under $Df(x)$ and, similarly, $T_x N$ is mapped to $T_{f(x)} N$. Then, given any small $\varepsilon_0 > 0$ we have that

$$\|\partial_u u_1(y) - Df|T_x N\|, \ \|\partial_s u_1(y)\|, \ \|\partial_u s_1(y)\|, \ \|\partial_s s_1(y) - Df|\tilde{E}^{cs}_x\|, \tag{11}$$

are all less than ε_0 for every $x \in V_0$ and $y \in U_x$, as long as δ_0 and V_0 are small. Taking the cone width a also small, we get

$$\begin{aligned} \left\| Df|T_y N - Df|\tilde{E}^{cu}_x \right\| \le \varepsilon_0 \quad \text{and} \\ \left\| Df^{-1}|T_{f(y)} f(N) - Df^{-1}|\tilde{E}^{cu}_{f(x)} \right\| \le \varepsilon_0 , \end{aligned} \tag{12}$$

for every $x \in V_0$ and $y \in U_x$. Since f is C^2, there is also some constant $K_2 > 0$ such that

$$\|\partial_s u_1(y)\| \le K_2 d_x(y) \quad \text{and} \quad \|\partial_u s_1(y)\| \le K_2 d_x(y). \tag{13}$$

For y_1 in $U_{f(x)}$, let $A_{f(x)}(y_1)$ be the linear map from $T_{f(x)} f(N)$ to $\tilde{E}^{cs}_{f(x)}$ whose graph is parallel to $T_{y_1} f(N)$. We are going to prove that, fixing ε_0 sufficiently small, then $A_{f(x)}(y_1)$ satisfies (8) for any $C > \lambda_0 \kappa(N) + C_0$, with convenient λ_0 and C_0. Let us begin by noting that $\|A_{f(x)}(y_1)\|$ is bounded by some uniform constant $K_1 > 0$, since $f(N)$ is tangent to the centre-unstable cone field. We will choose the constant $C_0 \ge K_1/(\delta_0/\|Df^{-1}\|)^\zeta$, so that (8) is immediate when $d_{f(x)}(y_1) \ge \delta_0/\|Df^{-1}\|$:

$$\|A_{f(x)}(y_1)\| \le K_1 \le C_0(\delta_0/\|Df^{-1}\|)^\zeta \le C_0 d_{f(x)}(y_1)^\zeta .$$

Here $\|Df^{-1}\|$ is the supremum of all $\|Df^{-1}(z)\|$ with $z \in U_w$, $w \in V_0$, where the norms are taken with respect to the Riemannian metrics in the local charts. This permits us to restrict to the case when $d_{f(x)}(y_1) < \delta_0/\|Df^{-1}\|$ in all that follows. Let Γ_1 be any curve on $f(N) \cap U_{f(x)}$ joining $f(x)$ to y_1 and whose length approximates $d_{f(x)}(y_1)$. Then $\Gamma = f^{-1}(\Gamma_1)$ is a curve in $N \cap U_x$ joining x to $y = f^{-1}(y_1)$, with length less than δ_0. In fact, cf. (12),

$$d_x(y) \le \text{length}(\Gamma) \le \left(\|Df^{-1}|\tilde{E}^{cu}_{f(x)}\| + \varepsilon_0\right) \text{length}(\Gamma_1).$$

This shows that $d_x(y) \le (\|Df^{-1}|\tilde{E}^{cu}_{f(x)}\| + \varepsilon_0) d_{f(x)}(y_1)$.

Now we observe that

$$A_{f(x)}(y_1) = \left[\partial_u s_1(y) + \partial_s s_1(y) \cdot A_x(y)\right] \cdot \left[\partial_u u_1(y) + \partial_s u_1(y) \cdot A_x(y)\right]^{-1}.$$

On the one hand, by (11) and (13),

$$\begin{aligned}\|\partial_u s_1(y) + \partial_s s_1(y) \cdot A_x(y)\| &\le K_2 d_x(y) + \big(\|Df|\tilde{E}^{cs}_x\| + \varepsilon_0\big)\kappa(N) d_x(y)^\zeta \\ &\le \big(K_2 + \big(\|Df|\tilde{E}^{cs}_x\| + \varepsilon_0\big)\kappa(N)\big) d_x(y)^\zeta.\end{aligned}$$

On the other hand, $\|\partial_s u_1(y) \cdot A_x(y)\| \le \epsilon_0 K_1$, which can be made much smaller than $1/\|(\partial_u u_1(y)^{-1}\|$. As a consequence, recall (12) and (13),

$$\left\| \left[\partial_u u_1(y) + \partial_s s_1(y) \cdot A_x(y)\right]^{-1} \right\| \le \left\| Df^{-1} \middle| \tilde{E}^{cu}_{f(x)} \right\| + \varepsilon_1 ,$$

where ε_1 can be made arbitrarily small by reducing ε_0. Putting these bounds together, we conclude that $\|A_{f(x)}(y_1)\|\, d_{f(x)}(y_1)^{-\zeta}$ is less than

$$\frac{\big(\|Df|\tilde{E}^{cs}_x\| + \varepsilon_0\big)\big(\|Df^{-1}|\tilde{E}^{cu}_{f(x)}\| + \varepsilon_1\big)}{\big(\|Df^{-1}|\tilde{E}^{cu}_{f(x)}\| + \varepsilon_0\big)^{-\zeta}}\kappa(N) + \frac{K_2\big(\|Df^{-1}|\tilde{E}^{cu}_{f(x)}\| + \varepsilon_1\big)}{\big(\|Df^{-1}|\tilde{E}^{cu}_{f(x)}\| + \varepsilon_0\big)^{-\zeta}}.$$

So, choosing δ_0, V_0, a sufficiently small, we can make ε_0, ε_1, sufficiently close to zero so that the factor multiplying $\kappa(N)$ is less than some $\lambda_0 \in (\lambda_1, 1)$; recall (9). Moreover, the second term in the expression above is bounded by some constant that depends only on f. We take C_0 larger than this constant. □

Remark 2.3. The proof remains valid if the diffeomorphism f is only of class $C^{1+\zeta}$: it suffices to replace (13) by $\|\partial_s u_1(y)\|, \|\partial_u s_1(y)\| \le K_2 d_x(y)^\zeta$, which is sufficient for the rest of the argument.

Corollary 2.4. *There exists $C_1 > 0$ such that, given any C^1 submanifold $N \subset V_0$ tangent to the centre-unstable cone field,*

a) there exists $n_0 \ge 1$ such that $\kappa(f^n(N)) \le C_1$ for every $n \ge n_0$ such that $f^k(N) \subset V_0$ for all $0 \le k \le n$;
b) if $\kappa(N) \le C_1$, then the same is true for every iterate $f^n(N)$, $n \ge 1$, such that $f^k(N) \subset V_0$ for all $0 \le k \le n$;
c) in particular, if N and n are as in b), then the functions

$$J_k : f^k(N) \ni x \mapsto \log\left|\det\left(Df \mid T_x f^k(N)\right)\right|, \quad 0 \le k \le n,$$

are (L_1, ζ)-Hölder continuous with $L_1 > 0$ depending only on C_1 and f.

Proof. It suffices to choose any $C_1 \ge C_0/(1 - \lambda_0)$. □

Remark 2.5. Suppose that we have the following stronger form of domination

$$\|Df|E^{cs}_x\| \cdot \|Df^{-1}|E^{cu}_{f(x)}\|^i \le \lambda \quad \text{for } i = 1, 2,$$

and any $x \in K$. Then, assuming that f is a C^2 diffeomorphism, we may take $\zeta = 1$ in the previous arguments. In that case, $\kappa(N)$ yields a bound on the curvature tensor of N. So, Corollary 2.4 asserts that if N is C^2 then the curvature of all its iterates $f^n(N)$, $n \ge 1$, is bounded by some constant that depends only on the curvature of N.

2.2. Hyperbolic times and distortion bounds

The following notion will allow us to derive *uniform behaviour* (expansion, distortion) from the hypothesis of non-uniform expansion in (2).

Definition 2.6. *Given $\sigma < 1$, we say that n is a σ-*hyperbolic time *for a point $x \in K$ if*

$$\prod_{j=n-k+1}^{n} \left\| Df^{-1} \mid E^{cu}_{f^j(x)} \right\| \le \sigma^k \qquad \textit{for all } 1 \le k \le n.$$

In particular, if n is a σ-hyperbolic time for x then $Df^{-k} \mid E^{cu}_{f^n(x)}$ is a contraction for every $1 \le k \le n$:

$$\left\| Df^{-k} \middle| E^{cu}_{f^n(x)} \right\| \le \prod_{j=n-k+1}^{n} \left\| Df^{-1} \middle| E^{cu}_{f^j(x)} \right\| \le \sigma^k.$$

Moreover, if a is taken sufficiently small in the definition of our cone fields, and we choose $\delta_1 > 0$ also small (in particular, the δ_1-neighbourhood of K should be contained in V_0), then, by continuity,

$$\|Df^{-1}(f(y))v\| \le \frac{1}{\sqrt{\sigma}} \|Df^{-1}|E^{cu}_{f(x)}\| \, \|v\| \tag{14}$$

whenever $x \in K$, $\mathrm{dist}(x, y) \le \delta_1$, and $v \in C^{cu}(y)$.

Let D be any C^1 disk contained in V_0 and tangent to the centre-unstable cone field. We use $\mathrm{dist}_D(\cdot\,,\cdot)$ to denote distance between two points in the disk, measured along D. The distance from a point $x \in D$ to the boundary of D is $\mathrm{dist}_D(x, \partial D) = \inf_{y \in \partial D} \mathrm{dist}_D(x, y)$.

Lemma 2.7. *Given any C^1 disk $D \subset V_0$ tangent to the centre-unstable cone field, $x \in D \cap K$, and $n \ge 1$ a σ-hyperbolic time for x,*

$$\mathrm{dist}_{f^{n-k}(D)}(f^{n-k}(y), f^{n-k}(x)) \le \sigma^{k/2} \,\mathrm{dist}_{f^n(D)}(f^n(y), f^n(x)).$$

for any point $y \in D$ with $\mathrm{dist}(f^n(x), f^n(y)) \le \delta_1$.

Proof. Let η_0 be a curve of minimal length in $f^n(D)$ connecting $f^n(x)$ to $f^n(y)$. For $1 \le k \le n$ write $\eta_k = f^{n-k}(\eta_0)$. We prove the lemma by induction. Let $1 \le k \le n$ and assume that

$$\mathrm{length}(\eta_j) \le \delta_1 \quad \text{for } 0 \le j \le k-1.$$

Denote by $\dot\eta_0(z)$ the tangent vector to the curve η_0 at the point z. Then, in view of the choice of δ_1 in (14) and the definition of σ-hyperbolic times,

$$\|Df^{-k}(z)\dot\eta_0(z)\| \le \sigma^{-k/2} \,\|\dot\eta_0(z)\| \prod_{j=n-k+1}^{n} \|Df^{-1}|E^{cu}_{f^j(x)}\| \le \sigma^{k/2}\|\dot\eta_0(z)\|.$$

As a consequence,

$$\operatorname{length}(\eta_k) \leq \sigma^{k/2} \operatorname{length}(\eta_0) = \sigma^{k/2} \operatorname{dist}_{f^{n-k}(D)}(f^{n-k}(y), f^{n-k}(x)) \leq \delta_1 .$$

This completes our induction, thus proving the lemma. □

Proposition 2.8. *There exists $C_2 > 1$ such that, given any C^1 disk D tangent to the centre-unstable cone field with $\kappa(D) \leq C_1$, and given any $x \in D \cap K$ and $n \geq 1$ a σ-hyperbolic time for x, then*

$$\frac{1}{C_2} \leq \frac{|\det Df^n \mid T_y D|}{|\det Df^n \mid T_x D|} \leq C_2$$

for every $y \in D$ such that $\operatorname{dist}(f^n(y), f^n(x)) \leq \delta_1$.

Proof. For $0 \leq i < n$ and $y \in D$, we denote $J_i(y) = |\det Df \mid T_{f^i(y)} f^i(D)|$. Then,

$$\log \frac{|\det Df^n \mid T_y D|}{|\det Df^n \mid T_x D|} = \sum_{i=0}^{n-1} \big(\log J_i(y) - \log J_i(x) \big)$$

By Corollary 2.4, $\log J_i$ is (L_1, ζ)-Hölder continuous, for some uniform $L_1 > 0$. Moreover, by Lemma 2.7, the sum of all $\operatorname{dist}_D(f^j(x), f^j(y))^\zeta$ over $0 \leq j \leq n$ is bounded by $\delta_1/(1 - \sigma^{\zeta/2})$. Now it suffices to take $C_2 = \exp(L_1\delta_1/(1 - \sigma^{\zeta/2}))$. □

2.3. *Curvature at hyperbolic times*

It is possible to obtain control of the curvature at hyperbolic times, without having to assume the stronger form of domination in Remark 2.5. As before, we assume that f is a C^2 diffeomorphism with a partially hyperbolic set K. Let $\sigma < 1$ be fixed, and $\delta_1 > 0$ be chosen as in (14).

Proposition 2.9. *Let D be a C^2 disk tangent to the centre-unstable cone field, $x \in D \cap K$, and $n \geq 1$ be a σ-hyperbolic time for x. Then, the curvature of the δ_1-neighbourhood of $f^n(x)$ in $f^n(D)$ is bounded by a constant $K_0 > 0$ that depends only on f, σ, and the curvature of D. In fact, if n is sufficiently large then K_0 may be taken depending only on f and σ.*

The main idea in the proof of this proposition is to show that, up to conformal changes of the Riemannian metric, we may suppose that $Df \mid E^{cu}$ is uniformly expanding at every point $f^j(x)$, $0 \leq j < n$. As a consequence, the domination condition 2. in Subsect. 1.1 implies the condition in Remark 2.5 (with respect to the modified metrics). In doing this, it is important that all metric changes can be done by dilation, which is due to the hyperbolic time condition.

Lemma 2.10. *Let $n \geq 1$ and $a_1, \ldots, a_n, c_0$, be real numbers such that*

$$\sum_{j=k+1}^{n} a_j \geq (n-k)c_0 \quad \textit{for all } 0 \leq k < n. \tag{15}$$

Then, there exist $b_1, \ldots, b_n$ such that

1. $|b_j| \leq \sup_{1 \leq j \leq n} |c_0 - a_j|$ *for all* $1 \leq j \leq n$;
2. $a_j + b_j \geq c_0$ *for all* $1 \leq j \leq n$;
3. $\sum_{j=1}^{k} b_j \geq 0$ *for* $1 \leq k < n$ *and* $\sum_{j=1}^{n} b_j = 0$.

Proof. Define b_j by recurrence, through

$$b_1 = \max\{0, c_0 - a_1\} \ \textit{ and } \ b_j = \max\{-\sum_{i=1}^{j-1} b_j, c_0 - a_j\} \qquad \text{for } j = 2, \ldots, n.$$

The first condition in the statement is clear, in the case when $b_j = c_0 - a_j$. Otherwise, $b_j = -\sum_{i=1}^{j-1} b_i$ which, by construction, is always non-positive. So, in this second case we must have $0 \geq b_j \geq c_0 - a_j$, so that the bound in 1. remains valid. The second condition follows immediately from the construction, and the same is true for the first statement in 3. To obtain the last claim, we begin by proving the following fact, by induction on j:

$$\sum_{i=1}^{j} b_i \leq \sum_{i=j+1}^{n} (a_i - c_0) \quad \text{for } j = 1, \ldots, n-1. \tag{16}$$

In view of (15),

$$\sum_{i=2}^{n}(a_i - c_0) \geq 0 \quad \text{and} \quad \sum_{i=2}^{n}(a_i - c_0) \geq \sum_{i=1}^{n}(a_i - c_0) + (c_0 - a_1) \geq (c_0 - a_1).$$

This gives $\sum_{i=2}^{n}(a_i - c_0) \geq \max\{0, c_0 - a_1\} = b_1$, corresponding to case $j = 1$. Now we suppose that, by recurrence, $\sum_{i=1}^{j-1} b_i \leq \sum_{i=j}^{n}(a_i - c_0)$. Then, either $b_j = c_0 - a_j$ in which case, adding b_j to both sides of the previous inequality immediately gives the conclusion. Or else, $b_j = -\sum_{i=1}^{j-1} b_i$ and then

$$\sum_{i=1}^{j} b_i = 0 \leq \sum_{i=j+1}^{n} (a_i - c_0),$$

due to our assumption (15). This completes the proof of (16). In particular, taking $j = n-1$, we get that $-\sum_{i=1}^{n-1} b_i \geq c_0 - a_n$, and so $b_n = -\sum_{j=1}^{n-1} b_i$. $\square$

Now we use this lemma to prove Proposition 2.9.

