Absolutely continuous invariant measures for the quadratic family

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1 Introduction

A main topic in Dynamical Systems is the study of the iterates of transformations $T : X \rightarrow X$, where $X$ has some special structure (e.g. $X$ is a topological space or a smooth manifold) and $T$ preserves this structure (e.g. $T$ is a homeomorphism, a diffeomorphism, or simply a continuous or differentiable transformation). One is particularly interested in describing the asymptotic behaviour of the orbits $\{T^n(x) : n \in \mathbb{Z}\}$ for a set as large as possible of points $x \in X$. The concept of large set depends on the additional structure of $X$: It may be understood either in the topological sense (dense, open, residual, . . .) or in the sense of measure theory (positive measure, full measure, . . .). In this last case one deals with a measure $m$ (defined on the Borel $\sigma$-algebra $B$ of $X$, say) which is invariant under $T$, meaning that $m(T^{-1}(A)) = m(A)$ for every measurable set $A$. If the measure $m$ satisfies $m(X) = 1$, we say that $(X, B, m)$ is a probability space.

The existence of invariant measures provides important information on the dynamical behaviour of the transformation. In this context we mention Poincaré’s Recurrence Theorem, which says that given $A \in B$ with $m(A) > 0$ in a probability space $(X, B, m)$, almost all the points of $A$ (i.e., all the points of $A$ except for a set of measure zero) return infinitely often to $A$ under positive iterations by $T$. Another basic result involving invariant measures is the very well known Birkhoff Ergodic Theorem: suppose $m$ is an invariant measure for the transformation $T : X \rightarrow X$ and $f$ is $m$-integrable. Then $(1/n) \sum_{i=0}^{n-1} f(T^i(x))$ converges for almost all the points $x \in X$. Denoting $f^*$ the limit function, also $f^* \circ T = f^*$.
Therefore, by the Ergodic Theorem and if \( m(X) < \infty \), then \( \int f^* dm = \int f dm \). If, in addition, the measure \( m \) is ergodic (i.e. \( T^{-1}(A) = A \) implies \( m(A) = 0 \) or \( m(X \setminus A) = 0 \)), then \( \lim_{n \to \infty} (1/n) \sum_{i=0}^{n-1} f(T^i(x)) = \int f dm \) for almost all the points \( x \in X \).

As an application of the Ergodic Theorem we have the following: suppose \( m \) is an invariant ergodic probability for the transformation \( f \) for almost all the points, \( \int T \) is a continuous with respect to Lebesgue measure (i.e., \( m(A) = 0 \) whenever the Lebesgue measure of \( A \) is equal to zero). With this notion, we have that sets with positive \( m \)-measure have also positive Lebesgue measure, and so, properties that are verified for a large set of points in the sense of positive \( m \)-measure, are also verified for a large set of points in the sense of positive Lebesgue measure.

For sets \( X \subset \mathbb{R}^n \) the Lebesgue measure has an important position among all the others, since, in some sense, it corresponds to our intuitive notion of measure, and is better fitted to experimental observations. Since for a large part of the transformations we are interested in, Lebesgue measure is not an invariant measure, we try at least to find an invariant measure \( m \) which is absolutely continuous with respect to Lebesgue measure (i.e., \( m(A) = 0 \) whenever the Lebesgue measure of \( A \) is equal to zero). With this notion, we have that sets with positive \( m \)-measure have also positive Lebesgue measure, and so, properties that are verified for a large set of points in the sense of positive \( m \)-measure, are also verified for a large set of points in the sense of positive Lebesgue measure.

The study of the quadratic family \( f_a : [-1,1] \to [-1,1] \) defined by \( f_a(x) = 1 - ax^2 \) with \( a \in [1,2] \), has deserved a special attention, since the dynamical behaviour exhibited by the elements of this family has relevant applications in other more general situations. We denote \( \lambda \) the Lebesgue measure on
and when we refer to the absolute continuity of some measure, we mean the absolute continuity with respect to $\lambda$.

Here, we are going to prove the abundance of parameters for which there is an absolutely continuous invariant probability measure. More precisely, we prove the following:

**Theorem** (Jakobson): There is a positive Lebesgue measure set of parameters $a \in [1, 2]$, for which $f_a$ has an ergodic absolutely continuous invariant probability measure.

An important contribution on the subject was also given in [R], where it was proved the abundance of rational transformations of the Riemann sphere for which there is an absolutely continuous invariant measure and strong growth of the derivative for a set of points with full Lebesgue measure.

In [BC1] it was shown that, under some assumptions on the set of parameters, one can have a positive Lebesgue measure set of parameters for which the orbit of the critical value has subexponential growth of the derivative, i.e., $|(f_a^n)'(1)| \geq e^{\sqrt{n}}$. From this, it was deduced the existence of an absolutely continuous invariant measure.

Jakobson’s proof of the theorem is based on the construction of a Bernoulli map associated to $f$ and to a carefully defined partition of $[-1,1]$ into intervals. More precisely, he shows that up to delicate assumptions on the parameters (satisfied by a positive Lebesgue measure set of values $a \in [1, 2]$) one can construct, in an inductive way, a partition $\mathcal{P}$ of $[-1,1]$ and a map $\tilde{f}$ such that for every $I \in \mathcal{P}$, $\tilde{f}_{\mid I} = f_{n(I)}$ for some $n(I) \geq 1$. Moreover, for each $I \in \mathcal{P}$, $\tilde{f}_{\mid I}$ is an expanding map onto $[-1,1]$ and has bounded distortion. A standard well-known result gives that such an $\tilde{f}$ has an absolutely continuous invariant measure, which can easily be seen to be invariant for $f$. From this argument it also follows that the measure is ergodic and has positive Liapounov exponent.

Usually, invariant measures for a continuous map defined on a compact set, are taken as a weak accumulation point for some suitable sequence of measures. By a convenient choice of the sequence we are going to prove that such an accumulation point is also absolutely continuous. Our plan, following [BY], is to prove that there is some $\lambda$-integrable map $\phi$, such that the density of all the elements of our sequence are dominated by $\phi$. From this, it is easy to check that the limit point is also absolutely continuous. Our exposition differs from [BY] in that we use
the assumptions of [BC2] leading to exponential growth of the derivative. More precisely, under a slight modification on the assumptions of [BC1], it was proved in [BC2] that for \( c > 0 \) there is a positive Lebesgue measure set of parameters \( E(c) \subset [1, 2] \) such that for \( a \in E(c) \) the critical orbit of \( f_a \) has exponential growth of the derivative, i.e., \( |(f^n_a)'(1)| \geq e^{cn} \). This set \( E(c) \) will be exactly our set of parameters that satisfy the statement of the theorem above.

In [BL] it is shown that for an unimodal map (i.e., a map with only one critical point which is non-degenerated) with negative Schwarzian derivative (which is the case of our family of maps) there exists at most one absolutely continuous invariant probability measure. So, in our case, the absolutely continuous invariant probability we obtain, is in fact unique. From the uniqueness it follows that the measure is also ergodic. In fact, if the measure \( \mu \) were not ergodic there would be some \( f_a \)-invariant set \( K \) for which \( 0 < \mu(K) < 1 \). Let \( \nu \) be the probability defined by \( \nu(A) = \mu(A \cap K)/\mu(K) \). From the fact that \( \mu \) is an absolutely continuous invariant probability, we easily deduce that \( \nu \) is also an absolutely continuous invariant probability. But \( \nu \neq \mu \), since \( \nu([-1, 1] \setminus K) = 0 \) and \( \mu([-1, 1] \setminus K) \neq 0 \) (recall that \( \mu(K) < 1 \)). This gives a contradiction with the unicity of \( \mu \). So \( \mu \) is in fact ergodic.

2 The set of good parameters

In this section we give a brief sketch of Benedicks-Carleson’s arguments to prove exponential growth of the derivative on the critical orbit of \( f_a \), for a positive Lebesgue measure set of parameter values, [BC1], [BC2]. For a complete and detailed exposition see also [M]. These are the parameter values for which we shall be proving, in the following sections, that \( f_a \) admits an absolutely continuous invariant measure.

Fix \( 0 < c < c_0 < \log 2 \), and let \( \alpha > 0 \) and \( \delta = e^{-\Delta} \) be small numbers chosen in this order. In [BC2] is shown that for a positive measure subset \( E = E(c) \) of \([1,2]\) we have

\[ |(f^n_a)'(1)| \geq e^{cn} \]

for every \( a \in E \). A first approximation to the set \( E \) is given by Lemma 2.2, that also plays an important role in our proof of the existence of the absolutely continuous invariant measure. For each positive integer \( m \), let \( U_m \) denote the interval \((-e^{-m}, e^{-m})\).
Lemma 2.1 For every $a \in [1, 2]$ we have that $|f_a(x)| \leq 1 - \delta^2/9$, whenever $x \not\in U_{\Delta+1}$ and $|x| \leq 1 - \delta^2/9$.

Proof: Since $x \not\in U_{\Delta+1}$, it follows that for $a \geq 1$

$$x^2 \geq e^{-2\Delta-2} = \frac{\delta^2}{e^2} \geq \frac{\delta^2}{9a},$$

which implies that

$$1 - ax^2 \leq 1 - \frac{\delta^2}{9}.$$ (1)

We easily verify that for $\delta < 1$

$$|x| \leq 1 - \frac{\delta^2}{9} \implies x^2 \leq \frac{2 - \frac{\delta^2}{9}}{2} \implies x^2 \leq \frac{2 - \frac{\delta^2}{9}}{a}, \text{ since } a \leq 2 \implies 1 - ax^2 \geq -1 + \frac{\delta^2}{9},$$

and from (1) above, it follows that

$$|f_a(x)| = |1 - ax^2| \leq 1 - \delta^2/9$$

as we wanted.

Lemma 2.2 Given $\Delta$ sufficiently large and $0 < c_0 < \log 2$, there are $\gamma = \gamma(\Delta) > 0$ and $a_0 = a_0(\Delta, c_0) < 2$ such that for $x \in [-1, 1]$, $a \in (a_0, 2]$ and $k \geq 1$ the following holds:

(a) If $x, f_a(x), \ldots, f_a^{k-1}(x) \not\in U_{\Delta+1}$ then $|(f_a^k)'(x)| \geq \gamma e^{c_0 k}$.

(b) If $x, f_a(x), \ldots, f_a^{k-1}(x) \not\in U_{\Delta+1}$ and $f_a^k(x) \in U_{\Delta-1}$ then $|(f_a^k)'(x)| \geq e^{c_0 k}$.

Proof: The arguments of the proof depend on whether the orbit of the point $x$ is close to 1 or not. So, before we give the proof in the general case we consider the following two particular cases:

(i) Suppose first that

$$\forall i \in \{0, \ldots, k-1\} |f_a^i(x)| > 1 - \frac{\delta^2}{9}$$

\[5\]
In this case we have
\[ |(f^k_a)'(x)| = \prod_{i=0}^{k-1} |f'_a(f^i_a(x))| = \prod_{i=0}^{k-1} 2a|f^i_a(x)| > \left[ 2a(1 - \frac{\delta^2}{9}) \right]^k. \]

If we choose \( a_0 \geq (1 - \frac{\delta^2}{9})^{-1} \) (which is less than 2, as long as \( \delta^2/9 < 1/2 \)) we have that \( 2a_0(1 - \frac{\delta^2}{9}) \geq 2 \). So, for \( a > a_0 \), it follows that
\[ |(f^k_a)'(x)| \geq \left[ 2a_0(1 - \frac{\delta^2}{9}) \right]^k = 2^k \geq e^{c_0 k}, \]
and Lemma 2.2 is proved in this case.