Proof. Let D_n be the neighbourhood of radius δ_1 around $f^n(x)$ in $f^n(D)$, and $D_j = f^{j-n}(D_n)$ for $0 \le j < n$. Take $c_0 = -\log\sigma$ and $a_j = -\log\|Df^{-1}|E^{cu}_{f^j(x)}\|$, $1 \le j \le n$, in Lemma 2.10. Let b_j, $1 \le j \le n$, be the corresponding sequence, and denote $t_j = \exp(\sum_{i=1}^{j} b_i)$ for $1 \le j \le n$, and $t_0 = 1$. Conclusion 3. in the lemma implies that

$$t_j \ge 1 \text{ for every } j \quad \text{and} \quad t_n = 1.$$

In all that follows $\|\cdot\|_j$ denotes the metric obtained by multiplying the initial Riemannian metric of M by t_j, $0 \le j \le n$. Accordingly, we denote $\|Df\|_{j-1,j}$ and $\|D^2 f\|_{j-1,j}$ the norms of the derivatives of f from $(M, \|\cdot\|_{j-1})$ to $(M, \|\cdot\|_j)$. We use similar notations for the restrictions of Df, Df^{-1} to the subbundles $\tilde{E}^{cu}$ and $\tilde{E}^{cs}$. Observe that

$$\|Df\|_{j-1,j} = \exp b_j \|Df\| \quad \text{and} \quad \|D^2 f\|_{j-1,j} = \frac{\exp b_j}{t_{j-1}} \|D^2 f\|.$$

Since the b_j are bounded, and the t_j are larger than 1, $\|Df\|_{j-1,j}$ and $\|D^2 f\|_{j-1,j}$ are bounded by constants that depend only on f and σ. Note also that the domination property is not affected by this conformal change of metrics:

$$\left\|Df\middle|\tilde{E}^{cs}_y\right\|_{j-1,j} \cdot \left\|Df^{-1}\middle|\tilde{E}^{cu}_{f(y)}\right\|_{j,j-1} = \left\|Df\middle|\tilde{E}^{cs}_y\right\| \cdot \left\|Df^{-1}\middle|\tilde{E}^{cu}_{f(y)}\right\| \le \lambda, \tag{17}$$

at every point y where these subbundles are defined.

Conclusion 2. in the lemma now means that

$$\left\|Df^{-1}\middle|E^{cu}_{f^j(x)}\right\|_{j,j-1} = \exp(-b_j - a_j) \le \sigma.$$

So, by (14) and Lemma 2.7, $\|Df^{-1}|\tilde{E}^{cu}_{f^j(y)}\|_{j,j-1} \le \sqrt{\sigma} < 1$ for every y in D_0 and $1 \le j \le n$. Together with (17), this gives

$$\left\|Df\middle|\tilde{E}^{cs}_{f^{j-1}(y)}\right\|_{j-1,j} \cdot \left\|Df^{-1}\middle|\tilde{E}^{cu}_{f^j(y)}\right\|^2_{j,j-1} \le \lambda\sqrt{\sigma} \le \lambda$$

for every $y \in D_0$ and $1 \le j \le n$. This means that a strong domination property as in Remark 2.5 is valid, with respect to the relevant modified metrics, at every point of $D_0 \cup f(D_0) \cup \cdots \cup f^{n-1}(D_0)$. Since we already checked that the first and second derivatives have uniformly bounded norms relative to these modified metrics, the arguments in Proposition 2.2 carry on completely to the present context to prove Proposition 2.9. □

Closing this section we observe that these arguments could also be used to give an alternative proof of the distortion bounds we obtained in the previous section.

3. Lebesgue measure at hyperbolic times

The following lemma, due to Pliss [Pli72], will permit us to prove that a point x satisfying assumption (2) has many (positive density at infinity) hyperbolic times.

Lemma 3.1. *Given $A \geq c_2 > c_1 > 0$, let $\theta_0 = (c_2 - c_1)/(A - c_1)$. Then, given any real numbers $a_1, \dots, a_N$ such that*

$$\sum_{j=1}^{N} a_j \geq c_2 N \qquad \text{and} \qquad a_j \leq A \ \text{for every } 1 \leq j \leq N,$$

there are $l > \theta_0 N$ and $1 < n_1 < \cdots < n_l \leq N$ so that

$$\sum_{j=n+1}^{n_i} a_j \geq c_1(n_i - n) \quad \text{for every } 0 \leq n < n_i \text{ and } i = 1, \dots, l.$$

Proof. (cf. [Mañ87, Section 2]) Define $S(n) = \sum_{j=1}^{n}(a_j - c_1)$, for each $1 \leq n \leq N$, and also $S(0) = 0$. Then define $1 < n_1 < \cdots < n_l \leq N$ to be the maximal sequence such that $S(n_i) \geq S(n)$ for every $0 \leq n < n_i$ and $i = 1, \dots, l$. Note that l can not be zero, since $S(N) > S(0)$. Moreover, the definition means that

$$\sum_{j=n+1}^{n_i} a_j \geq c_1(n_i - n) \quad \text{for } \ 0 \leq n < n_i \ \text{ and } \ i = 1, \dots, l.$$

So, we only have to check that $l > \theta_0 N$. Observe that, by definition,

$$S(n_i - 1) < S(n_{i-1}) \quad \text{and so} \quad S(n_i) < S(n_{i-1}) + (A - c_1)$$

for every $1 < i \leq l$. Moreover, $S(n_1) \leq (A - c_1)$ and $S(n_l) \geq S(N) \geq N(c_2 - c_1)$. This gives,

$$N(c_2 - c_1) \leq S(n_l) = \sum_{i=2}^{l} \big(S(n_i) - S(n_{i-1})\big) + S(n_1) < l(A - c_1),$$

which completes the proof. □

Clearly, the set H in the statement of Theorem A may be taken positively invariant under f. Given any $\sigma < 1$, let $H(\sigma)$ be the set of points in H for which the limit in (2) is smaller than $3 \log \sigma$. Then $H(\sigma)$ is positively invariant and, since we are assuming that H has positive Lebesgue measure, $H(\sigma)$ must also have positive Lebesgue measure if σ is close enough to 1. Then, there exists some small C^2 disk D transverse to the centre-stable subbundle, and intersecting $H(\sigma)$ in a set with positive Lebesgue measure inside D. Up to replacing it by some small disk contained in $f^l(D)$ for some large enough $l \geq 1$, we may suppose that D is tangent to the centre-unstable cone field and $\kappa(D) \leq C_1$. Here $\kappa(\cdot)$ is the Hölder constant defined by (10), recall Corollary 2.4. We fix such a disk D, once and for all.

Corollary 3.2. *There is $\theta > 0$, depending only on σ and f, such that, given any $x \in D \cap H(\sigma)$ and any sufficiently large $N \geq 1$, there exist σ-hyperbolic times $1 \leq n_1 < \cdots < n_l \leq N$ for x, with $l \geq \theta N$.*

Proof. Since $x \in H(\sigma)$, we have

$$\sum_{j=1}^{N} \log \left\| Df^{-1} \middle| E^{cu}_{f^j(x)} \right\| \leq 2N \log \sigma,$$

for all $N \geq 1$ sufficiently large. Now it suffices to take $c_1 = |\log \sigma|$, $c_2 = 2c_1$, $A = \sup \left| \log \|Df^{-1}|E^{cu}\| \right|$, and $a_j = -\log \|Df^{-1}|E^{cu}_{f^j(x)}\|$ in the previous lemma. □

Let $\delta_1 > 0$ be the small number introduced prior to Lemma 2.7. In particular, we requested that the δ_1-neighbourhood of the set K be contained in the domain V_0 of the invariant cone fields C^{cs} and C^{cu}. Reducing δ_1 if necessary, we may suppose that the subset $A = A(D, \sigma, \delta_1)$ of points $x \in D \cap H(\sigma)$ such that $\mathrm{dist}_D(x, \partial D) \geq \delta_1$ still has positive Lebesgue measure in D.

We consider the sequence

$$\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} f_*^j \, \mathrm{Leb}_D \tag{18}$$

of averages of forward iterates of Lebesgue measure on D. A main idea is to decompose μ_n as a sum of two measures, to be denoted ν_n and η_n, such that ν_n is uniformly absolutely continuous on iterates of the disk D (uniformly bounded density with respect to Lebesgue measure) and has total mass uniformly bounded away from zero for all large n. This is done as follows.

Given integers $n \geq 1$, define the following subset of $D \cap H(\sigma)$

$$H_n = \{x \in A : n \text{ is a } \sigma\text{-hyperbolic time for } x\}.$$

It follows from Lemma 2.7 that if $x \in H_n$ then the distance from $f^n(x)$ to the boundary of $f^n(D)$ is larger than δ_1. For $\delta > 0$, we denote $\Delta_n(x, \delta)$ the δ-neighbourhood of $f^n(x)$ inside $f^n(D)$. Clearly, $(f_*^n \, \mathrm{Leb}_D) \mid \Delta_n(x, \delta_1)$ is absolutely continuous with respect to Lebesgue measure on $\Delta_n(x, \delta_1)$. Moreover, if $x \in H_n$ and one normalizes both measures, then Proposition 2.8 means that the density of the former with respect to the latter is uniformly bounded from below and above.

Proposition 3.3. *There exists a constant $\tau > 0$ such that for any n there exists a finite subset $\widehat{H}_n$ of H_n such that the balls of radius $\delta_1/4$ in $f^n(D)$ around the points $x \in f^n(\widehat{H}_n)$ are two-by-two disjoint, and their union Δ_n satisfies*

$$f_*^n \, \mathrm{Leb}_D(\Delta_n \cap H(\sigma)) \geq f_*^n \, \mathrm{Leb}_D(\Delta_n \cap f^n(H_n)) \geq \tau \, \mathrm{Leb}_D(H_n) \; .$$

This is obtained taking $N = f^n(D) \cap V_0$, $\omega = f_*^n \operatorname{Leb}_D$, $r = \delta_1$, $\Omega = f^n(H_n)$, and $\widehat{H}_n = I$ in the following abstract result.

Lemma 3.4. *There exist $\tau > 0$ and $r_0 > 0$ such that the following holds. Let $N \subset V_0$ be a C^1 embedded submanifold of M tangent to the centre-unstable cone field, and ω be a finite Borelian measure in N. Let $0 < r \le r_0$ and $\Omega \subset N$ be a measurable subset with compact closure, whose distance to the boundary of N is larger than $r > 0$.*

Then there exists a finite subset $I \subset \Omega$ such that the balls $\Delta(x, r/4)$ in N around the points of I are two-by-two disjoint, and their union Δ satisfies

$$\omega(\Delta \cap \Omega) \ge \tau\omega(\Omega) .$$

Proof. By the continuity of the centre-unstable cone field, we may fix $r_0 > 0$ small enough so that the connected component of the intersection of N with the ball of radius r_0 around each point $z \in N$ that contains z coincides with the graph of a map g_z from a neighbourhood of 0 in E_z^{cu} to E_z^{cs} (or, to be more precise, with the image of such a graph under the exponential map $\exp_z$). Moreover, as long as r_0 is small enough, then g_z is a Lipschitz continuous map, with Lipschitz constant depending only on the constant a in the definition of our cone fields. As a consequence, there exists a constant $R > 0$ such that, given $0 < r_1 < r_2 \le r_0$, any ball of radius r_2 in N can be covered by at most $(r_2/r_1)^d R$ balls of radius r_1, with $d = \dim E^{cu}$. We assume that $\omega(\Omega) > 0$, since otherwise there is nothing to prove.

Let $0 < r \le r_0$ and $z_1 \in N$ be such that $\omega(\Omega \cap \Delta(z_1, r))$ is larger than $\omega(\Omega \cap \Delta(z, r))/2$ for any other point $z \in N$. By the previous remarks, we may find a point $y_1 \in N$ such that the ball $\Delta(y_1, r/8)$ of radius $r/8$ around y_1 intersects $\Delta(z_1, r)$ and

$$\omega(\Delta(y_1, r/8) \cap \Omega) \ge \frac{1}{R8^d}\, \omega(\Omega \cap \Delta(z_1, r)).$$

In particular $\Delta(y_1, r/8)$ contains some point $x_1 \in \Omega$. We take it to be the first point in our set I. Observe that, due to the choice of z_1,

$$\begin{aligned} \omega(\Delta(x_1, r/4) \cap \Omega) &\ge \omega(\Delta(y_1, r/8) \cap \Omega) \\ &\ge \frac{1}{R8^d}\, \omega(\Omega \cap \Delta(z_1, r)) \\ &\ge \frac{1}{2R8^d}\, \omega(\Omega \cap \Delta(x_1, r)). \end{aligned} \tag{19}$$

Now we consider $\Omega_1 = \Omega \setminus \Delta(x_1, r)$. Either this set has zero ω measure, in which case we stop, or we may apply the same construction as before to determine a second point $x_2 \in \Omega$. Observe that the balls of radius $r/4$ around x_1 and x_2 are disjoint, since x_2 belongs in Ω_1. Repeating this procedure, we find a sequence x_i, $i \ge 1$, of points in Ω whose balls of radius $r/4$ are two-by-two disjoint. By compactness, this sequence is necessarily finite.

Moreover, by construction, Ω is contained in the union of the balls of radius r around these x_i. So, using the bound in (19),

$$\omega(\Omega) \leq 2R8^d \sum_i \omega(\Delta(x_i, r/4)) \cap \Omega) = 2R8^d \omega(\Delta \cap \Omega).$$

This means that we may take $\tau = 1/(2R8^d)$. $\square$

In the sequel we shall denote $\mathcal{D}_n$ the family of balls of radius $\delta_1/4$ in $f^n(D)$ around the points $x \in f^n(\widehat{H}_n)$, that form Δ_n. Now we define

$$\nu_n = \frac{1}{n} \sum_{j=0}^{n-1} (f_*^j \operatorname{Leb}_D) \mid \Delta_j\,, \tag{20}$$

and $\eta_n = \mu_n - \nu_n$.

Proposition 3.5. *There is $\alpha > 0$ such that $\nu_n(H(\sigma)) \geq \alpha$ for all n large enough.*

Proof. Recall that we took H and $H(\sigma)$ positively invariant under f. By Proposition 3.3 we have that $\nu_n(H(\sigma))$ is bounded from below by the product of τ by $n^{-1} \sum_{i=0}^{n-1} \operatorname{Leb}_D(H_i)$. So, it suffices to prove that this last expression is larger than some positive constant, for n large.

For all $k > 0$, denote A_k the set of points $x \in A$ such that, for any $n \geq k$ the sum $\sum_{j=1}^{n} \log \|Df^{-1}|E^{cu}_{f^j(x)}\|$ is smaller than $2n \log \sigma$. As A is the increasing union of the A_n, there is $k_0 \geq 1$ such that the Lebesgue measure of A_k is nonzero for all $k \geq k_0$. Given any $k \geq k_0$, let ξ_n be the measure in $\{1, \ldots, n\}$ defined by $\xi_n(B) = \#B/n$, for each subset B. Then, using Fubini's theorem

$$\begin{aligned} \frac{1}{n} \sum_{i=0}^{n-1} \operatorname{Leb}_D(H_n) &= \int \left(\int \chi(x, i)\, d \operatorname{Leb}_D(x) \right) d\xi_n(i) \\ &= \int \left(\int \chi(x, i)\, d\xi_n(i) \right) d \operatorname{Leb}_D(x), \end{aligned}$$

where $\chi(x, i) = 1$ if $x \in H_i$ and $\chi(x, i) = 0$ otherwise. Now, Corollary 3.2 means that the integral with respect to $d\xi_n$ is larger than $\theta > 0$, as long as $k \geq k_0$. So, the expression on the right hand side is bounded from below by $\theta \operatorname{Leb}_D(D)$. $\square$

Remark 3.6. We proved a slightly stronger fact, that will be useful in Sect. 4: $\nu_n\big(\cup_{i=0}^{n-1} f^i(D \cap H(\sigma))\big) \geq \alpha$ for every large n.