(ii) Suppose now that
\[ \forall i \in \{0, \cdots, k\} \ |f^i_a(x)| \leq 1 - \frac{\delta^2}{9} \]

Consider the homeomorphism
\[ h: [-1, 1] \rightarrow [-1, 1] \]
\[ y \mapsto \sin \left( \frac{\pi}{2} y \right) \]

and define \( g_a = h^{-1} \circ f_a \circ h \). We have that
\[ g_2(y) = \frac{2}{\pi} \arcsin \left( 1 - 2 \sin^2 \left( \frac{\pi}{2} y \right) \right) = \frac{2}{\pi} \arcsin \left( \cos(\pi y) \right) \]
\[ = \frac{2}{\pi} \left( \frac{\pi}{2} - |y| \right) = 1 - 2|y| \]
and so \( |g_2'(y)| = 2 \) for all \( y \neq 0 \). Let \( y = h^{-1}(x) \). Since \( g_a = h^{-1} \circ f_a \circ h \) we have
that for number $\epsilon$ we have that for some constant $C$

This, altogether with $\|f'(x)\| \leq 2\|a - 2\|$, we have

$$|g'(y) - g'_2(y)| \leq 2\pi \left| (h^{-1})'(f_a(x)) - (h^{-1})'(f_2(x)) \right| + \pi \left| (h^{-1})'(f_2(x)) \right| \cdot |a - 2|.$$ 

Taking into account that $|f'_a(x)| \leq 4$, $|h'(y)| \leq \frac{\pi}{2}$ and $|f'_a(x) - f'_2(x)| \leq 2|a - 2|$, we have

$$|g'(y) - g'_2(y)| \leq 2\pi \left| (h^{-1})'(f_a(x)) - (h^{-1})'(f_2(x)) \right| + \pi \left| (h^{-1})'(f_2(x)) \right| \cdot |a - 2|.$$ 

Since $|f_a(x)| \leq 1 - \frac{\delta_1}{9}$ and $|f_2(x)| \leq 1 - \frac{\delta_1}{9}$ by Lemma 2.1, there are constants $C'_\delta$ and $C''_\delta$ for which

$$\left| (h^{-1})'(f_2(x)) \right| \leq C'_\delta$$

and

$$\left| (h^{-1})'(f_a(x)) - (h^{-1})'(f_2(x)) \right| \leq C''_\delta |f_a(x) - f_2(x)|.$$ 

This, altogether with $|f_a(x) - f_2(x)| \leq |a - 2| x^2$, leads to

$$|g'(y) - g'_2(y)| \leq 2\pi C''_\delta |f_a(x) - f_2(x)| + \pi C'_\delta |a - 2|$$

$$\leq 2\pi C''_\delta |a - 2| x^2 + \pi C'_\delta |a - 2|$$

$$\leq C_\delta |a - 2|$$

for some constant $C_\delta$. It follows from the last inequality, that given a small number $\epsilon$, $0 < \epsilon < \log 2 - c_0$, and since $e^{c_0 + \epsilon} < 2 = |g'_2(y)|$, there is $a_0 < 2$ such that for $a \in (a_0, 2]$

$$|g'_a(y)| \geq e^{c_0 + \epsilon}.$$ 

(1)

We have that

$$|(f^k_a)'(x)| = \left| h'(g^k_a(h^{-1}(x))) \cdot (g^k_a)'(h^{-1}(x)) \right| \cdot \left| (h^{-1})'(x) \right|$$

$$= \left| h'(h^{-1}(f^k_a(x))) \right| \cdot \prod_{i=0}^{k-1} |g'_a \left( g^i_a(h^{-1}(x)) \right) | \cdot \left| (h^{-1})'(x) \right|$$

$$= \frac{|(h^{-1})'(x)|}{|(h^{-1})'(f^k_a(x))|} \cdot \prod_{i=0}^{k-1} |g'_a \left( h^{-1}(f^i_a(x)) \right) |.$$
Taking into account that \( h^{-1}(f^i_a(x)) \neq 0 \) for \( i = 0, \cdots, k - 1 \) and (1) above we, easily, get
\[
\left| (f^k_a)'(x) \right| \geq \left| \frac{(h^{-1})'(x)}{(h^{-1})'(f^k_a(x))} \right| \cdot e^{(c_0 + \epsilon)k} \geq \left| \frac{(h^{-1})'(x)}{(h^{-1})'(f^k_a(x))} \right| \cdot e^{c_0k + \epsilon}
\]

Now, we only need to study the quotient involved in the last inequality, which will be done separately for (a) and (b).

(a) Since \( x \not\in U_{\Delta+1} \) we have that
\[
x^2 \geq e^{-2\Delta - 2} > \frac{\delta^2}{9}.
\]
In this case we also have
\[
\left| f^k_a(x) \right| \leq 1 - \frac{\delta^2}{9}.
\]
Hence
\[
\left| \frac{(h^{-1})'(x)}{(h^{-1})'(f^k_a(x))} \right| = \sqrt{\frac{1 - (f^k_a(x))^2}{1 - x^2}} \geq \sqrt{\frac{1 - (1 - \frac{\delta^2}{9})^2}{1 - \frac{\delta^2}{9}}} = \gamma(\Delta) > 0.
\]
Finally, we have
\[
\left| (f^k_a)'(x) \right| \geq \gamma \cdot e^{c_0k + \epsilon}.
\]
(Here we did not need the extra factor \( e^\epsilon \), which will be useful in part (b) ).

(b) This time, we have that \( x \not\in U_{\Delta+1} \) and \( f^k_a(x) \in U_{\Delta - 1} \), which implies that \( |x| \geq e^{-\Delta - 1} \) and \( |f^k_a(x)| \leq e^{-\Delta + 1} \), leading to
\[
\left| \frac{(h^{-1})'(x)}{(h^{-1})'(f^k_a(x))} \right| = \sqrt{\frac{1 - (f^k_a(x))^2}{1 - x^2}} \geq \sqrt{\frac{1 - (e^{-\Delta + 1})^2}{1 - (e^{-\Delta - 1})^2}} = A(\Delta)
\]
Since \( A(\Delta) \) tends to 1 when \( \Delta \) tends to infinity, we can make \( A(\Delta) > e^{-\epsilon} \) if we choose \( \Delta \) sufficiently large. Thus,
\[
\left| (f^k_a)'(x) \right| \geq A(\Delta) \cdot e^{c_0k + \epsilon} \geq e^{c_0k}.
\]
and Lemma 2.2 is also proved in this case.
In the general case, let $1 \leq p \leq k$ be the minimum such that $|f_a^p(x)| \leq 1 - \delta^2/9$. Under the hypothesis of the lemma, and taking into account Lemma 2.1, we have that $|f_a^i(x)| \leq 1 - \delta^2/9$ for $i = p + 1, \ldots, k$. Since

$$(f_a^k)'(x) = (f_a^{k-p})' (f_a^p(x)) \cdot (f_a^p)'(x),$$

and from the previous particular cases we get

$$|(f_a^{k-p})'(f_a^p(x))| \cdot |(f_a^p)'(x)| \geq \left\{ \begin{array}{ll} \gamma e^{c_0(k-p)} \cdot e^{c_0 p} & \text{for (a)} \\ e^{c_0(k-p)} \cdot e^{c_0 p} & \text{for (b)} \end{array} \right.$$  

which gives the result.

The previous lemma means that for the parameters in $[a_0, 2]$, we have expanding behaviour for orbits remaining outside the neighbourhood of the critical point, $(-\delta, \delta)$. However, in [BC1] it was shown that for almost all the points $a \in [a_0, 2]$, the orbit of the critical point cannot be kept away from $(-\delta, \delta)$. Since

$$|(f_a^n)'(1)| = \prod_{j=1}^n |2a f_a^j(0)|, \quad (1)$$

we introduce some small factors on the derivative of the orbit at the returns, i.e., the iterates $r$ for which $f_a^r(0) \in (-\delta, \delta)$, and, consequently, we have a loss on the expansion of the derivative. So, we must compensate that loss on the expansion. For this, we make some exclusions on the set of parameters and retain only the parameters $a$ satisfying the basic assumption

$$|f_a^n(0)| \geq e^{-|\alpha n|}. \quad (BA)$$

At this point is introduced the notion of bound period associated to the return $r$, which consists of the $p$ iterates after the return $r$ for which the orbits of zero and $f_a^r(0)$ behave similarly. Since for early iterates of the critical value there is a large growth of the derivative, it is expected that the same hold for the orbit of $f_a^r(0)$ during the bound period. In fact, using (BA) one can show that

$$|(f_a^{p+1})'(f_a^r(0))| > 1 \quad (2)$$

and so, the small factors introduced in (1) at the returns, can be compensated.
After the bound period associated to a return, the critical orbit spends its time in a free period until the next return. From Lemma 2.2 we have exponential growth of the derivative during those free periods.

Based on a large deviations argument, new exclusions are made in the set of parameters, in order to guarantee that only a small fraction $\epsilon n$ of the iterates until $n$ belong to a bound period. During the complementar $(1-\epsilon)n$ iterates the derivative of the orbit of the critical value increases with exponential constant $c_0$. From this, and taking into account (2), it follows that

$$| (f_n^a)'(1) | \geq e^{(1-\epsilon)c_0 n} \geq e^{cn},$$

if $\epsilon > 0$ is small enough.

From now on, we consider $f = f_a$ where $a \in E(c)$, and so, we have that $f$ verifies the basic assumption (BA), and the orbit of the critical value has exponential growth of the derivative.

### 3 Auxiliary lemmas

In this section we are going to give some definitions and results that will be important both in the construction of the measure and in the estimates of the density. First, we introduce some useful notation:

$$I_m = (e^{-m-1}, e^{-m}) \text{ for } m > 0,$$

$$I_m = -I_{-m} \text{ for } m < 0,$$

$$I_m^+ = I_{m-1} \cup I_m \cup I_{m+1}.$$

For each $n \in \mathbb{Z}^+$ let $\xi_n = f^n(0)$. In the next definition we use a small number $\beta > \alpha$; for our computations this can be taken equal to $2\alpha$.

**Definition 3.1** Let $p(m)$ be the largest $p$ such that

$$| f^j(x) - \xi_j | \leq e^{-\beta j}$$

for all $j = 1, \ldots, p - 1$ and $x \in I_m^+$. The time interval $1, \ldots, p(m) - 1$ is called the bound period for $I_m^+$. 

10
Lemma 3.2  For each $|m| \geq \Delta$, $p(m)$ has the following properties:

(a) There is a constant $C_1 = C_1(\alpha - \beta)$ (not depending on $\delta$) such that for all $y \in f(U_{|m|-1})$,

$$\frac{1}{C_1} \leq \frac{(f^j)'(y)}{(f^j)'(\xi_1)} \leq C_1$$

for $j = 0, \ldots, p(m) - 1$.

(b) $p(m) \leq \frac{3}{e}|m|$.

(c) $|(f^p)'(x)| \geq e^{(1 - \frac{4\beta}{e})|m|}$ for $x \in I_m^+$ and $p = p(m)$.

Proof: (a) This part will be proved inductively. The result is obvious for $j = 0$. Supposing it is valid for $0 \leq k \leq j - 1$, we have in particular that $f^k|f(U_{|m|-1})$ is a diffeomorphism. We first verify that

$$\left| \frac{(f^j)'(y)}{(f^j)'(\xi_1)} \right| = j \prod_{k=1}^{j} \left| \frac{f'(f^{k-1}(y))}{f'(\xi_k)} \right| = j \prod_{k=1}^{j} \left| 1 + \frac{f^{k-1}(y) - \xi_k}{\xi_k} \right|.$$

Since

$$\left| \frac{f^{k-1}(y) - \xi_k}{\xi_k} \right| = \left| \frac{f^{k-1}(y) - f^{k-1}(\xi_1)}{\xi_k} \right| \leq \left| \frac{f^{k-1}(f(e^{-|m|+1)}) - f^{k-1}(\xi_1)}{\xi_k} \right|, \text{ by the monotonicity of } f^{k-1}|f(U_{|m|-1})$$

$$= \left| \frac{f^k(e^{-|m|+1}) - \xi_k}{\xi_k} \right| \leq \frac{e^{-\beta k}}{e^{-\alpha k}}$$

it follows that

$$C_- \leq \left| \frac{(f^j)'(y)}{(f^j)'(\xi_1)} \right| \leq C_+,$$

where

$$C_\pm = \prod_{k=1}^{\infty} \left( 1 \pm e^{(\alpha - \beta)k} \right).$$
The result follows by choosing $C_1 = \max\{C_+, 1/C_\}$. 