We consider some subsequence $(n_k)_k$ such that μ_{n_k} and ν_{n_k} converge to measures μ and ν, respectively. It is easy to see that μ is a probability and f-invariant. Moreover $\nu(K) \geq \limsup_k \nu_k(K) \geq \alpha > 0$. We shall prove, in the next section, that ν has a property of absolute continuity along certain (fairly large) disks contained in its support. Here are some preparatory comments.

Recall that each ν_n is supported on a finite union $\cup_{j=0}^{n-1}\Delta_j$ of disks whose size is bounded from below and from above. Then the support of ν is contained in the set

$$\Delta_\infty = \cap_{n=1}^{\infty} \text{closure}\left(\cup_{j\geq n}\Delta_j\right)$$

of accumulation points of such Δ_j. Given $y \in \Delta_\infty$ then there exist $(j_i)_i \to \infty$, disks $D_i = \Delta_{j_i}(x_i, \delta_1/4) \subset \Delta_{j_i}$, and points $y_i \in D_i$ converging to y as $i \to \infty$. Up to considering subsequences, we may suppose that the centers x_i converge to some point x and, using the theorem of Ascoli-Arzela, the D_i converge to a disk $D(x)$ of size $\delta_1/4$ around x. Then y is in the closure $\bar{D}(x)$ of $D(x)$, and $\bar{D}(x) \subset \Delta_\infty$.

We shall denote $\mathcal{D}_\infty$ the family of disks $D(x)$ obtained in this way. Observe that these points x are in $\hat{H}_\infty = \cap_{n=1}^{\infty} \text{closure}\left(\cup_{j\geq n} f^j(\hat{H}_j)\right)$. Since every $\hat{H}_j$ is contained in K, which is compact and positively invariant, $\hat{H}_\infty$ is a subset of $\cap_{n=1}^{\infty} f^n(K)$. According to the next lemma, $D(x)$ depends only on x and not on the various choices we made in the construction.

Lemma 3.7. *The subspace E_x^{cu} is uniformly expanding: $\|Df^{-k} \mid E_x^{cu}\| \leq \sigma^{k/2}$ for all $k \geq 1$. The disk $D(x)$ is contained in the corresponding strong-unstable manifold $W^{uu}(x)$, and so it is uniquely defined by x. Moreover, $D(x)$ is tangent to the centre-unstable subbundle at every point of $\cap_{n=1}^{\infty} f^n(K) \cap D(x)$.*

Proof. Let $j_i \to \infty$, $x_i \to x$, and $D_i \to D(x)$ be as in the construction of $D(x)$ above. Note that D_i is contained in the j_i-iterate of D, which was taken tangent to the centre-unstable cone field. So, the domination property implies that the angle between D_i and E^{cu} goes to zero as $i \to \infty$, uniformly on $\cap_{n=1}^{\infty} f^n(K)$. By Lemma 3.7, given any $k \geq 1$ then f^{-k} is a $\sigma^{k/2}$-contraction on D_i for every large i. Passing to the limit, we get that every f^k is a $\sigma^{k/2}$-contraction on $D(x)$, and $D(x)$ is tangent to the centre-unstable subbundle at every point in $\cap_{n=1}^{\infty} f^n(K) \cap D(x)$, including x.

In particular, we have shown that the subspace E_x^{cu} is indeed uniformly expanding for Df. The domination property means that any expansion Df may exhibit along the complementary direction E^{cs} is weaker than this. Then, see [Pes76], there exists a unique *strong-unstable* manifold $W_{loc}^{uu}(x)$ tangent to E^{cu} and which is contracted by the negative iterates of f: for every $y \in W^{uu}(x)$, $\text{dist}(f^{-k}(x), f^{-k}(y))$ decreases at least as $\|Df^{-k} \mid E_x^{cu}\| \leq \sigma^{k/2}$ when k gets large. To see that $D(x)$ is contained in $W^{uu}(x)$ is suffices to recall that it is contracted by every f^{-k}, and that all its negative iterates are tangent to centre-unstable cone field. $\square$

4. Existence and finiteness of SRB measures

This section contains the proofs of Theorem A and Corollary B. First we show that the measure ν obtained in the previous section has an absolute continuity property on disks as in Lemma 3.7. Then, using the uniformly contracting bundle, we conclude that some ergodic component of the invariant measure μ is an SRB measure.

4.1. *Absolute continuity*

We write $u = \dim E^{cu}$ and $s = \dim E^{cs}$, and use B^u, B^s to represent the unit compact balls in the Euclidean space of dimension u, s, respectively. In what follows we call cylinder any diffeomorphic image of $B^u \times B^s$.

Proposition 4.1. *There exists a cylinder $\mathcal{C} \subset M$, and there exists a family $\mathcal{K}_\infty$ of disjoint disks contained in $\mathcal{C} \cap \Delta_\infty$ and which are graphs over B^u, such that*

1. *the union of all the disks in $\mathcal{K}_\infty$ has positive ν-measure; in fact, the intersection of K with this union also has positive ν-measure;*
2. *the restriction of ν to that union has absolutely continuous conditional measures along the disks in $\mathcal{K}_\infty$.*

The first step of the proof is to construct a covering of the support of ν by cylinders, one of which will be the $\mathcal{C}$ in the statement. Let us point out that these cylinders we shall obtain are not small: each one contains some ball with radius uniformly bounded away from zero, depending only on the diffeomorphism f.

As we have seen, given any $y \in \Delta_\infty$ there exists a point $x \in \hat{H}_\infty$ and a disk $D(x)$ of size $\delta_1/4$ around x such that $y \in \bar{D}(x) \subset \Delta_\infty$. For any such x and $r > 0$ small, let $C_r(x)$ be the tubular neighbourhood of $\bar{D}(x)$, defined as the union of the images under the exponential map at each point $z \in \bar{D}(x)$ of all vectors orthogonal to $\bar{D}(x)$ at z and with norm less or equal than r. We take r to be sufficiently small, so that $C_r(x)$ is a cylinder and it is endowed with a canonical projection $\pi : C_r(x) \to D(x)$. Slightly adjusting r if necessary, we may also suppose that the boundary of $C_r(x)$ has zero ν-measure.

The covering of the support of ν by cylinders that we mentioned above will be obtained decomposing each of these $C_r(x)$ into a sufficient number of domains with small diameter in the centre-unstable direction, as we now explain.

Recall that each set Δ_j, $j \geq 0$, consists of a finite union of disks of radius $\delta_1/4$ inside $f^j(D)$. For any small $\varepsilon > 0$, we denote $\Delta_{j,\varepsilon}$ the subset of Δ_j obtained by removing the ε-neighbourhood of the boundary from each one of these disks. Moreover, for $n \geq 1$, we denote $\nu_{n,\varepsilon}$ the restriction of ν_n to $\cup_{j=0}^{n-1}\Delta_{j,\varepsilon}$. Let $\alpha > 0$ be as in Proposition 3.5.

Lemma 4.2. *If $\varepsilon > 0$ is sufficiently small then $\nu_{n,\varepsilon}(K) \geq \alpha/2$ for every large n.*

Proof. This is a simple consequence of Proposition 3.5. Indeed, the proposition implies that the ν_n-measure of K is greater or equal than α, for all large n. On the other hand, if ε is small then the Lebesgue measure of the ε-neighbourhood of the boundary of each disk in Δ_j is a small fraction of the Lebesgue measure of that disk. Then, in view of the distortion bound given by Proposition 2.8, the same is true with $f_*^j \operatorname{Leb}_D$ in the place of Lebesgue measure. So, taking ε small enough, we are certain to have $(f_*^j \operatorname{Leb}_D)\big(\Delta_j \setminus \Delta_{j,\varepsilon}\big) \leq \alpha/2$ for every $j \geq 0$. Then, by the definitions of ν_n and $\nu_{n,\varepsilon}$,

$$\nu_n(K) - \nu_{n,\varepsilon}(K) \leq \frac{1}{n}\sum_{j=0}^{n-1}(f_*^j \operatorname{Leb}_D)\big(\Delta_j \setminus \Delta_{j,\varepsilon}\big) \leq \frac{\alpha}{2}.$$

This completes the proof. □

In the sequel, we fix $\varepsilon > 0$ as in the lemma. Let $x \in \hat{H}_\infty$ and $\pi : C_r(x) \to \bar{D}(x)$ be as above. We fix a covering of $\bar{D}(x)$ by finitely many domains $D_{x,l} \subset \bar{D}(x), l = 1, \ldots, N$, small enough so that the intersection of each $C_{x,l} = \pi^{-1}(D_{x,l})$ with any smooth disk γ tangent to the centre-unstable cone has diameter less than ε inside γ. We take the $D_{x,l}$ diffeomorphic to the compact ball B^u, so that every $C_{x,l}$ is a cylinder.

We say that a disk γ *crosses* $C_{x,l}$ if π maps $\gamma \cap C_{x,l}$ diffeomorphically onto $D_{x,l}$. For each $j \geq 0$, let $K_j(x,l)$ be the union of the intersections of $C_{x,l}$ with all the disks in $\mathcal{D}_j$ (the disks in the support Δ_j of $f_*^j \operatorname{Leb}_D$) that cross $C_{x,l}$. Similarly, let $K_\infty(x,l)$ be the union of the intersections of $C_{x,l}$ with all the disks $D(x)$ in $\mathcal{D}_\infty$ that cross $C_{x,l}$.

The next lemma asserts that, for at least one of the cylinders $C_{x,l}$, the part of the measure ν that is carried by the disks in $K_\infty(x,l)$ gives positive weight to the set K. Recall that $\nu = \lim \nu_{n_k}$ for some $(n_k)_k$.

Lemma 4.3. *There exist (x,l) and $\alpha_1 > 0$ such that $\nu\big(K \cap K_\infty(x,l)\big) \geq \alpha_1$, and $\nu_n\big(K \cap \cup_{j=0}^{n-1} K_j(x,l)\big) \geq \alpha_1$ for n in some subsequence of $(n_k)_k$.*

Proof. For each $n \geq 1$, let $\tilde{\nu}_{n,\varepsilon}$ be the restriction of $\nu_{n,\varepsilon}$ to K, i.e., the measure defined by $\tilde{\nu}_{n,\varepsilon}(E) = \nu_{n,\varepsilon}(K \cap E)$ for every measurable subset E of M. Up to replacing $(n_k)_k$ by some subsequence, we may suppose that $\tilde{\nu}_{n_k,\varepsilon}$ converges to some measure $\tilde{\nu}_\varepsilon$. On the one hand, Lemma 4.2 means that $\tilde{\nu}_{n,\varepsilon}(M) \geq \alpha/2$ for every large n, and so $\tilde{\nu}_\varepsilon(M) \geq \alpha/2$. On the other hand, the support of $\tilde{\nu}_\varepsilon$ is contained in $\cap_{n=1}^{\infty}$ closure $\big(\cup_{j \geq n} \Delta_{j,\varepsilon}\big)$, and this set is covered by the interiors of the cylinders $C_r(x)$. By compactness, the support of $\tilde{\nu}_\varepsilon$ is contained in the union of a finite number of these $C_r(x)$, and so it is also contained in the union of finitely many cylinders $C_{x,l}$.

As a consequence, there must be (x, l) such that $\tilde{\nu}_\varepsilon(C_{x,l}) > 0$. We are going to show that any such (x, l) satisfies the conclusion of the lemma, if $\alpha_1 < \tilde{\nu}_\varepsilon(C_{x,l})$.

Given any disk D_j in $\mathcal{D}_j$, $j \geq 1$, let $D_{j,\varepsilon}$ be the subset obtained by removing from D_j the ε-neighbourhood of the boundary. As a consequence of the way we chose these cylinders, we have that if $D_{j,\varepsilon}$ intersects $C_{x,l}$ then D_j must cross $C_{x,l}$. This implies that

$$\tilde{\nu}_{n,\varepsilon}\big(C_{x,l}\big) = \nu_{n,\varepsilon}\big(K \cap C_{x,l}\big) \leq \nu_n\big(K \cap \cup_{j=0}^{n-1} K_j(x, l)\big)$$

for every $n \geq 1$. Since the boundary of $C_{x,l}$ has zero measure for ν, and $\tilde{\nu}_\varepsilon \leq \nu$,

$$\lim_k \tilde{\nu}_{n_k,\varepsilon}\big(C_{x,l}\big) = \tilde{\nu}_\varepsilon(C_{x,l}) > \alpha_1\,.$$

Combining this with the previous inequality, we get the second part of the lemma. To get the first part, we observe that the accumulation set of $\cup_{j=0}^{n-1} K_j(x, l)$, as $n \to \infty$, is contained in $K_\infty(x, l)$. So, since K is compact,

$$\limsup_k \nu_n\big(K \cap \cup_{j=0}^{n-1} K_j(x, l)\big) \leq \nu(K \cap K_\infty(x, l)).$$

Thus, $\nu(K \cap K_\infty(x, l)) \geq \tilde{\nu}_\varepsilon(C_{x,l}) > \alpha_1$ as we claimed. □

In what follows, we fix (x, l) as in the lemma. We take the cylinder $\mathcal{C}$ in Proposition 4.1 to be $C_{x,l}$, and we let $\mathcal{K}_\infty$ be the family of disks forming $K_\infty(x, l)$. To complete the proof of the proposition, we now show that the restriction of ν to $K_\infty(x, l)$ has absolutely continuous conditional measures along the disks in $\mathcal{K}_\infty$.

Lemma 4.4. *There exists $C_3 > 1$ and a family of conditional measures $(\nu_\gamma)_\gamma$ of $\nu \mid K_\infty(x, l)$ along the disks $\gamma \in \mathcal{K}_\infty$, such that ν_γ is absolutely continuous with respect to Lebesgue measure Leb_γ on γ, with $1/C_3\,\mathrm{Leb}_\gamma(B) \leq \nu_\gamma(B) \leq C_3\,\mathrm{Leb}_\gamma(B)$ for any Borel set $B \subset \gamma$.*

Proof. Let us introduce $\widehat{K}(x, l) = \cup_{0\leq j\leq\infty} K_j(x, l) \times \{j\}$. In this space, we consider the sequence of (finite) measures $\hat{\nu}_n$ defined by

$$\hat{\nu}_n\,(B_0 \times \{0\} \cup \cdots \cup B_{n-1} \times \{n-1\}) = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j\,\mathrm{Leb}_D(B_j),$$

and $\hat{\nu}_n(B) = 0$ whenever B is contained in $\cup_{n\leq j\leq\infty} K_j(x, l) \times \{j\}$. We also consider a sequence of partitions $\mathcal{P}_k$ in $\widehat{K}(x, l)$ constructed as follows. Fix an arbitrary point z in $D_{x,l}$, and let V be the inverse image $\pi^{-1}(z)$ under the canonical projection. Fix also a sequence $\mathcal{V}_k$, $k \geq 1$, of increasing partitions of V with diameter going to zero. Then, by definition, two points (x, m), $(y, n) \in \widehat{K}(x, l)$ are in a same atom of the partition $\mathcal{P}_k$ if

1. the disk in $\mathcal{D}_m$ containing x, and the disk in $\mathcal{D}_n$ containing y intersect a same element of $\mathcal{V}_k$;
2. either $m \geq k$ and $n \geq k$ or $m = n < k$.