(b) We have that
\[ 2 \geq \left| f^p(e^{-|m|}) - \xi_p \right| = \left| f^{p-1}\left(f(e^{-|m|})\right) - f^{p-1}(\xi_1) \right|. \]

Let, on the other hand, $y \in f(U_{|m|-1})$ be such that
\[ \left| f^{p-1}\left(f(e^{-|m|})\right) - f^{p-1}(\xi_1) \right| = \left| (f^{p-1})'(y) \cdot \left| f(e^{-|m|}) - \xi_1 \right| \right| \geq \left| (f^{p-1})'(y) \right| \cdot e^{-2|m|}. \]

It follows from (a) that
\[ \left| (f^{p-1})'(y) \right| \cdot e^{-2|m|} \geq \frac{1}{C_1} \cdot (f^{p-1})'(\xi_1) \cdot e^{-2|m|} \]
\[ \geq \frac{1}{C_1} \cdot e^{c(p-1)} \cdot e^{-2|m|}. \]

Hence
\[ \log 2 \geq \log \left( \frac{1}{C_1} \right) + c(p - 1) - 2|m| \]
and so
\[ p \leq \frac{c + 2|m| - \log \left( \frac{1}{C_1} \right) + \log 2}{c} \leq \frac{3|m|}{c} \]
(as long as $\Delta$ is sufficiently large).

(c) It follows from the definition of $p(m)$ and the Mean Value theorem, that for some $z \in I^+_m$ and $y \in f(U_{|m|-1})$
\[ e^{-\beta p} \leq \left| f^p(z) - \xi_p \right| = \left| (f^{p-1})'(y) \right| \cdot \left| f(z) - \xi_1 \right|. \]

Hence
\[ e^{-\beta p} \leq \left| (f^{p-1})'(y) \right| \cdot 2z^2 \leq \left| (f^{p-1})'(y) \right| \cdot 2e^{-2|m|+2} \]
and so
\[ \left| (f^{p-1})'(y) \right| \geq \frac{1}{2} e^{2|m|-2} \cdot e^{-\beta p}. \] (1)
For $x \in I_m^+$ we have

\[
|(f^p)'(x)| = |f'(x)| \cdot |(f^{p-1})'(x)| \\
= |2ax| \cdot \left| \frac{(f^{p-1})'(f(x))}{(f^{p-1})'(y)} \right| \cdot |(f^{p-1})'(y)| \\
\geq 2ae^{-|m|-2} \cdot \left| \frac{(f^{p-1})'(f(x))}{(f^{p-1})'(y)} \right| \cdot |(f^{p-1})'(y)|.
\]

By (a) and (1) above, we have that

\[
|(f^p)'(x)| \geq 2ae^{-|m|-2} \cdot \frac{1}{C_1^2} \cdot \frac{1}{2} e^{2|m|-2 - \beta p} \\
\geq \frac{a}{C_1^2} e^{m-\beta \frac{2}{4}|m|-4} \\
\geq e^{(1-4\beta)|m|}
\]

if $\Delta$ is sufficiently large.

In order to prove that $f$ has bounded distortion, we consider for each $I_m$, the collection of $m^2$ equal length intervals $I_{m,1}, I_{m,2}, \ldots, I_{m,m^2}$ whose union is $I_m$. It is convenient to consider these $I_{m,i}$ ordered in the following way: if $i > j$, then dist($I_{m,i}, 0$) < dist($I_{m,j}, 0$). By $I^+_{m,i}$ we denote the union of $I_{m,i}$ with the two adjacent intervals of the same type. In the following sections we shall also make use of the intervals $U_m^+$ which are, by definition, the union of $U_m$ with $I^{-1}_{m+1,(m-1)^2}$ and $I^{-1}_{-m+1,(m-1)^2}$.

**Definition 3.3** For a point $x \in U_\Delta$ we define the return times to $U_\Delta$, $t_0(x) < t_1(x) < t_2(x) < \cdots$, in the following way: first we set $t_0(x) = 0$; assuming that $t_k$ is defined and $f^{t_k}(x) \in I_{m_k,i_k}$, we take $t_{k+1}$ to be the smallest $j \geq t_k + p(m_k)$ such that $f^j(x) \in U_\Delta$.

**Lemma 3.4** (Bounded Distortion): There is a constant $C_2$ satisfying:

Let $\sigma_0 = [x, y]$ be an interval in $U_\Delta$ such that all the points of $\sigma_0$ have the same return times $0 = t_0 < t_1 < \cdots < t_q \leq N$ before $N$. Suppose, moreover, that for all $k$, $0 \leq k \leq q$, $\sigma_k = f^{t_k}(\sigma_0) \subset I^+_{m_k,i_k}$ for some $m_k \geq \Delta$ and $1 \leq i_k \leq m_k^2$. Then we have:

\[
\left| \frac{(f^N)'(x)}{(f^N)'(y)} \right| \leq C_2
\]
Proof: Let \( x_j = f^j(x) \) and \( y_j = f^j(y) \).

\[
\left| \frac{(f^N)'(x)}{(f^N)'(y)} \right| = \prod_{j=0}^{N-1} \left| \frac{f'(x_j)}{f'(y_j)} \right| \leq \prod_{j=0}^{N-1} \left( 1 + \left| \frac{x_j - y_j}{y_j} \right| \right)
\]

It is sufficient to show that

\[
S = \sum_{j=0}^{N-1} \left| \frac{x_j - y_j}{y_j} \right|
\]

is uniformly bounded. We first estimate the contribution of the free period \([t_k-1, t_k - 1]\) for the sum \( S \)

\[
F_k = \sum_{j=t_{k-1}+p_{k-1}}^{t_k-1} \left| \frac{x_j - y_j}{y_j} \right| \leq \sum_{j=t_{k-1}+p_{k-1}}^{t_k-1} \left| \frac{x_j - y_j}{\delta} \right|
\]

For \( j = t_{k-1} + p_{k-1}, \ldots, t_k - 1 \) we have

\[
\lambda(\sigma_k) \geq \left| f^{t_k-j}(x_j) - f^{t_k-j}(y_j) \right|
\]

\[
= \left| (f^{t_k-j})'(z) \right| \cdot |x_j - y_j|, \text{ for some } z \text{ between } x_j \text{ and } y_j
\]

\[
\geq e^{c_0(t_k-j)}|x_j - y_j|, \text{ by Lemma 2.2}
\]

and so

\[
F_k \leq \sum_{j=t_{k-1}+p_{k-1}}^{t_k-1} e^{-c_0(t_k-j)} \cdot \frac{\lambda(\sigma_k)}{\delta}
\]

\[
\leq \sum_{j=1}^{\infty} e^{-c_0j} \cdot \frac{\lambda(I_{m_k})}{\delta} \cdot \frac{\lambda(\sigma_k)}{\lambda(I_{m_k})}
\]

\[
\leq a_1 \cdot \frac{\lambda(\sigma_k)}{\lambda(I_{m_k})}, \text{ for some constant } a_1 = a_1(c_0).
\]

The contribution of the return \( t_k \) is

\[
\left| \frac{x_{t_k} - y_{t_k}}{y_{t_k}} \right| \leq \frac{\lambda(\sigma_k)}{e^{-|m_k|^{-2}}} \leq a_2 \cdot \frac{\lambda(\sigma_k)}{\lambda(I_{m_k})}, \text{ where } a_2 \text{ is a constant.}
\]

Finally, let us compute the contribution of bound periods

\[
B_k = \sum_{j=1}^{p_{k-1}} \left| \frac{x_{t_k+j} - y_{t_k+j}}{y_{t_k+j}} \right|
\]
We have that
\[ |x_{tk+j} - y_{tk+j}| = |(f^j)'(z)| \cdot |x_{tk} - y_{tk}|, \text{ for some } z \text{ between } x_{tk} \text{ and } y_{tk} \]
\[ = |(f^{j-1})'(f(z))| \cdot |f'(z)| \cdot |x_{tk} - y_{tk}| \]
\[ = |(f^{j-1})'(f(z))| \cdot 2a|z| \cdot |x_{tk} - y_{tk}| \]
\[ \leq C_1 |(f^{j-1})'(\xi_1)| \cdot 2ae^{-m_k} \cdot \lambda(\sigma_k). \]

On the other hand, we have
\[ |y_{tk+j} - \xi_j| = |(f^{j-1})'(\theta)| \cdot |y_{tk+1} - \xi_1| \]
for some \( \theta \in [y_{tk+1}, \xi_1]. \) Noting that \([y_{tk+1}, \xi_1] \subset f(f^t_k(\sigma_0)) \subset f(I^+_m),\) we apply Lemma 3.2 and get
\[ |y_{tk+j} - \xi_j| \geq \frac{1}{C_1} |(f^{j-1})'(\xi_1)| \cdot |y_{tk+1} - \xi_1| \]
\[ = \frac{1}{C_1} |(f^{j-1})'(\xi_1)| \cdot 2ay_{tk}^2 \]
\[ \geq \frac{1}{C_1} |(f^{j-1})'(\xi_1)| \cdot 2ae^{-2|m_k| - 4}. \]

Combining what we know about \( |x_{tk+j} - y_{tk+j}| \) and \( |y_{tk+j} - \xi_j| \) we obtain
\[ \frac{|x_{tk+j} - y_{tk+j}|}{|y_{tk+j}|} = \frac{|x_{tk+j} - y_{tk+j}|}{|y_{tk+j} - \xi_j|} \cdot \frac{|y_{tk+j} - \xi_j|}{|y_{tk+j}|} \]
\[ \leq C_2^2 \frac{e^5}{e^{-m_k}} \cdot \lambda(\sigma_k) \cdot \frac{|y_{tk+j} - \xi_j|}{|y_{tk+j}|} \]
\[ \leq C_1^2 \cdot e^5 \cdot \frac{\lambda(\sigma_k)}{\lambda(I^+_m)} \cdot \frac{e^{-\beta_j}}{e^{-\alpha_j} - e^{-\beta_j}} \]
since
\[ |y_{tk+j}| \geq |\xi_j| - |y_{tk+j} - \xi_j| \geq e^{-\alpha_j} - e^{-\beta_j}. \]

Clearly,
\[ \sum_{j=1}^{\infty} \frac{e^{-\beta_j}}{e^{-\alpha_j} - e^{-\beta_j}} < \infty \]
and so
\[ B_k \leq a_3 \cdot \frac{\lambda(\sigma_k)}{\lambda(I^+_m)}. \]
for some constant $a_3 = a_3(\alpha - \beta)$.

If $t_q + p_q \leq N$ we have to consider a last piece of free period

$$F_{q+1} = \sum_{j=t_q+p_q}^{N} \frac{|x_j - y_j|}{y_j}.$$  

Clearly, for $j = t_q + p_q, \ldots, N$

$$|f^N(x) - f^N(y)| = |(f^{N-j})'(z)||x_j - y_j|$$

for some $z$ between $x_j$ and $y_j$. Since $z, f(z), \ldots, f^{N-j-1}(z) \notin U_\Delta$, it follows from Lemma 2.2 that

$$|(f^{N-j})'(z)| \geq \gamma e^{c_0(N-j)}.$$  

Hence

$$|x_j - y_j| \leq \gamma^{-1} e^{-c_0(N-j)} \cdot |f^N(x) - f^N(y)| \leq 2\gamma^{-1} e^{-c_0(N-j)},$$

which leads to

$$F_{q+1} \leq \sum_{j=t_q+p_q}^{N} 2\gamma^{-1} e^{-c_0(N-j)} e^{-\Delta}$$

$$\leq 2\gamma^{-1} e^{\Delta} \sum_{k=0}^{\infty} e^{-c_0k}$$

$$\leq a_4, \text{ for some constant } a_4 = a_4(\Delta, c_0).$$

From the estimates obtained above, we get

$$S \leq a_4 + a_5 \cdot \sum_{k=0}^{q} \frac{\lambda(\sigma_k)}{\lambda(I_{m_k})}, \text{ where } a_5 = a_1 + a_2 + a_3.$$