It is clear from the construction that for any point $\xi \in K_j(x,l)$, $0 \le j \le \infty$,

$$\mathcal{P}_1(\xi) \supset \cdots \supset \mathcal{P}_k(\xi) \supset \cdots$$

and $\cap_{k=1}^{\infty} \mathcal{P}_k(\xi)$ coincides with the intersection of the cylinder $C_{x,l}$ with the disk in $\mathcal{D}_j$ that contains ξ. Let $\hat{\pi} : \widehat{K}(x,l) \to D_{x,l}$ be defined by $\hat{\pi}(x, j) = \pi(x)$. We claim that there exits $C_3 > 1$ such that, given any Borel subset B of $D_{x,l}$, $k \ge 1$, and $\xi \in \widehat{K}(x,l)$,

$$\frac{1}{C_3} \nu_n(\mathcal{P}_k(\xi)) \operatorname{Leb}(B) \le \hat{\nu}_n \left(\hat{\pi}^{-1}(B) \cap \mathcal{P}_k(\xi)\right) \le C_3 \nu_n(\mathcal{P}_k(\xi)) \operatorname{Leb}(B)\,. \tag{21}$$

Indeed, by definition each atom $\mathcal{P}_k(\xi)$ is a union of sets $\gamma \times \{j\}$, where γ is the intersection of the cylinder with a disk in $\mathcal{D}_j$. Since the projection π maps γ diffeomorphically onto $D_{x,l}$,

$$\begin{aligned} \frac{1}{C_4} \frac{\operatorname{Leb}(B)}{\operatorname{Leb}(D_{x,l})} \ &\le\ \frac{\operatorname{Leb}(\hat{\pi}^{-1}(B) \cap (\gamma \times \{j\}))}{\operatorname{Leb}(\gamma \times \{j\})} \\ &=\ \frac{\operatorname{Leb}(\pi^{-1}(B) \cap \gamma)}{\operatorname{Leb}(\gamma)} \ \le\ C_4 \frac{\operatorname{Leb}(B)}{\operatorname{Leb}(D_{x,l})}\,, \end{aligned}$$

for some uniform constant C_4. By Proposition 2.8, the density of $f_*^j \operatorname{Leb}_D$ with respect to Lebesgue measure on each disk in Δ_j is bounded from below and from above. So, the previous inequality implies

$$\frac{1}{C_2^2 C_4} \frac{\operatorname{Leb}(B)}{\operatorname{Leb}(D_{x,l})} \le \frac{(f_*^j \operatorname{Leb}_D)(\hat{\pi}^{-1}(B) \cap (\gamma \times \{j\}))}{(f_*^l \operatorname{Leb}_D)(\gamma \times \{j\})} \le C_2^2 C_4 \frac{\operatorname{Leb}(B)}{\operatorname{Leb}(D_{x,l})}\,,$$

Since this holds for every γ, we get (21) with $C_3 = C_2^2 C_4 / \operatorname{Leb}(D_{x,l})$.

Clearly, any accumulation measure of the sequence $\hat{\nu}_n$ must be supported in $K_\infty(x,l) \times \{\infty\}$. We have chosen a sequence n_k such that ν_{n_k} converges to some measure ν, and it is easy to see that this is just the same as saying that $\hat{\nu}_k$ converges to the measure $\hat{\nu}_\infty$ defined by $\hat{\nu}_\infty(B \times \{\infty\}) = \nu(B)$, for any Borel set $B \subset C_{x,l}$. Then, by (21) and the theorem of Radon-Nikodym, the disintegration of $\hat{\nu}$ along the disks $\cap_{k=1}^{\infty} \mathcal{P}_k(\xi)$ is absolutely continuous with respect to Lebesgue measure on those disks, with densities almost everywhere bounded from above by C_3 and from below by $1/C_3$. Since $\hat{\nu}$ is naturally identified with ν, this gives the conclusion of the lemma. $\square$

At this point we completed the proof of Proposition 4.1.

4.2. Ergodicity and basin of attraction

Let us introduce some notations that are useful for the proof of the next lemma. We denote by R the set of *regular points* z of f: this means that, given any continuous function $\varphi : M \to \mathbb{R}$, both forward and backward time averages

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(z)) \quad \text{and} \quad \lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^{-j}(z))$$

exist and they coincide. The ergodic theorem ensures that R has full measure, with respect to any invariant probability. We say that two points $z, w \in R$ are in a same *accessibility class*, see [BP74,PS89], if there exist $N \geq 1$ and points $z = z_0, z_1, \ldots, z_{N-1}, z_N = w$ in R such that $f^{t_i}(z_i)$ is in the union $W^s(z_{i-1}) \cup W^u(z_{i-1})$ of the stable and the unstable sets of z_{i-1}, for some integer t_i and every $1 \leq i \leq N$. It follows from the definition, that accessibility classes are invariant sets. Since forward averages are constant on stable sets, and backward averages are constant on unstable sets, the restriction of any invariant probability measure to an accessibility class is an ergodic measure (possibly identically zero).

Lemma 4.5. *The invariant measure $\mu = \nu + \eta$ has some ergodic component μ_* whose Lyapunov exponents are all non-zero, and whose conditional measures along local unstable manifolds are absolutely continuous with respect to Lebesgue measure. Moreover, we may choose μ_* with support contained in $\cap_{j=0}^{\infty} f^j(K)$ and* $\mathrm{Leb}_D(B(\mu_*) \cap H) > 0$.

Proof. Since R has full μ-measure, it must also have full ν-measure. In particular, up to replacing $\mathcal{K}_\infty$ by a convenient sub-family of disks γ, whose union has full ν measure in K_∞, we may suppose that ν_γ almost every point in γ is regular, for every $\gamma \in \mathcal{K}_\infty$. In particular, Leb_γ almost every point in any disk γ is regular. Using the fact that the strong-stable foliation is absolutely continuous, cf. [BP74, Section 2], we conclude that all such regular points are in a same accessibility class. Moreover, this accessibility class $\mathcal{A}$ has positive ν-measure, and so also positive μ-measure, by Lemma 4.3.

We let μ_* be the normalized restriction of μ to $\mathcal{A}$: $\mu_*(B) = \mu(B \cap \mathcal{A})/\mu(\mathcal{A})$ for every Borel set B. Then μ_* is an invariant ergodic probability measure. It follows from Lemma 3.7 that the Lyapunov exponents of μ_* along the tangent space of the disks γ are positive. Of course, the Lyapunov exponents along the strong-stable direction are all negative. Since μ_* is ergodic, its conditional measures along local unstable manifolds in $\mathcal{K}_\infty$ are either almost everywhere singular or almost everywhere absolutely continuous (with respect to Lebesgue measure). This is a well known fact, whose proof can be sketched as follows.

Suppose there is $A \subset K_\infty(x, l)$ such that $m_\gamma(A \cap \gamma) = 0$ for all $\gamma \in \mathcal{K}_\infty$, and yet $\mu_*(A) > 0$ (hence $\mu_{*\gamma}(A \cap \gamma) > 0$ for many $\gamma \in \mathcal{K}_\infty$). Let

$B = \cup_{j=-\infty}^{+\infty} f^j(A)$. By ergodicity, $\mu_*(B) = 1$, and so $\mu_{*,\gamma}(B \cap \gamma) = 1$ for $\hat{\mu}_*$-almost all γ in $\mathcal{K}_\infty$. On the other hand, $m_\gamma(B \cap \gamma) = 0$ for every $\gamma \in \mathcal{K}_\infty$. This is because f is a diffeomorphism, unstable manifolds are an invariant family of submanifolds, and B is given by a countable union. So, in this case, $\mu_{*\gamma}$ is singular with respect to m_γ for $\hat{\mu}_*$-almost all $\gamma \in \mathcal{K}_\infty$. Now suppose that, on the contrary, every measurable set $A \subset K_\infty(x, l)$ satisfying $m_\gamma(A \cap \gamma) = 0$ for all $\gamma \in \mathcal{K}_\infty$ has zero μ_*-measure. Then, restricted to $K_\infty(x, l)$, μ_* is absolutely continuous with respect to the product measure $m_\gamma \times \hat{\mu}_*$. Consequently, in this second case, the conditional measures $\mu_{*,\gamma}$ are absolutely continuous with respect to Lebesgue measure m_γ for $\hat{\mu}_*$-almost every γ in $\mathcal{K}_\infty$.

In the setting we are dealing with, the singular case is easily excluded: the conditional measures of μ_* can be written as the sum of the conditional measures of the restrictions of η and ν to the accessibility class, and the latter are equivalent to Lebesgue measure at least on $K_\infty(x, l)$. So, the conditional measures of μ_* must be almost everywhere absolutely continuous.

Since $\nu(K \cap K_\infty(x, l))$ is positive, by Lemma 4.3, $\mu(K \cap \mathcal{A}) > 0$ and so $\mu_*(K) > 0$. As K is compact and positively invariant, the ergodicity of μ_* implies that $\mu_*(\cap_{n=0}^{\infty} f^n(K)) = 1$, and the support of μ_* is contained in $\cap_{j=0}^{\infty} f^j(K)$.

To prove the last statement in the lemma we need the following fact:

Claim: *There exists a disk D_∞ inside $K_\infty(x, l)$ such that Lebesgue almost all the points in D_∞ are in the basin of μ_*, and there exists a sequence D_k of disks in $K_{j_k}(x, l)$ accumulating on D_∞ and such that* $\mathrm{Leb}_{D_k}(D_k \cap f^{j_k}(H(\sigma) \cap D))$ *is uniformly bounded away from zero.*

We assume this for a while, and explain how to conclude the proof of the lemma from it. Since forward averages of continuous functions are constant on strong-stable leaves, the basin of μ_* contains the union of all strong-stable leaves through Lebesgue almost all points in D_∞. As this foliation is absolutely continuous, and the D_k accumulate on D_∞, such union intersects D_k in a subset whose relative Lebesgue measure inside D_k goes to 1 when k goes to infinity. In particular, $\mathrm{Leb}_{D_k}(D_k \cap f^{j_k}(H(\sigma) \cap D) \cap B(\mu_*))$ is positive for every large k. Of course, the basin is invariant by f, so we may conclude that $\mathrm{Leb}_D(H(\sigma) \cap B(\mu_*)) > 0$. □

All that is left to do is to prove the Claim above.

Proof. We use Remark 3.6:

$$\nu_n\Big(\cup_{i=0}^{n-1} \ f^i(D \cap H(\sigma))\Big) \geq \alpha.$$

It follows that if $\epsilon > 0$ is fixed sufficiently small then there exists a subset of disks in the support of ν_n with total ν_n-mass larger than $\alpha/2$ and such that a fraction larger than ϵ of any such disk corresponds to points coming from $D \cap H(\sigma)$. Then, by the same argument as at the end of the proof of Lemma 4.3, the union E of the disks in $\mathcal{D}_\infty$ that are accumulated by disks as above has ν-mass larger than $\alpha/2$. Then some of these disks must be such that Lebesgue almost all points in it are in the basin of μ_*. Indeed, since μ_* is ergodic, its basin has full μ_*-measure. Then, a full measure subset of E consists of disks where almost all points, with respect to the conditional measure of μ_* on the disk, are in $B(\mu_*)$. Since we know that the conditional measures of μ_* along the disks in $\mathcal{K}_\infty$ are bounded away from zero (because the same is true for ν, cf. Lemma 4.4), we conclude that Lebesgue almost all points in some disk in E is in $B(\mu_*)$. $\square$

As a consequence, we also get that

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n-1} \log \left\| Df^{-1} \middle| E^{cu}_{f^j(x)} \right\| = \int \log \left\| Df^{-1} \middle| E^{cu}_y \right\| d\mu_*(y) < 2\log\sigma \tag{22}$$

for μ_*-almost every $x \in M$ (and for every $x \in B(\mu_*)$ that remains in the neighbourhood V_0 where E^{cu} has a meaning). This is a simple consequence of the ergodicity of μ_*, and the fact that its basin intersects $H(\sigma)$. In particular, all the Lyapunov exponents of μ_* in the centre-unstable direction are larger than $-\log\sigma$.

Finally, we deduce the following result which completes the proof of Theorem A and also gives Corollary B.

Corollary 4.6. *For any $\sigma < 1$, a full Lebesgue measure subset of $H(\sigma)$ is contained in the union of finitely many SRB measures supported in $\cap_{j=0}^\infty f^j(K)$.*

Proof. First we observe that the set of points in $H(\sigma)$ which do not belong in the basin of some SRB measure as in the statement must have zero Lebesgue measure. Indeed, otherwise we could apply the previous arguments with this set in the place of $H(\sigma)$: we would get, cf. Lemma 4.5, an extra positive Lebesgue measure subset in the basin of some SRB measure, contradicting the definition.

The main point to obtain the finiteness statement is to note that the choice of δ_1 in the context of (14) depends only on σ. Using this remark, we can deduce that for any SRB measure μ_* as we constructed above there exists a disk $D(\mu_*)$ of fixed radius δ_1, tangent to the centre-unstable cone field and such that Lebesgue almost every point in $D(\mu_*)$ is in the basin of μ_*. For this, we recall that for the SRB measures we constructed above there exist disks D_0 containing some point $x \in H(\sigma) \cap B(\mu_*)$ and on which Lebesgue almost every point is in the basin $B(\mu_*)$. In view of (22), x has many σ-hyperbolic times n. We may take $D(\mu_*)$ to be the disk of radius δ_1 around $f^n(x)$ inside $f^n(D_0)$, for any such n sufficiently large.

Then, the union of all strong-stable leaves through the points in $D(\mu_*) \cap B(\mu_*)$ is contained in $B(\mu_*)$. Using the absolute continuity property of the strong-stable foliation, we may conclude that this union contains a subset of a neighbourhood of $D(\mu_*)$ with volume bounded away from zero by some constant that depends only on δ_1 and f. For this, we fix some neighbourhood of $D(\mu_*)$ with size uniformly bounded from below, as well as a smooth foliation of it by disks C^1 close to $D(\mu_*)$. For instance, $D(\mu_*)$ could be contained in one of the leaves of this foliation. Given any leaf D, the strong-stable manifolds through the points of $D(\mu_*)$ intersect D on a subdisk D' whose Lebesgue measure inside D is bounded away from zero, by a constant that depends only on the size δ_1 of D, and on the map f. This is just by continuity of the strong-stable foliation. Moreover, absolute continuity implies that Lebesgue almost every point of D' is in the strong-stable manifold of a point of $D(\mu_*) \cap B(\mu_*)$. Now, our claim that the Lebesgue measure of the union of these strong-unstable leaves is uniformly bounded away from zero follows from Fubini's theorem.

Of course, basins of different SRB measures are two-by-two disjoint. So the conclusion of the previous paragraph implies that there can only be finitely many such measures (even if we do not assume M to be compact), since small neighbourhoods of the compact set K have finite volume. □

5. Maps with singular or critical points

Here we explain how the previous arguments can be adapted to prove Theorem C and Corollary D. A main difference concerns the notion of hyperbolic times. A key point in the previous sections was that *if n is a hyperbolic time for a point x then there exists a neighbourhood of x, in the disk D, which is mapped onto a ball of fixed radius around $f^n(x)$, in $f^n(D)$, diffeomorphically and with uniformly bounded distortion*. This was a consequence of the contraction property in Definition 2.6. Now, in the presence of a singular set $\mathcal{S}$, in order to have a similar property we must also ensure that iterates $f^j(x)$ with $0 \le j < n$ are not too close to $\mathcal{S}$.

Let $B > 1$ and $\beta > 0$ be as in the hypotheses (S1), (S2), (S3). In what follows b is any fixed constant such that $0 < b < \min\{1/2, 1/(2\beta)\}$.