We have that

$$\lambda(\sigma_{k+1}) = (f^{t_{k+1}-t_k})'(z) \cdot \lambda(\sigma_k), \text{ for some } z \in \sigma_k$$

$$= (f^{p_k})'(z) \cdot (f^{t_{k+1}-t_k-p_k})' \cdot (f^{p_k}(z)) \cdot \lambda(\sigma_k)$$

$$\geq e^{(1-\frac{3\beta}{2})(m_k)} \cdot e^{c_0(t_{k+1} - t_k - p_k)} \cdot \lambda(\sigma_k), \text{ by Lemmas 2.2 and 3.2}$$

$$\geq 2\lambda(\sigma_k), \text{ for small } \beta.$$  

Defining $k(m) = \max\{k : m_k = m\}$ and using the fact that $\lambda(\sigma_{k+1}) \geq 2\lambda(\sigma_k)$, we can easily see that

$$\sum_{\{k : m_k = m\}} \lambda(\sigma_k) \leq 2\lambda(\sigma_{k(m)}),$$
and so
\[ \sum_{k=0}^{q} \frac{\lambda(\sigma_k)}{\lambda(I_{m_k})} \leq \sum_{m \geq \Delta} \frac{1}{\lambda(I_m)} \sum_{\{k: m_k = m\}} \lambda(\sigma_k) \leq \sum_{m \geq \Delta} \frac{2\lambda(\sigma_{k(m)})}{\lambda(I_m)}. \]
Since
\[ \frac{\lambda(\sigma_{k(m)})}{\lambda(I_m)} \leq \frac{10}{m^2}, \]
it follows that
\[ \sum_{m \geq \Delta} \frac{2\lambda(\sigma_{k(m)})}{\lambda(I_m)} \leq 20 \sum_{m \geq \Delta} \frac{1}{m^2}, \]
which proves that \( S \) is uniformly bounded.

**Lemma 3.5** There are \( \epsilon \in I_{\Delta, \Delta^2} \) and constants \( \eta = \eta(\epsilon) < 1, C_3 = C_3(\epsilon) > 0 \) such that for all \( n \geq 1 \)
\[ \lambda(\{x : x, f(x), \ldots, f^{n-1}(x) \notin (-\epsilon, \epsilon)\}) \leq C_3 \eta^n. \]

**Proof:** Our strategy is to choose \( \epsilon \in I_{\Delta, \Delta^2} \) such that \( f^r(\epsilon) \in (-\epsilon, \epsilon) \) for some \( r \geq 1 \). It follows from the exponential growth of the derivative for the points whose orbit does not intersect \( U_{\Delta+1} \) (given by Lemma 2.2), that there is \( j > 0 \) such that \( f^j(I_{\Delta, \Delta^2}) \cap U_{\Delta+1} \neq \emptyset \). Choosing a point \( \epsilon \) in \( I_{\Delta, \Delta^2} \) such that \( f^j(\epsilon) \in U_{\Delta+1} \) we guarantee that \( f^j(\epsilon) \in (-\epsilon, \epsilon) \). Now, let \( r \) be the smallest positive integer for which \( f^r(\epsilon) \in (-\epsilon, \epsilon) \) and let \( \rho = \min\{ |\epsilon - f^r(\epsilon)|, | - \epsilon - f^r(\epsilon)| \} \). Define
\[ R_n = \{x : x, f(x), \ldots, f^{n-1}(x) \notin (-\epsilon, \epsilon)\}. \]
For any \( j \leq n \) and \( x \in f^j(R_n) \subset R_{n-j} \), we have from Lemma 2.2 that
\[ \left| (f^{n-j})'(x) \right| \geq \gamma e^{\epsilon_0(n-j)}. \]
Therefore, if \( K \) is a connected component of \( R_n \)
\[ 2 \geq \lambda(f^n(K)) \geq \gamma e^{\epsilon_0(n-j)} \lambda(f^j(K)) \] (1)
and so
\[ \lambda(f^j(K)) \leq 2\gamma^{-1} e^{-\epsilon_0(n-j)}. \]

17
Let $L$ be a Lipschitz constant for $\log |f'|$ in $[-1, 1] \setminus (-\epsilon, \epsilon)$. For every $x, y \in K$ we have

$$
\log \left( \frac{|(f^n)'(x)|}{|(f^n)'(y)|} \right) = \sum_{j=0}^{n-1} \left( \log |f'(f^j(x))| - \log |f'(f^j(y))| \right)
\leq \sum_{j=0}^{n-1} L |f^j(x) - f^j(y)|
\leq \sum_{j=0}^{n-1} L \lambda(f^j(K))
\leq \sum_{j=0}^{n-1} 2L \gamma^{-1} e^{-c_0(n-j)}, \text{ according to (1)}
\leq \sum_{j=0}^{\infty} 2L \gamma^{-1} e^{-c_0 j}
\equiv \log M.
$$

Thus, we have proved that there is a constant $M$, independent of $n$, such that for every connected component $K$ of $R_n$

$$
\sup_{x \in K} |(f^n)'(x)| / \inf_{x \in K} |(f^n)'(x)| \leq M.
$$

On the other hand, one can easily see that if $u$ and $v$ are the endpoints of $K$, then there are $i, j < n$ such that $f^j(u) = \pm \epsilon$ and $f^j(v) = \pm \epsilon$. Thus, there is $p$, $n \leq p < r + n$, for which $f^p(K)$ first hits $(-\epsilon, \epsilon)$. Our aim now is to prove that $\lambda(f^p(K \setminus R_{p+1})) \geq \rho$. From the choice of $p$ we have that

$$
K \setminus R_{p+1} = \{ x \in [u, v] : f^p(x) \in (-\epsilon, \epsilon) \}.
$$

We distinguish the following two cases:

(i) $f^p(u), f^p(v) \notin (-\epsilon, \epsilon)$.