Definition 5.1. *Given $\sigma < 1$ and $\delta > 0$, we say that n is a* (σ, δ)-hyperbolic time *for a point $x \in M \setminus \mathcal{S}_\infty$ if, for all $1 \le k \le n$,*

$$\prod_{j=n-k}^{n-1} \|Df(f^j(x))^{-1}\| \le \sigma^k \quad \textit{and} \quad \operatorname{dist}_\delta(f^{n-k}(x), \mathcal{S}) \ge \sigma^{bk}.$$

Let us begin by proving that this notion does imply the key property above:

Lemma 5.2. *Given $\sigma < 1$ and $\delta > 0$, there exists $\delta_1 > 0$ such that if n is a (σ, δ)-hyperbolic time for a point $x \in M \setminus \mathcal{S}_\infty$, then there exists a neighbourhood V_x of x such that*

1. *f^n maps V_x diffeomorphically onto the ball of radius δ_1 around $f^n(x)$;*
2. *for every $1 \le k < n$ and $y, z \in V_x$,*

$$\operatorname{dist}(f^{n-k}(y), f^{n-k}(z)) \le \sigma^{k/2} \operatorname{dist}(f^n(y), f^n(z)).$$

Proof. We shall prove, by induction on $j \geq 1$, that if δ_1 is chosen small enough then there exists a well defined branch of f^{-j} on the ball of radius δ_1 around $f^n(x)$, mapping $f^n(x)$ to $f^{n-j}(x)$. In addition, this branch is a $\sigma^{j/2}$-contraction. The precise condition δ_1 should satisfy is given by the following statement:

Claim: *Fix $\delta_1 > 0$ so that $4\delta_1 < \delta$ and $4B\delta_1 < \delta^\beta |\log \sigma|$. Then,*

$$\|Df(y)^{-1}\| \leq \sigma^{-1/2} \|Df(f^{n-j}(x))^{-1}\| \tag{23}$$

for any $1 \leq j < n$ and any point y in the ball of radius $2\delta_1\sigma^{j/2}$ around $f^{n-j}(x)$.

Proof. By hypothesis $\text{dist}_\delta(f^{n-j}(x), \mathcal{S}) \geq \sigma^j$. According to the definition of the truncated distance, this means that

$$\text{dist}(f^{n-j}(x), \mathcal{S}) = \text{dist}_\delta(f^{n-j}(x), \mathcal{S}) \geq \sigma^{bj}$$
$$\text{or else} \quad \text{dist}\ (f^{n-j}(x), \mathcal{S}) \geq \delta.$$

In either case, $\text{dist}(y, f^{n-j}(x)) < \text{dist}(f^{n-j}(x), \mathcal{S})/2$ because we chose $b < 1/2$ and $\delta_1 < \delta/4 < 1/4$. Therefore, we may use (S2) to conclude that

$$\log \frac{\|Df(y)^{-1}\|}{\|Df(f^{n-j}(x))^{-1}\|} \leq B \frac{\text{dist}(y, f^{n-j}(x))}{\text{dist}(f^{n-j}(x)), \mathcal{S})^\beta} \leq B \frac{2\delta_1\sigma^{j/2}}{\min\{\sigma^{b\beta j}, \delta^\beta\}}.$$

Since δ and σ are smaller than 1, and we took $b\beta < 1/2$, the term on the right hand side is bounded by $2B\delta_1\delta^{-\beta}$. Moreover, our second condition on δ_1 means that this last expression is smaller than $\log \sigma^{-1/2}$. □

Starting the induction argument to prove Lemma 5.2, we note that for $j = 1$ the Claim gives

$$\|Df(y)^{-1}\| \leq \sigma^{-1/2} \|Df(f^{n-1}(x))^{-1}\| \leq \sigma^{1/2},$$

since n is a hyperbolic time for x. This means that f is a $\sigma^{-1/2}$-dilation in the ball of radius $2\delta_1\sigma^{1/2}$ around $f^{n-1}(x)$. As a consequence, there exists some neighbourhood $V(n-1)$ of $f^{n-1}(x)$ contained in that ball of radius $2\delta_1\sigma^{1/2}$, that is mapped diffeomorphically onto the ball of radius δ_1 around $f^n(x)$.

Now, given any $j > 1$, let us suppose that we have constructed a neighbourhood $V(n-j+1)$ of $f^{n-j+1}(x)$ such that the restriction of f^{j-1} to $V(n-j+1)$ is a diffeomorphism onto the ball of radius δ_1 around $f^n(x)$, with

$$\|Df(f^i(z))^{-1}\| \leq \sigma^{-1/2} \|Df(f^{n-j+i+1}(x))^{-1}\| \tag{24}$$

for all z in $V(n-j+1)$ and $0 \leq i < j$. Then, by the Claim and the hypothesis that n is a hyperbolic time for x,

$$\|Df^j(y)^{-1}\| \leq \prod_{i=0}^{j-1} \|Df(f^i(y))^{-1}\| \leq \prod_{i=0}^{j-1} \sigma^{-1/2} \|Df(f^{n-j+i}(x))^{-1}\| \leq \sigma^{j/2}$$

for any point y in the ball of radius $2\delta_1\sigma^{j/2}$ whose image $z = f(y)$ is in $V(n-j+1)$.

Now we can construct an inverse branch of f^j on the ball of radius δ_1 around $f^n(x)$, by lifting geodesics in the following way. Given a geodesic γ connecting $f^n(x)$ to a point in the boundary of the ball, there is a well defined lift of the restriction of γ to a small neighbourhood of $f^n(x)$, into a curve starting at $f^{n-j}(x)$. Moreover, as far as this curve does not leave the ball of radius $2\delta_1\sigma^{j/2}$, the derivative on it is a $\sigma^{-j/2}$-dilation. This means that the length of the lifted curve is less than $\delta_1\sigma^{j/2}$, and so the curve is actually contained in a smaller ball. This proves that the lift is well defined on the whole geodesic γ. Thus, we have a well defined branch of f^{-j} on the ball of radius δ_1 around $f^n(x)$ as we claimed. We call $V(n-j)$ the image of that inverse branch. By construction, $V(n-j)$ is contained in the $2\delta_1\sigma^{j/2}$-ball around $f^{n-j}(x)$ and its image under f coincides with $V(n-j+1)$. So, in view of the Claim, we also recovered the induction assumption (24) for points in $V(n-j)$ and times $0 \le i \le j$.

In this way, we construct neighbourhoods $V(n-j)$ of $f^{n-j}(x)$ as above, for all $1 \le j \le n$. The lemma follows taking $V_x = V(0)$. $\square$

Corollary 5.3. *There exists $C_5 > 0$ such that for every $x \in M \setminus \mathcal{S}_\infty$, any n that is a (σ, δ)-hyperbolic time for x, and every $y, z \in V_x$*

$$\frac{1}{C_5} \le \frac{|\det Df^n(y)|}{|\det Df^n(z)|} \le C_5\,.$$

Proof. By construction, for $0 \le k < n$, the distance from $f^k(x)$ to either $f^k(y)$ or $f^k(z)$ is less than $\delta_1\sigma^{(n-k)/2}$, which is much smaller than $\sigma^{b(n-k)} \le \operatorname{dist}(f^k(x), \mathcal{S})$. So, assumption (S3) implies

$$\log\frac{|\det Df^n(y)|}{|\det Df^n(z)|} = \sum_{k=0}^{n-1}\log\frac{|\det Df(f^k(y))|}{|\det Df(f^k(z))|} \le \sum_{k=0}^{n-1} 2B\frac{\delta_1\sigma^{(n-k)/2}}{\sigma^{b\beta(n-k)}}.$$

Now, it suffices to take $C_5 \ge \exp\left(\sum_{i=1}^{\infty} 2B\delta_1\sigma^{(1/2-b\beta)i}\right)$, recall that $b\beta < 1/2$. $\square$

Let $\sigma < 1$ be fixed. The assumptions of Theorem C imply that, if σ is close enough to 1, then the set $H(\sigma)$ of points $x \in M \setminus \mathcal{S}_\infty$ for which the limit in (5) is less than $3\log\sigma$ has positive Lebesgue measure. The next lemma asserts that points in $H(\sigma)$ have many (σ, δ)-hyperbolic times.

Lemma 5.4. *There are $\theta > 0$ and $\delta > 0$, depending only on σ and on the map f, such that given any $x \in H(\sigma)$ and any sufficiently large $N \ge 1$ there exist (σ, δ)-hyperbolic times $1 \le n_1 < \cdots < n_l \le N$ for x, with $l \ge \theta N$.*

Proof. The strategy is to use Lemma 3.1 twice, first for the sequence given by $a_j = -\log\|Df(f^{j-1}(x))^{-1}\|$ (up to a cut off that makes it bounded from above), and then with $a_j = \log\operatorname{dist}_\delta(f^{j-1}(x), \mathcal{S})$ for a convenient $\delta > 0$.

We prove that there exist many times n_i for which the conclusion of Pliss' Lemma 3.1 holds, simultaneously, for both sequences. Then we check that any such n_i is a (σ, δ)-hyperbolic time for x.

Let $x \in H(\sigma)$. By definition of $H(\sigma)$, for every large N we have

$$\sum_{j=0}^{N-1} -\log \|Df(f^j(x))^{-1}\| \geq 2|\log \sigma| N .$$

Fix any $\rho > \beta$. Then (S1) implies that

$$\Big| \log \|Df(x)^{-1}\| \Big| \leq \rho \,|\log \operatorname{dist}(x, \mathcal{S})| \tag{25}$$

for every x in a neighbourhood V of $\mathcal{S}$. Fix $\varepsilon_1 > 0$ so that $\rho\varepsilon_1 \leq |\log \sigma|/2$, and let $r_1 > 0$ be small enough so that

$$\sum_{j=0}^{N-1} -\log \operatorname{dist}_{r_1}(f^j(x), \mathcal{S}) \leq \varepsilon_1 N . \tag{26}$$

Assumption (6) ensures that this is possible. Fix any $H_1 \geq \rho\,|\log r_1|$ large enough so that it is also an upper bound for $-\log \|Df^{-1}\|$ on the complement of V. Then let E be the subset of times $1 \leq j \leq N$ such that $-\log \|Df(f^{j-1}(x))^{-1}\| > H_1$, and define

$$a_j = \begin{cases} -\log \|Df(f^{j-1}(x))^{-1}\| & \text{if } j \notin E \\ 0 & \text{if } j \in E. \end{cases}$$

By construction, $a_j \leq H_1$ for $1 \leq j \leq N$. Note that if $j \in E$ then $f^{j-1}(x) \in V$. Moreover, for $j \in E$ we have $\operatorname{dist}(f^{j-1}(x), \mathcal{S}) < r_1$:

$$\rho\,|\log r_1| \leq H_1 < -\log \|Df(f^{j-1}(x))^{-1}\| < \rho\,|\log \operatorname{dist}(f^{j-1}(x), \mathcal{S})|.$$

In particular, $\operatorname{dist}_{r_1}(f^{j-1}(x), \mathcal{S}) = \operatorname{dist}(f^{j-1}(x), \mathcal{S}) < r_1$ for all $j \in E$. Therefore, by (25) and (26),

$$\sum_{j\in E} -\log \|Df(f^{j-1}(x))^{-1}\| \leq \rho \sum_{j\in E} |\log \operatorname{dist}(f^{j-1}(x), \mathcal{S})| \leq \rho\,\varepsilon_1 N.$$

We have chosen ε_1 in such a way that the last term is less than $|\log \sigma| N/2$. As a consequence,

$$\sum_{j=1}^{N} a_j = \sum_{j=1}^{N} -\log \|Df(f^{j-1}(x))^{-1}\| - \sum_{j\in E} -\log \|Df(f^{j-1}(x))^{-1}\| \geq \frac{3}{2}\,|\log \sigma| N .$$

Thus, we have checked that we may apply Lemma 3.1 to a_j, with $c_1 = |\log\sigma|$, $c_2 = 3|\log\sigma|/2$, and $A = H_1$. The lemma provides $\theta_1 > 0$ and $l_1 \geq \theta_1 N$ times $1 \leq p_1 < \cdots < p_{l_1} \leq N$ such that

$$\sum_{j=n+1}^{p_i} -\log\|Df(f^{j-1}(x))^{-1}\| \geq \sum_{j=n+1}^{p_i} a_j \geq (p_i - n)\,|\log\sigma| \tag{27}$$

for every $0 \leq n < p_i$ and $1 \leq i \leq l_1$.

Now fix $\varepsilon_2 > 0$ small enough so that $\varepsilon_2/(b|\log\sigma|) < \theta_1$, and let $r_2 > 0$ be such that, cf. condition (6),

$$\sum_{j=0}^{N-1} \log \operatorname{dist}_{r_2}(f^j(x), \mathcal{S}) \geq -\varepsilon_2 N\,.$$

Let $c_1 = b\log\sigma$, $c_2 = -\varepsilon_2$, $A = 0$, and

$$\theta_2 = \frac{c_2 - c_1}{A - c_1} = 1 - \frac{\varepsilon_2}{b\,|\log\sigma|}\,.$$

Applying Lemma 3.1 to the sequence $a_j = \log\operatorname{dist}_{r_2}(f^{j-1}(x), \mathcal{S})$, we conclude that there are $l_2 \geq \theta_2 N$ times $1 \leq q_1 < \cdots < q_{l_2} \leq N$ such that

$$\sum_{j=n}^{q_i-1} \log\operatorname{dist}_{r_2}(f^j(x), \mathcal{S}) \geq b\log\sigma\,(q_i - n) \tag{28}$$

for every $0 \leq n < q_i$ and $1 \leq i \leq l_2$.

Finally, our condition on ε_2 means that $\theta_1 + \theta_2 > 1$. Let $\theta = \theta_1 + \theta_2 - 1$. Then there exist $l = (l_1 + l_2 - N) \geq \theta N$ times $1 \leq n_1 < \cdots < n_l \leq N$ at which (27) and (28) occur simultaneously:

$$\sum_{j=n}^{n_i-1} \log\|Df(f^j(x))^{-1}\| \leq (n_i - n)\log\sigma$$

and

$$\sum_{j=n}^{n_i-1} \log\operatorname{dist}_{r_2}(f^j(x), \mathcal{S}) \geq b\log\sigma(n_i - n),$$

for every $0 \leq n < n_i$ and $1 \leq i \leq l$. Therefore, given $1 \leq i \leq l$ and $1 \leq k \leq n_i$,

$$\prod_{j=n_i-k+1}^{n_i} \|Df^{-1}(f^j(x))\| \leq \sigma^k \quad \text{and} \quad \operatorname{dist}_{r_2}(f^{n_i-k}(x), \mathcal{S}) \geq \sigma^{bk}.$$

In other words, all those n_i are (σ, δ)-hyperbolic times for x, for $\delta = r_2$. $\square$

Now we prove Theorem C, the same argument gives Corollary D.

Proof. Let H be the positive Lebesgue measure set in the statement, and $H(\sigma)$ be as above.

Lemma 5.5. *Suppose σ is close enough to 1 so that $H(\sigma)$ has positive Lebesgue measure. Then f admits some invariant probability measure μ_0 absolutely continuous with respect to Lebesgue measure and giving positive weight to $H(\sigma)$.*

Proof. According to Lemma 5.4, there exists $\delta > 0$ depending only on σ, such that for any point x in $H(\sigma)$ there exist many (σ, δ)-hyperbolic times. We let μ_n be the averages of the positive iterates of Lebesgue measure on M, and ν_n be part of μ_n carried on disks of radius δ_1 around points $f^j(x)$ such that $1 \le j \le n$ is a (σ, δ)-hyperbolic time for x. More precisely, arguing as in Lemma 3.4 and Proposition 3.3, we may find for each $j \ge 1$ a finite set of points $x_1, \dots, x_N$ admitting j as a (σ, δ)-hyperbolic time, such that

1. $V_{x_1}, \dots, V_{x_N}$ are two-by-two disjoint;
2. the Lebesgue measure of $W_j = V_{x_1} \cup \dots \cup V_{x_N}$ is larger than the Lebesgue measure of the set of points in $H(\sigma)$ having j as a (σ, δ)-hyperbolic time, up to a uniform multiplicative constant $\tau > 0$.

Then we take

$$\nu_n = \frac{1}{n} \sum_{j=0}^{n-1} f_*^j(\mathrm{Leb} \mid W_j).$$

As before in Proposition 3.5, each of these ν_n has total mass bounded away from zero, in fact, $\nu_n(H(\sigma)) \ge \alpha$ for some uniform $\alpha > 0$. Moreover, as a consequence of the distortion Corollary 5.3, every $f_*^j(\mathrm{Leb} \mid W_j)$ is absolutely continuous with respect to Lebesgue measure, with density uniformly bounded from above, and so the same is true for every ν_n.

Now take $n_k \to \infty$ such that both μ_{n_k} and ν_{n_k} converge to measures μ and ν, respectively, in the weak* sense. Then μ is an invariant probability measure, $\mu = \nu + \eta$ for some measure η, ν is absolutely continuous with respect to Lebesgue measure, and $\nu(H(\sigma)) \ge \alpha > 0$. Now, if $\eta = \eta_{ac} + \eta_s$ denotes the Lebesgue decomposition of η (as the sum of an absolutely continuous and a completely singular measure, with respect to Lebesgue measure), then $\mu_{ac} = \nu + \eta_{ac}$ gives the absolutely continuous component in the corresponding decomposition of μ. By uniqueness of the Lebesgue decomposition, and the fact that the push-forward under f preserves the class of absolutely continuous measures, we may conclude that μ_{ac} is an invariant measure. Clearly, $\mu_{ac}(H(\sigma)) \ge \nu(H(\sigma)) > 0$. □

Up to replacing μ_0 by its normalized restriction to the (positively invariant) set $H(\sigma)$ we may suppose that $\mu_0(H(\sigma)) = 1$. The next lemma will allow us to show that $H(\sigma)$ is covered by the basins of finitely many ergodic absolutely continuous invariant measures.

Lemma 5.6. *For any positively invariant set $G \subset H(\sigma)$ there exists some disk Δ with radius $\delta_1/4$ such that* $\operatorname{Leb}(\Delta \setminus G) = 0$.

Proof. It suffices to prove that there exist disks of radius $\delta_1/4$ where the relative measure of G is arbitrarily close to 1.

Let ϵ be some small number, G_c be a compact subset of G, and G_o be a neighbourhood of G_c such that both $G \setminus G_c$ and $G_o \setminus G_c$ have Lebesgue measure less than $\epsilon \operatorname{Leb}(G)$. By Lemma 5.4 and Fubini's theorem, there exist arbitrarily large values of $j \geq 1$ such that the Lebesgue measure of the subset G_j of points in G for which j is a (σ, δ)-hyperbolic time has Lebesgue measure is at least $\theta \operatorname{Leb}(G)$. So, as long as ϵ is fixed sufficiently small, the Lebesgue measure of $G_c \cap G_j$ is larger than $(\theta/2) \operatorname{Leb}(G)$. Assume that j is large enough so that for any point x in $G_c \cap G_j$, the neighbourhood V_x is contained in G_o. Here V_x is the neighbourhood of x constructed in Lemma 5.2: it is mapped diffeomorphically onto the ball of radius δ_1 around $f^j(x)$ by f^j. Let $W_x \subset V_x$ be the pre-image of the ball of radius $\delta_1/4$ under this diffeomorphism. Let $x_1, \ldots, x_N \in G_c \cap G_j$ be such that $W_{x_1}, \ldots, W_{x_N}$ cover the compact set $G_c \cap G_j$. Up to reordering, we may suppose that $W_{x_1}, \ldots, W_{x_n}$, some $n \leq N$, is a maximal sub-family whose elements are two-by-two disjoint. Notice that the $V_{x_1}, \ldots, V_{x_n}$ cover $G_c \cap G_j$, since their union contains every W_{x_i}, $1 \leq i \leq N$. Indeed, every W_{x_i} must intersect some W_{x_k} with $k \leq n$. Then its image under f^j intersects the ball of radius $\delta_1/4$ around $f^j(x_k)$ and so it is contained in the corresponding ball of radius δ_1. This means, precisely, that W_{x_i} is contained in V_{x_k}.

By the bounded distortion property, $\operatorname{Leb}(W_x)$ is larger than the product of $\operatorname{Leb}(V_x)$ by some uniform constant $\tau > 0$ (independent of x or j). So, the Lebesgue measure of $W_{x_1} \cup \cdots \cup W_{x_n}$ is larger than $\tau \operatorname{Leb}(G_c \cap G_j)$. If $\xi > 0$ is such that $\operatorname{Leb}(W_{x_i} \setminus (G_c \cap G_j)) \geq \xi \operatorname{Leb}(W_{x_i})$ for every $1 \leq i \leq n$, then

$$\operatorname{Leb}\left(W_{x_1} \cup \cdots \cup W_{x_n}) \setminus (G_c \cap G_j)\right) \geq \xi\tau \operatorname{Leb}(G_c \cap G_j) \geq \xi\tau\theta \operatorname{Leb}(G) \; .$$

On the other hand, since the W_{x_i} are contained in G_o and $G_c \cap G_j \subset G$, this measure must be smaller than $\epsilon \operatorname{Leb}(G)$. This means that by reducing ϵ (which we may, by increasing j), we can force ξ to be arbitrarily small. In other words, we may find j and W_{x_i} such that the relative Lebesgue measure of $W_{x_i} \cap G_c \cap G_j$ in W_{x_i} is arbitrarily close to 1. Then, by bounded distortion, the relative Lebesgue measure of $G \supset f^j(G_c \cap G_j)$ in the ball of radius $\delta_1/4$ around $f^j(x_i)$ is also arbitrarily close to 1. So the proof of the lemma is complete. $\square$

Finally, we may conclude the proof of Theorem C and Corollary D.

Let μ_0 be any absolutely continuous invariant measure with $\mu_0(H(\sigma)) = 1$. If μ_0 is not ergodic then we may decompose $H(\sigma)$ into two disjoint invariant sets H_1, H_2 both with positive μ_0-measure. In particular, both H_1 and H_2 have positive Lebesgue measure. Let μ_1, μ_2 be the normalized restrictions of μ_0 to H_1, H_2, respectively. Clearly, they are also absolutely continuous invariant measures. If they are not ergodic, we continue decomposing them, in the same way as we did for μ_0. On the other hand, by Lemma 5.6, each one of the invariant sets we find in this decomposition has full Lebesgue measure in some disk with fixed radius. Since these disks must be disjoint, and the ambient manifold is compact, there can only be finitely many of them. So, the decomposition must stop after a finite number of steps, giving that μ_0 can be written $\mu_0 = \sum_{i=1}^{s} \mu_0(H_i)\mu_i$ where $H_1, \ldots, H_s$ is a partition of $H(\sigma)$ into invariant sets with positive measure and each $\mu_i(\cdot) = \mu_0(\cdot \cap H_i)/\mu_0(H_i)$ is an ergodic measure.

This completes the proof of Theorem C and Corollary D, and it also gives the finiteness result stated in Subsect. 1.1 right after the corollary. □

Having in mind important classes of maps with singularities, such as Poincaré return maps of singular (or generalized Lorenz) attractors, [ABS77,GW79,Rov93,MPP98,BPV97] we now propose a natural extension of the previous results for *partially hyperbolic maps with singularities*.

Let us a manifold M, a compact subset $\mathcal{S}$, and a C^2 diffeomorphism (onto its image) $f : M \setminus \mathcal{S} \to M$. We suppose f has a compact positively invariant subset K, in the sense that $f(K \setminus \mathcal{S}) \subset K$, such that the tangent bundle of M restricted to $K \setminus \mathcal{S}$ has a Df-invariant dominated splitting $T_{K\setminus\mathcal{S}}M = E^{ss} \oplus E^{cu}$ such that E^{ss} is uniformly contracting.

We assume that f behaves like a power of the distance to $\mathcal{S}$ *along the centre-unstable* direction: for every $x \in K \setminus \mathcal{S}$ and $v \in E^{cu}_x$

(R1) $\dfrac{1}{B}\,\mathrm{dist}(x,\mathcal{S})^{\beta} \le \dfrac{\|Df(x)v\|}{\|v\|} \le B\,\mathrm{dist}(x,\mathcal{S})^{-\beta}$;

(R2) $\|D(Df(x))\| \le B\,\mathrm{dist}^{cu}(x,\mathcal{S})^{-\beta}$ and
$\|D(Df(x)^{-1})\| \le B\,\mathrm{dist}^{cu}(x,\mathcal{S})^{-\beta}$

(R3) $|\log\|Df^{-1} \mid E^{cu}_{f(x)}\| - \log\|Df^{-1} \mid E^{cu}_{f(y)}\|| \le B\frac{\mathrm{dist}^{cu}(x,y)}{\mathrm{dist}^{cu}(x,\mathcal{S})^{\beta}}$, if x and y are in a same disk tangent to the centre-unstable cone field, and $\mathrm{dist}^{cu}(x,y) < \mathrm{dist}^{cu}(x,\mathcal{S})/2$.

Here dist^{cu} denotes the shortest distance measured along curves tangent to the centre-unstable cone field, and we also define the truncated version $\mathrm{dist}^{cu}_{\delta}$ of dist^{cu}, in the same way as in (4). Let $\mathcal{S}_\infty = \cup_{n=-\infty}^{+\infty} f^n(\mathcal{S})$.

Although we did not try to check all the details, it seems that the following statement can be obtained by combining the arguments in the proofs of Theorems A and C:

Let f be as above, and assume that it is non-uniformly expanding along the centre-unstable direction, in the sense that (2) holds for all x in a positive Lebesgue measure set $H \subset M \setminus \mathcal{S}_\infty$ Assume moreover that, given any $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in H$

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} -\log \mathrm{dist}^{cu}_{\delta}(f^j(x),\mathcal{S}) \le \varepsilon.$$

Then Lebesgue almost every point in H is in the basin of some SRB measure.

The main technical point in giving a full proof, is to control the curvature of iterates of disks tangent to the centre-unstable cone field, similarly to what we did in Subsect. 2.1.

6. Diffeomorphisms with a dominated splitting

Let $f : M \to M$ be a $C^{1+\zeta}$ diffeomorphism on a manifold M. Here we suppose that f has a compact positively invariant set $K \subset M$ with a continuous invariant *dominated splitting* $T_K M = E^{cs} \oplus E^{cu}$: there exists a constant $\lambda < 1$ and some choice of a Riemannian metric on M such that

- $\|Df \mid E^{cs}_x\| \cdot \|Df^{-1} \mid E^{cu}_{f(x)}\| \le \lambda$ for all $x \in K$.

We call E^{cs} centre-stable subbundle and E^{cu} centre-unstable subbundle.

As we did at the beginning of Sect. 2, we can extend the subbundles continuously to a neighbourhood V_0 of K, and then consider cone fields C^{cs}_a and C^{cu}_a with small width $a > 0$ around these extended subbundles. As before, we assume that f is non-uniformly expanding along the centre-unstable direction:

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \log \left\| Df^{-1} \middle| E^{cu}_{f^j(x)} \right\| < 0 \tag{29}$$

for every x in a positive Lebesgue measure subset H of $D \cap K$, where D is some C^2 disk tangent to the centre-unstable cone field. We fix any $\sigma < 1$ such that $H(\sigma)$, defined as in Sect. 3, has positive Lebesgue measure in D.

Everything we did in Sect. 2 through Subsect. 4.1 applies immediately in this context. So, cf. Proposition 4.1, there exist measures μ, ν, η, a cylinder $\mathcal{C}$ and a family $\mathcal{K}_\infty$ of disjoint disks crossing $\mathcal{C}$ such that

1. μ is an invariant probability measure and $\mu = \nu + \eta$;
2. the disks in $\mathcal{K}_\infty$ are accumulated by sub-disks of radius δ_1 in $f^n(D)$, around points $f^n(x)$ such that n is a σ-hyperbolic time for $x \in H(\sigma)$;
3. the union K_∞ of all the disks in $\mathcal{K}_\infty$ intersects K in a set with positive ν-measure;
4. the restriction of ν to K_∞ has absolutely continuous conditional measures along the disks in $\mathcal{K}_\infty$.

Let us recall a few well-known notions and facts that are useful for the proof of the next lemma. Given a point x, let us denote μ_x the probability measure given by the time average along the orbit of x:

$$\int \varphi \, d\mu_x = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) \quad \text{for every continuous } \varphi : M \to \mathbb{R}. \tag{30}$$

According to the ergodic decomposition theorem, see [Mañ87, Section 2.6], μ_x is well-defined and ergodic for every x in a set $\Sigma \subset M$ that has full measure with respect to any invariant measure ξ. Moreover, $x \mapsto \int g\, d\mu_x$ is measurable and

$$\int g\, d\xi = \int \left(\int g\, d\mu_x \right) d\xi(x)$$

for every measurable bounded function $g : M \to \mathbb{R}$. In fact, for any such g the integral $\int g\, d\mu_x$ coincides almost everywhere with the time average of g over the orbit of x.

Let R be the set of regular points of f, as introduced in Subsect. 4.2: $z \in R$ if and only if the forward and backward time averages of each continuous function over the orbit of z exist and coincide. R has full measure for any f-invariant probability measure ξ, as a consequence of Birkhoff's ergodic theorem. Let us point out that μ_x is constant on the intersection of R with every disk γ of $\mathcal{K}_\infty$. This is because these disks are (exponentially) contracted by negative iterates, cf. property 2. above and Lemmas 2.7 and 3.7, and so points in a same $\gamma \in \mathcal{K}_\infty$ have the same backward average (hence points in $R \cap \gamma$ also have the same forward average) for each continuous function φ.

Lemma 6.1. *There exists $z \in K_\infty \cap K \cap \Sigma \cap R$ such that $\mu_z(K_\infty \cap K) > 0$ and μ_z has absolutely continuous conditional measures along the disks in $\mathcal{K}_\infty$. In particular, the support of μ_z is contained in $\cap_{j=0}^{\infty} f^j(K)$.*

Proof. Fix B to be some measurable subset of M such that

$$m_\gamma(B \cap \gamma) = 0 \quad \text{for every} \quad \gamma \in \mathcal{K}_\infty, \tag{31}$$

and $\mu(B)$ is maximal among all measurable sets with this property. For instance, $B = \cup_n B_n$ where the $B_n, n \geq 1$, are measurable sets with property (31) such that $\mu(B_n)$ converges to the largest value compatible with that property. Observe that $\nu(B) = 0$, because ν is absolutely continuous along the leaves of $\mathcal{K}_\infty$. Let $Z_\infty = K_\infty \cap K \cap \Sigma \cap R \setminus B$. Then,

$$\mu(Z_\infty) \geq \nu(Z_\infty) = \nu(K_\infty \cap K \cap \Sigma \cap R) = \nu(K_\infty \cap K) > 0.$$

Let $(\mu \mid Z_\infty)$ be the restriction of μ to Z_∞: by definition $(\mu \mid Z_\infty)(E) = \mu(E \cap Z_\infty)$ for any measurable set E in M.

Let A be any measurable subset of Z_∞ such that $m_\gamma(A \cap \gamma) = 0$ for every $\gamma \in \mathcal{K}_\infty$. Then $\mu(A)$ must be zero, since we took $\mu(B)$ maximal. This means that $(\mu \mid Z_\infty)$ is absolutely continuous with respect to the product $m_\gamma \times \hat{\mu}$, where $\hat{\mu}$ stands for the quotient measure induced by $(\mu \mid Z_\infty)$ on $\mathcal{K}_\infty$. As a consequence, the conditional measures $\tilde{\mu}_\gamma$ of $(\mu \mid Z_\infty)$ on the disks $\gamma \in \mathcal{K}_\infty$ are absolutely continuous with respect to Lebesgue measure

m_γ for $\hat{\mu}$-almost all $\gamma \in \mathcal{K}_\infty$. On the other hand, for any measurable set $A \subset Z_\infty$,

$$\mu(A) = \int \mu_x(A)\, d\mu(x) \tag{32}$$

where the integral is taken over M or, more precisely, over the full measure subset Σ. We want to express this in terms of an integral over Z_∞. As we mentioned before,

$$\mu_x(A) = \int \mathcal{X}_A\, d\mu_x = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathcal{X}_A(f^j(x))$$

almost everywhere, with respect to any invariant measure. So, up to disregarding a zero μ-measure set of points, $\mu_x(A)$ can be non-zero only if x has some iterate in $A \subset Z_\infty$. Let $k(z)$ denote the first backward return time of a point $z \in Z_\infty$, that is, the smallest positive integer such that $f^{-k(z)}(z) \in Z_\infty$. This is defined μ-almost everywhere, by Poincaré's recurrence theorem. Observe also that $\mu_z = \mu_{f^j(z)}$ for any z and any integer j. Thus, we can rewrite (32) as

$$\mu(A) = \int_{Z_\infty} k(z)\mu_z(A)\, d\mu(z)$$

for any measurable subset A of Z_∞. The next lemma can be inferred from §3 of Rokhlin [Rok62]. For the reader's convenience, we state it explicitly and prove it, after completing the proof of Lemma 6.1.

Lemma 6.2. *Let λ be a finite measure on a measure space Z, with $\lambda(Z) > 0$. Let $\mathcal{K}$ be a measurable partition of Z, and $(\lambda_z)_{z\in Z}$ be a family of finite measures on Z such that*

1. *the function $z \mapsto \lambda_z(A)$ is measurable, and it is constant on each element of $\mathcal{K}$, for any measurable set $A \subset Z$*
2. *$\{w : \lambda_z = \lambda_w\}$ is a measurable set with full λ_z-measure, for every $z \in Z$.*

Assume that $\lambda(A) = \int \ell(z)\lambda_z(A)\, d\lambda$ for some measurable function $\ell : Z \to \mathbb{R}_+$ and any measurable subset A of Z. Let $\{\tilde{\lambda}_\gamma, \gamma \in \mathcal{K}\}$, and $\{\tilde{\lambda}_{z,\gamma}, \gamma \in \mathcal{K}\}$, be disintegrations of λ and λ_z, respectively, into conditional probability measures along the elements of the partition $\mathcal{K}$. Then

$$\tilde{\lambda}_{z,\gamma} = \tilde{\lambda}_\gamma$$

for λ-almost every $z \in Z$ and $\hat{\lambda}_z$-almost every γ, where $\hat{\lambda}_z$ is the quotient measure induced by λ_z on $\mathcal{K}$.

We take $Z = Z_\infty$, $\lambda = (\mu \mid Z_\infty)$, $\mathcal{K} = \mathcal{K}_\infty$, $\lambda_z = (\mu_z \mid Z_\infty)$, and $\ell(z) = k(z)$, for each $z \in Z_\infty$. It is easy to check that the hypotheses of Lemma 6.1 are satisfied. The first part of assumption 1 is contained in the ergodic decomposition theorem, and the second part follows from (30), as we explained before. Let $\mathcal{D}$ be any countable dense subset of the space of continuous functions on M. Given $z, w \in Z_\infty$, then $\mu_z = \mu_w$ if and only for every $\varphi \in \mathcal{D}$ and every $p \geq 1$ there exists $q \geq 1$ such that

$$\left| \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^{-j}(z)) - \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^{-j}(w)) \right| < \frac{1}{p} \qquad \text{for any } n \geq p\,.$$

This gives the measurability condition in assumption 2. In this case the last part of assumption 2 is just a restatement of the fact that $\lambda_z = \mu_z$ is ergodic.

Then, according to Lemma 6.2, the conditional probability measures $\tilde{\mu}_{z,\gamma}$ of $(\mu_z \mid Z_\infty)$ along the disks $\gamma \in \mathcal{K}_\infty$ coincide almost everywhere with the corresponding conditional measures $\tilde{\mu}_\gamma$ of $(\mu \mid Z_\infty)$. Recall that we had already shown that the latter are almost everywhere absolutely continuous with respect to Lebesgue measure m_γ. We also have that

$$\int_{Z_\infty} k(z)\mu_z(Z_\infty)\, d\mu = \mu(Z_\infty) > 0.$$

It follows that there exists a positive μ-measure subset of points $z \in Z_\infty$ such that $\mu_z(K_\infty \cap K) \geq \mu_z(Z_\infty) > 0$, and the restriction of μ_z to Z_∞ has conditional measures with respect to $\mathcal{K}$ that are μ_z-almost everywhere absolutely continuous with respect to Lebesgue measure on the corresponding disk $\gamma \in \mathcal{K}$. Thus, any such z satisfies the first two claims in the statement of the lemma.

Finally, since K is compact and positively invariant, ergodicity implies that the support of μ_z is contained in $\cap_{j=0}^{\infty} f^j(K)$. □

Now we prove Lemma 6.2:

Proof. The idea is quite simple. Let $\mathcal{E}$ be the partition of the set Z into equivalence classes for the equivalence relation $z \sim w \Leftrightarrow \lambda_z = \lambda_w$. Assumption 2 ensures that the elements of $\mathcal{E}$ are measurable sets, and assumption 1 implies that every $\gamma \in \mathcal{K}$ is contained in some element of $\mathcal{E}$. Given any $e \in \mathcal{E}$ we define $\lambda_e = \lambda_z$, where z is an arbitrary point in e. We show that, up to normalization, $\{\lambda_e,\, e \in \mathcal{E}\}$ is a disintegration of λ with respect to the partition $\mathcal{E}$. Now, $\{\tilde{\lambda}_{e,\gamma} = \tilde{\lambda}_{z,\gamma},\, \gamma \in \mathcal{K},\, \gamma \subset e\}$ is a disintegration of $\lambda_e = \lambda_z$, with respect to the partition induced by $\mathcal{K}$ on each $e \in \mathcal{E}$. It follows that $\{\tilde{\lambda}_{e,\gamma},\, \gamma \in \mathcal{K},\, \gamma \subset e,\, e \in \mathcal{E}\}$ is a disintegration of λ with respect to $\mathcal{K}$ (obtained by conditioning first to $\mathcal{E}$, then to $\mathcal{K}$). By (essential) uniqueness of the disintegration into conditional probability measures, we must have $\tilde{\lambda}_{z,\gamma} = \tilde{\lambda}_\gamma$ almost everywhere.

Now we give the detailed argument. Let $\pi_\gamma : Z \to \mathcal{K}$ and $\pi_e : \mathcal{K} \to \mathcal{E}$ be the canonical projections. We represent by $\mathcal{B}(\mathcal{E})$ the σ-algebra generated by $\mathcal{E}$. Let $g : Z \to \mathbb{R}_+$ be a conditional expectation of ℓ relative to $\mathcal{E}$, that is, a Radon-Nikodym derivative, with respect to the restriction of λ to $\mathcal{B}(\mathcal{E})$, of the measure defined by

$$\mathcal{B}(\mathcal{E}) \ni E \mapsto \int_E \ell \, d\lambda.$$

In other words, g is a $\mathcal{B}(\mathcal{E})$-measurable function satisfying

$$\int_E \ell \, d\lambda = \int_E g \, d\lambda \quad \text{for every } E \in \mathcal{B}(\mathcal{E}). \tag{33}$$

$\mathcal{B}(\mathcal{E})$-measurability implies that g is constant on elements of $\mathcal{E}$. Set $g(e) = g(z)$ for any $e \in \mathcal{E}$ and $z \in e$. Let us consider the set

$$\{h : Z \to \mathbb{R} \text{ such that } \int \ell h \, d\lambda = \int g h \, d\lambda\}.$$

By (33), every characteristic function of an element of $\mathcal{B}(\mathcal{E})$ is in this set. Using linearity of the integral and the dominated convergence theorem, we conclude that the set contains any bounded $\mathcal{B}(\mathcal{E})$-measurable function. In particular, it contains $h(z) = \lambda_z(A)$, for any measurable set $A \subset Z$. Therefore,

$$\lambda(A) = \int \ell(z)\lambda_z(A) \, d\lambda(z) = \int g(z)\lambda_z(A) \, d\lambda(z) = \int g(e)\lambda_e(A) \, de, \tag{34}$$

where $de = (\pi_e \circ \pi_\gamma)_* \lambda$ is the quotient measure induced by λ on $\mathcal{E}$. Assumption 2 implies that

$$g(e)\lambda_e(Z \setminus e) = 0 \quad \text{for any } e \in \mathcal{E}. \tag{35}$$

Then,

$$g(e)\lambda_e(Z) = g(e)\lambda_e(e) = 1 \quad \text{for } de\text{-almost all } e \in \mathcal{E}. \tag{36}$$

Indeed, let $\delta > 0$ and F_δ be the set of all $e \in \mathcal{E}$ for which $g(e)\lambda_e(Z) \geq 1 + \delta$. Denote $E_\delta = (\pi_e \circ \pi_\gamma)^{-1}(F_\delta)$. Then, using (34) and (35),

$$de(F_\delta) = \lambda(E_\delta) = \int g(e)\lambda_e(E_\delta) \, de = \int_{F_\delta} g(e)\lambda_e(Z) \, de \geq (1 + \delta) de(F_\delta),$$

which implies $de(F_\delta) = 0$. Analogously, the set of $e \in \mathcal{E}$ for which $g(e)\lambda_e(Z)$ is less than $1 - \delta$ has zero de-measure for any $\delta > 0$. This proves (36). In this way, we have shown that $\{g(e)\lambda_e, e \in \mathcal{E}\}$ is a disintegration of λ with respect to the partition $\mathcal{E}$.

Now, for each $e \in \mathcal{E}$, let $\hat{\lambda}_e = (\pi_\gamma \mid e)_* \lambda_e$ be the quotient measure of λ_e on $(\mathcal{K} \mid e)$. Moreover, let $\{\tilde{\lambda}_{e,\gamma}, \gamma \in \mathcal{K}, \gamma \subset e\}$ be a disintegration of λ_e with respect to $(\mathcal{K} \mid e)$. Of course, $\tilde{\lambda}_{e,\gamma} = \tilde{\lambda}_{z,e}$ for any $z \in e$. Then

$$\lambda_e(A) = \int \tilde{\lambda}_{e,\gamma}(A)\, d\hat{\lambda}_e(\gamma) \quad \text{and so}$$
$$g(e)\lambda_e(A) = \int \tilde{\lambda}_{e,\gamma}(A)\, d(g(e)\hat{\lambda}_e)(\gamma)$$

for any measurable set $A \subset Z$. Replacing this in (34), we find

$$\lambda(A) = \int \int \tilde{\lambda}_{e,\gamma}(A)\, d(g(e)\hat{\lambda}_e)(\gamma)\, de. \tag{37}$$

Denote $d\gamma = (\pi_\gamma)_* \lambda$, the quotient measure of λ on $\mathcal{K}$. Note that $de = (\pi_e)_* d\gamma$, that is, de coincides with the quotient measure of $d\gamma$ on $\mathcal{E}$. Moreover,

$$d\gamma(\Gamma) = \lambda\big(\pi_\gamma^{-1}(\Gamma)\big) = \int g(e)\lambda_e\big(\pi_\gamma^{-1}(\Gamma)\big)\, de = \int g(e)\hat{\lambda}_e(\Gamma)\, de$$

for every measurable set $\Gamma \subset \mathcal{K}$. This means that $\{g(e)\hat{\lambda}_e, e \in \mathcal{E}\}$, is a disintegration of $d\gamma$ with respect to the partition $\mathcal{E}$. Thus (37) gives

$$\lambda(A) = \int \tilde{\lambda}_{e,\gamma}(A)\, d\gamma,$$

and so $\{\tilde{\lambda}_{e,\gamma}, \gamma \in \mathcal{K}\}$, is a disintegration of λ with respect to the partition $\mathcal{K}$. Since disintegrations into conditional probability measures, when they exist, are uniquely defined almost everywhere, it follows that $\tilde{\lambda}_{e,\gamma} = \tilde{\lambda}_\gamma$ for $d\gamma$-almost every $\gamma \in \mathcal{K}$. Equivalently, this holds for de-almost every $e \in \mathcal{E}$ and $d\hat{\lambda}_e$-almost every $\gamma \in (\mathcal{K} \mid e)$, which is just the same as the conclusion of the lemma. $\square$

Let $z \in K_\infty \cap K$ be as in Lemma 6.1. Property 2. above implies that the measure $\mu_* = \mu_z$ has $\dim E^{cu}$ Lyapunov exponents larger than $-\log\sigma$. The domination condition implies that all the other exponents are less than $-\log\sigma + \log\lambda < -\log\sigma$. So, by Pesin theory [Pes76], μ_z-almost every point x has a local strong-unstable manifold which is an embedded disk whose backward orbits approach the backward of x at the exponential rate $\log\sigma$. Moreover, the disks $\gamma \in \mathcal{K}_\infty$ contain the local strong-unstable manifolds of points in their interior.

Combining these remarks with Lemma 6.1, we get

Theorem 6.3. *Let f be a C^2 diffeomorphism admitting a positively invariant compact set with a dominated splitting $E^{cs} \oplus E^{cu}$. Assume that f is non-uniformly expanding along the centre-unstable direction, cf. (29). Then f has some ergodic Gibbs cu-state μ_* supported in $\cap_{j=0}^{\infty} f^j(K)$: μ_* is an* *invariant probability measure whose* $\dim E^{cu}$ *larger Lyapunov exponents are positive and whose conditional measures along the corresponding local strong-unstable manifolds are almost everywhere absolutely continuous with respect to Lebesgue measure on these manifolds.*

If the remaining $\dim E^{cs}$ Lyapunov exponents of μ_* are all negative then μ_* is an SRB measure. This is a well known consequence of the absolute continuity property of μ_* and absolute continuity of the stable lamination [Pes76]: the union of the stable manifolds through the points whose time averages are given by μ_* is a positive Lebesgue measure set contained in the basin of μ_*.

Clearly, the centre-stable Lyapunov exponents are indeed negative whenever the subbundle E^{cs} is uniformly contracting, which was precisely our setting in Sects. 2 through 4. In general, if one assumes that E^{cs} is non-uniformly contracting

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left\| Df \big| E^{cs}_{f^j(x)} \right\| < 0 \tag{38}$$

on a positive Lebesgue measure subset of $H \subset D$, it is not clear whether this information can be passed to a limit Gibbs *cu*-state. There are however some cases where this can be done, and so our methods do yield SRB measures supported in K.

A sufficient condition is that there exist a positive Lebesgue measure set of points in H with many (positive density at infinity) *simultaneous* σ*-hyperbolic times* with respect to the two subbundles:

$$\prod_{j=n-k+1}^{n} \left\| Df^{-1} \big| E^{cu}_{f^j(x)} \right\| \le \sigma^k \quad \text{and} \quad \prod_{j=n-k}^{n-1} \left\| Df \big| E^{cs}_{f^j(x)} \right\| \le \sigma^k$$

for every $0 \le k \le n$, for some $\sigma < 1$.

Proposition 6.4. *In the setting of Theorem 6.3, suppose that every point* H *has many (positive density at infinity) simultaneous* σ*-hyperbolic times for some* $\sigma < 1$. *Then ergodic Gibbs cu-states can be constructed as in the theorem which are SRB measures, and whose basins cover a full Lebesgue measure subset of* H.

Proof. Let $\sigma < 1$ be fixed such that the subset $H(\sigma)$ of points with many simultaneous σ-hyperbolic times has positive Lebesgue measure. Cf. (14) if n is a σ-hyperbolic time (with respect to E^{cu}) then the tangent space at every point in the ball of radius δ_1 around $f^n(x)$ in $f^n(D)$ is uniformly contracted by the first n negative iterates of f. Up to reducing δ_1, we may also suppose that, whenever n is a simultaneous σ-hyperbolic time then the centre-stable subbundle is $\sigma^{j/2}$-expanded by these iterates f^{-j}, $1 \le j \le n$, at every point in that ball. We construct measures ν_n as before, except that we take into account only simultaneous hyperbolic times. As a consequence, E^{cs} is $\sigma^{j/2}$-expanded by all negative iterates f^{-j}, $j \geq 1$, at every point in the support of the limit measure ν. In particular, any ergodic Gibbs *cu*-state for which the support of ν has positive measure must have $\dim E^{cs}$ negative Lyapunov exponents, and so it is an SRB measure.

Moreover, as we have seen in the proof of the Claim in Lemma 4.5, a definite fraction of measures ν_n is carried by disks where points in the corresponding iterate of $H(\sigma)$ occupy a subset with relative Lebesgue measure bounded away from zero. Then it is easy to verify that the content of that Claim is valid for, at least, some Gibbs *cu*-state μ_* charging the support of ν: there exists a sequence D_k of disks in $f^{j_k}(D)$ in which $f^{j_k}(H(\sigma))$ has relative Lebesgue measure bounded away from zero, converging to some disk D_∞ in the support of μ_*, tangent to the centre-unstable direction and such that almost every point in D_∞ is in the basin of μ_*. By Pesin theory (absolute continuity of the stable lamination [Pes76]) the union of the stable manifolds of these points in $D_\infty \cap B(\mu_*)$ cuts D_k in a subset with relative Lebesgue measure going to 1 as D_k approaches D_∞. In particular, since these stable manifolds are contained in $B(\mu_*)$, the basin must contain a positive Lebesgue measure subset of $H(\sigma)$. This proves that Lebesgue almost every point in $H(\sigma)$ is in the basin of some SRB measure, for every $\sigma < 1$.

$\square$

Finally, we describe a simple condition on the diffeomorphism f implying existence of many simultaneous hyperbolic times. This condition is satisfied by a non-empty C^1 open set of diffeomorphisms of the 4-torus admitting an invariant set with a dominated splitting (without uniformly hyperbolic subbundles), as will be shown in the Appendix.

Proposition 6.5. *Let f be a C^2 diffeomorphism admitting a dominated splitting $E^{cs} \oplus E^{cu}$ on some positively invariant compact set K. Let*

$$A^u = \sup_{f(K)} -\log \|Df^{-1} \mid E^{cu}\| \quad \textit{and} \quad A^s = \sup_{K} -\log \|Df \mid E^{cs}\|.$$

Suppose that there exist positive constants c^u and c^s such that

$$\frac{c^u}{A^u} + \frac{c^s}{A^s} > 1 \tag{39}$$

and

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \log \left\| Df \middle| E^{cu}_{f^j(x)} \right\| \leq -c^u,$$

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left\| Df \middle| E^{cs}_{f^j(x)} \right\| \leq -c^s$$

for some point $x \in K$. Then there exists $\sigma < 1$ such that the simultaneous σ-hyperbolic times of x have positive density at infinity.

Proof. This is a direct consequence of the expression of the density bound θ in Lemma 3.1: given σ close to 1 then, for every large N, there exist $\theta^s N$ σ-hyperbolic times with respect to E^{cs} and $\theta^u N$ σ-hyperbolic times with respect to E^{cu} in the time interval $\{1, \dots, N\}$, with

$$\theta^u = \frac{c^u + \log\sigma}{A^u + \log\sigma} \quad \text{and} \quad \theta^s = \frac{c^s + \log\sigma}{A^s + \log\sigma}.$$

If σ is close enough to 1 then $\theta^u + \theta^s > 1$ and the simultaneous hyperbolic times have density $\theta^u + \theta^s - 1 > 0$. □

A. Appendix: Applications

Here we present a few simple conditions implying the assumptions of Theorem A, Corollary D, and Propositions 6.4 and 6.5. They allow us to exhibit some robust (C^1 open) classes of maps to which these results apply.

Lemma A.1. *Given a real number σ_1 and integers $p, q \geq 1$ with $\sigma_1 > q$, there exists $\varepsilon_0 > 0$ such that the following holds. Let M be a manifold with finite volume, $f : M \to M$ be a C^1 map, and $\{B_1, \dots, B_p, B_{p+1}, \dots, B_{p+q}\}$ be a covering of M by measurable sets, such that*

1. *$|\det Df(x)| \geq \sigma_1$ for every x in $B_{p+1} \cup \cdots B_{p+q}$;*
2. *$(f \mid B_i)$ is injective for all $1 \leq i \leq p+q$.*

Then the orbit of Lebesgue almost every point $x \in M$ spends a fraction ε_0 of the time in $B_1 \cup \cdots \cup B_p$: that is, $\#\{0 \leq j < n : f^j(x) \in B_1 \cup \cdots \cup B_p\} \geq \varepsilon_0 n$ for every large n.

Proof. Let n be fixed, for the time being. Given a sequence $\underline{i} = (i_0, i_1, \dots, i_{n-1})$ in $\{1, \dots, p+q\}$, we denote

$$[\underline{i}] = B_{i_0} \cap f^{-1}(B_{i_1}) \cap \cdots \cap f^{-n+1}(B_{i_{n-1}}).$$

Moreover, we define $g(\underline{i})$ to be the number of values of $0 \leq j \leq n-1$ for which $i_j \leq p$. We begin by noting that, given any $\varepsilon_0 > 0$, the total number of sequences $\underline{i}$ for which $g(\underline{i}) < \varepsilon_0 n$ is bounded by

$$\sum_{k<\varepsilon_0 n} \binom{n}{k} p^k q^{n-k} \leq \sum_{k \leq \varepsilon_0 n} \binom{n}{k} p^{\varepsilon_0 n} q^n .$$

A standard application of Stirling's formula (see e.g. [BV99, Section 6.3]) gives that the last expression is bounded by $e^{\gamma_0 n} p^{\varepsilon_0 n} q^n$, where γ_0 depends only on ε_0 and goes to zero when ε_0 goes to zero. On the other hand, as a consequence of assumptions 1 and 2, $\mathrm{Leb}([\underline{i}]) \leq \mathrm{Leb}(M)\, \sigma_1^{-(1-\varepsilon_0)n}$. Then the measure of the union I_n of all the sets $[\underline{i}]$ with $g(\underline{i}) < \varepsilon_0 n$ is less than

$$\mathrm{Leb}(M) \sigma_1^{-(1-\varepsilon_0)n} e^{\gamma_0 n} p^{\varepsilon_0 n} q^n.$$

Since we supposed $q < \sigma_1$, we may fix ε_0 small so that $e^{\gamma_0} p^{\varepsilon_0} q < \sigma_1^{1-\varepsilon_0}$. This means that the Lebesgue measure of I_n goes to zero exponentially fast as $n \to \infty$. Thus, by the lemma of Borel-Cantelli, Lebesgue almost every point $x \in M$ belongs in only finitely many sets I_n. Clearly, any such point x satisfies the conclusion of the lemma. □

Proposition A.2. *Given real numbers $\sigma_1, \sigma_2 > 0$ and integers $p, q \geq 1$ such that $\sigma_1 > q \geq 1 > \sigma_2$, there exist $\delta_0 > 0$ and $c_0 > 0$ such that the following holds. Let M, $f : M \to M$, and $B_1, \ldots, B_{p+q}$ be as in Lemma A.1, and assume that*

1. *$\|Df(x)^{-1}\| \leq \sigma_2$ if $x \in B_i$, $1 \leq i \leq p$, and*
2. *$\|Df(x)^{-1}\| \leq 1 + \delta_0$ if $x \in B_i$, $p+1 \leq i \leq p+q$.*

Then f is non-uniformly expanding: for Lebesgue almost every point $x \in M$

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \leq -c_0\,.$$

Proof. Let $\varepsilon_0 > 0$ be the constant given by Lemma A.1. Then, fix $\delta_0 > 0$ small enough so that $\sigma_2^{\varepsilon_0}(1+\delta_0) \leq e^{-c_0}$ for some $c_0 > 0$. Let x be any point satisfying the conclusion of the lemma. Then

$$\prod_{j=0}^{n-1} \|Df(f^j(x))^{-1}\| \leq \sigma_2^{\varepsilon_0 n}(1+\delta_0)^{(1-\varepsilon_0)n} \leq e^{-c_0 n}$$

for every large enough n, This means that x satisfies the conclusion of the proposition, so the proof is complete. □

Remark A.3. We also proved that, for f as in Proposition A.2, the Lebesgue measure of the set

$$\left\{x \in M : \|Df^j(x)^{-1}\| > e^{-c_0 j} \text{ for some } j \geq n\right\}$$

goes to zero exponentially fast when $n \to \infty$.

With the aid of this proposition we can exhibit an explicit construction of a C^1 open class of maps satisfying the hypotheses of Corollary D. Let M be any compact manifold supporting some uniformly expanding map f_0: there exists $\sigma_2 < 1$ such that

$$\|v\| < \sigma_2 \|Df_0(x)v\| \quad \text{for every } x \in M \text{ and } v \in T_x M.$$

For instance, M could be the d-dimensional torus T^d. Let $V \subset M$ be some small compact domain, so that the restriction of f_0 to V is injective.

Corollary A.4. *Let f_1 be any C^1 map coinciding with f_0 outside V, and such that f_1 is volume expanding everywhere, $|\det Df_1(x)| > 1$ for every $x \in M$, and f is not too contracting on V: $\|Df_1(x)^{-1}\| \leq 1 + \delta_0$ for every $x \in V$ and some small enough $\delta_0 > 0$. Then every map f in a C^1-neighbourhood $\mathcal{N}$ of f_1 is non-uniformly expanding.*

Proof. Taking the C^1-neighbourhood sufficiently small, we may assume that there exists $\sigma_1 > 1$ such that the Jacobian of every f in it is bounded from below by σ_1. Moreover, $\|Df(x)^{-1}\| \le \sigma_2$ for every x outside V. Let $B_1, \dots, B_p, B_{p+1} = V$ be any partition of M into domains such that f is injective on each B_j, $1 \le j \le p+1$. The claim follows from Proposition A.2, with $q = 1$. □

Remark A.5. Maps f_1 as in the statement can be obtained, e.g. through deformation of f_0 by isotopy inside V. In general, these maps are not expanding: deformation can be made in such way that f_1 have periodic saddles.

Using very similar ideas one can also construct robust classes of partially hyperbolic diffeomorphisms (or, more generally, diffeomorphisms with a dominated splitting) whose centre-unstable direction is non-uniformly expanding. We just sketch the main points.

This time we start with a linear Anosov diffeomorphism f_0 on the d-dimensional torus $M = T^d$, $d \ge 2$. We write $TM = E^u \oplus E^s$ the corresponding hyperbolic decomposition. Let V be a small closed domain in M, in the following sense: there exist unit open cubes K^0 and K^1 in $\mathbb{R}^d$ such that $V \subset \pi(K^0)$ and $f_0(V) \subset \pi(K^1)$, where $\pi : \mathbb{R}^d \to T^d$ is the canonical projection. Now, let f be a diffeomorphism on T^d such that

(a) f admits invariant cone fields C^{cu} and C^{cs}, with small width $\alpha > 0$ and containing, respectively, the unstable bundle E^u and the stable bundle E^s of the Anosov diffeomorphism f_0;
(b) there is $\sigma_1 > 1$ so that $|\det(Df \mid T_x\mathcal{D}^{cu})| > \sigma_1$ and $|\det(Df \mid T_x\mathcal{D}^{cs})| < \sigma_1^{-1}$ for any $x \in M$ and any disks $\mathcal{D}^{cu}$ and $\mathcal{D}^{cs}$ through x tangent, respectively, to the centre-unstable cone field C^{cu} and to centre-stable cone field C^{cs}.
(c) f is C^1-close to f_0 in the complement of V, so that there exists $\sigma_2 < 1$ satisfying

$$\|(Df \mid T_x\mathcal{D}^{cu})^{-1}\| < \sigma_2 \quad \text{and} \quad \|(Df \mid T_x\mathcal{D}^{cs})\| < \sigma_2$$

for $x \in (M \setminus V)$ and any disks $\mathcal{D}^{cu}$, $\mathcal{D}^{cs}$ tangent to C^{cu}, C^{cs}, respectively.
(d) there exists some small $\delta_0 > 0$ satisfying

$$\|(Df \mid T_x\mathcal{D}^{cu})^{-1}\| < (1+\delta_0) \quad \text{and} \quad \|(Df \mid T_x\mathcal{D}^{cs})\| < (1+\delta_0)$$

for any $x \in V$ and any disks $\mathcal{D}^{cu}$, $\mathcal{D}^{cs}$ tangent to C^{cu}, C^{cs}, respectively.

Closeness in (c) should be enough to ensure that $f(V)$ is also contained in the projection of a unit open cube.

For instance, if f_1 is a torus diffeomorphism satisfying (a), (b), (d), and coinciding with f_0 outside V, then any map f in a C^1 neighbourhood of f_1 satisfies all the previous conditions. The C^1 open classes of transitive non-Anosov diffeomorphisms presented in [BV99, Section 6], as well as other robust examples from [Mañ78], are constructed in this way and they fit in the present setting: both these diffeomorphisms and their inverses satisfy (a)–(d).

In what is left of this appendix, we argue that any f satisfying (a)–(d) is non-uniformly expanding along its centre-unstable direction. More precisely, condition (2) in Theorem A holds, with limit bounded away from zero, on a full Lebesgue set of points $x \in M$.

To explain this, let $B_1, \dots, B_p, B_{p+1} = V$ be any partition of T^d into small domains, in the same sense as before: there exist open unit cubes K_i^0 and K_i^1 in $\mathbb{R}^d$ such that

$$B_i \subset \pi(K_i^0) \quad \text{and} \quad f(B_i) \subset \pi(K_i^1). \tag{40}$$

Let $\mathcal{F}_0^u$ be the unstable foliation of f_0, and $\mathcal{F}_j = f^j(\mathcal{F}_0^u)$ for every $j \geq 0$. By (a), each $\mathcal{F}_j$ is a foliation of T^d tangent to the centre-unstable cone field C^{cu}. For any subset E of a leaf of $\mathcal{F}_j$, $j \geq 0$, we denote $\mathrm{Leb}_j(E)$ the Lebesgue measure of E inside that leaf.

Fix any small disk D_0 contained in a leaf of the foliation $\mathcal{F}_0$. Then, for any sequence $\underline{i} = (i_0, \dots, i_{n-1})$ in $\{1, \dots, p, p+1\}$, define

$$[\underline{i}] = \{x \in D_0 : f^j(x) \in B_{i_j} \text{ for } 0 \leq j < n\}.$$

Claim: *There exists* $C_0 > 0$ *depending only on* f *such that* $\mathrm{Leb}_0([\underline{i}]) \leq C_0\sigma_1^{-n}$ *for every sequence* $\underline{i}$ *as above.*

Proof. Indeed, let $\tilde{\mathcal{F}}_j$ be the lift to $\mathbb{R}^d$ of $\mathcal{F}_j$, for $j \geq 0$. Using (40) one can easily conclude, by induction on j, that $f^j([\underline{i}])$ is contained in the image $\pi(K_{j-1}^1 \cap \tilde{F}_j)$ of the intersection of K_{j-1}^1 with some leaf $\tilde{F}_j$ of $\tilde{\mathcal{F}}_j$, for every $0 \leq j \leq n$. So, using (b) and the fact that $(\pi \mid K_{n-1}^1)$ is a diffeomorphism and an isometry onto its image,

$$\mathrm{Leb}_0([\underline{i}]) \leq \sigma_1^{-n}\, \mathrm{Leb}_n(f^n([\underline{i}])) \leq \sigma_1^{-n}\, \mathrm{Leb}_n(F_n \cap K_{n-1}^1). \tag{41}$$

Recall that we took f_0 linear, so that its unstable foliation $\mathcal{F}_0^u$ lifts to a foliation $\tilde{\mathcal{F}}_0^u$ of $\mathbb{R}^d$ by affine hyperplanes. The leaves of every $\tilde{\mathcal{F}}_n$ are C^1 submanifolds of $\mathbb{R}^d$ transverse to these hyperplanes, with angles uniformly bounded away from zero at every intersection point. Consequently, the intersection of a leaf of $\tilde{\mathcal{F}}_n$ with any unit cube in $\mathbb{R}^d$ has Lebesgue measure (inside the leaf) bounded by some uniform constant C_0. In particular, the last factor in (41) is bounded by C_0. □

Now, using the same arguments as in Lemma A.1, we may conclude that Leb_0-almost every point $x \in D_0$ spends a positive fraction ε_0 of the time outside the domain V. Then, using assumptions (c) and (d) above, there exists $c_0 > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left\| \left(Df \middle| E^{cu}_{f^j(x)}\right)^{-1} \right\| \leq -c_0$$

for Leb_0-almost every point $x \in D_0$. Since D_0 was an arbitrary disk inside a leaf of $\mathcal{F}^s_0$, and the latter is an absolutely continuous foliation, we conclude that f is non-uniformly expanding along E^{cu}, Lebesgue almost everywhere in $M = T^d$.

Remark A.6. These arguments also show that f is non-uniformly contracting along the centre-stable direction, if it satisfies (a)–(d): Lebesgue almost every $x \in M$ has

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left\| Df \middle| E^{cs}_{f^j(x)} \right\| \le -c_0 \,.$$

Finally, reducing δ_0 if necessary (this can be done without changing c_0), we can make $A^u = \sup - \log \|Df^{-1} \mid E^{cu}\|$ and $A^s = \sup \log \|Df \mid E^{cs}\|$ arbitrarily close to zero. In particular, $c_0/A^u + c_0/A^s > 1$, as in Proposition 6.5.

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