Since $f^p_K$ is a diffeomorphism and $f^p(K)$ hits $(-\epsilon, \epsilon)$, $f^p(K)$ must contain $(-\epsilon, \epsilon)$ and hence $\lambda(f^p(K \setminus R_{p+1})) \geq 2\epsilon \geq \rho.$
(ii) $f^p(u) \in (-\epsilon, \epsilon)$ or $f^p(v) \in (-\epsilon, \epsilon)$.

Suppose $f^p(u) \in (-\epsilon, \epsilon)$. We remark that in this case $p = i + r$. If $p < i + r$, and since $f^{p-i}(\epsilon) = f^{p-i}(f^i(u)) = f^p(u)$, we would have $f^{p-i}(\epsilon) \in (-\epsilon, \epsilon)$ with $p-i < r$, in contradiction with the definition of $r$. With this argument we also prove that $f^p(v) \notin (-\epsilon, \epsilon)$: otherwise we would have $p = j + r$, giving $i = j$ and then $f^i(u) = \pm \epsilon$, $f^i(v) = \pm \epsilon$, which is impossible since $f^i|_K$ is a diffeomorphism and $f^i(K)$ does not intersect $(-\epsilon, \epsilon)$ for $i < n$. From $f^p(u) = f^r(\epsilon)$ and $f^p(v) \notin (-\epsilon, \epsilon)$ it follows that $\lambda(f^p(K \setminus R_{p+1})) \geq \rho$.

Now from $R_{r+n} \subset R_{p+1}$ we have

$$\lambda(K \setminus R_{r+n}) \geq \lambda(K \setminus R_{p+1}) \geq \rho \left( \sup_{x \in K} |(f^p)'(x)| \right)^{-1}. $$

Since $f^p|_K$ is a diffeomorphism and $\lambda(f^p(K)) \leq 2$ we also have

$$\lambda(K) \leq 2 \left( \inf_{x \in K} |(f^p)'(x)| \right)^{-1}. $$

Thus,

$$\frac{\lambda(K \cap R_{r+n})}{\lambda(K)} \leq 1 - \frac{\lambda(K \setminus R_{r+n})}{\lambda(K)} \leq 1 - \frac{\rho \inf_{x \in K} |(f^p)'(x)|}{2 \sup_{x \in K} |(f^p)'(x)|} \leq 1 - \frac{\rho}{2M}. $$

Defining $\eta^r = 1 - \rho/(2M)$ and summing over all $K$ we get

$$\lambda(R_{r+n}) \leq \eta^r \lambda(R_n). $$

For the sake of notational simplicity we take $a_j = \lambda(R_j)$ for $j \geq 1$ and $a_0 = 1$. Writing $n = qr + s$ for some positive integer $q$ and $0 \leq s < r$ we get

$$a_n = \frac{a_{qr+s}}{a_{q-1}r+s} \cdot \frac{a_{(q-1)r+s}}{a_{q-2}r+s} \cdots \frac{a_{r+s}}{a_s} \cdot a_s \leq \eta^{qr} \cdot a_s. $$

Taking

$$C_3 = \max_{0 \leq r \leq r-1} \left\{ \frac{a_i}{\eta^r} \right\}$$

19
we have that $a_s \leq C_3 \eta^s$ for every $s = 0, \ldots, r - 1$. Hence
\[
\lambda(R_n) = a_n \leq \eta^{qr} \cdot C_3 \eta^s = C_3 \eta^n
\]
and the assertion follows.

4 Construction of the measure

Let $(\mu_n)$ be the sequence of finite measures defined by

\[
\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} f_j^*(\lambda|U_\Delta),
\]

where $f_j^*(\lambda|U_\Delta)$ denotes the pull-back, by $f^j$, of the restriction of the Lebesgue measure to $U_\Delta$: $f_j^*(\lambda|U_\Delta)(A) = \lambda(f^{-j}(A) \cap U_\Delta)$, for every borelean $A$. From the fact that the space of probability measures on $[-1, 1]$ is compact (with respect to the weak topology), it follows that the sequence $(\mu_n)$ must have accumulation points $\mu$. We easily have from the definition of $\mu_n$, that any such $\mu$ is an invariant measure for $f$.

In order to prove absolute continuity of $\mu$ we shall consider sequences of measures $(\mathcal{X}_n)$, $(\mathcal{Y}_n)$, $(\mathcal{Z}_n)$ and $(\mathcal{W}_n)$ such that

\[
\mu_n \leq \frac{1}{n} \sum_{k=0}^{n-1} (\mathcal{X}_k + \mathcal{Y}_k + \mathcal{Z}_k + \mathcal{W}_k)
\]

and prove (see the next section) that there is $\phi \in L^1([-1, 1], \lambda)$ such that

\[
\frac{d}{d\lambda} (\mathcal{X}_k + \mathcal{Y}_k + \mathcal{Z}_k + \mathcal{W}_k) \leq \phi \quad \forall k \in \mathbb{Z}^+,
\]

which implies that

\[
\frac{d\mu_n}{d\lambda} \leq \phi \quad \forall n \in \mathbb{Z}^+.
\]

The fact that any accumulation point $\mu$ is absolutely continuous with respect to Lebesgue measure then follows from the following well known result:

\[
\mu_n \text{ weakly converges to } \mu \Rightarrow \lim \inf \mu_n(B) \geq \mu(B) \quad \forall B \text{ open}.
\]
Indeed, let us consider $A$ such that $\lambda(A) = 0$. Since $\lambda$ is a regular measure we have that

$$
\lambda(A) = \inf \{ \lambda(B) : A \subset B, \ B \text{ open} \},
$$

and so, there is a decreasing sequence $(B_n)$ of open sets containing $A$ such that $\lim \lambda(B_n) = \lambda(A) = 0$. It follows from (1) and (2) that

$$
\mu(B_k) \leq \liminf_{n \to \infty} \mu_n(B_k) \leq \int_{B_k} \phi d\lambda.
$$

Since $\lim_{k \to \infty} \int_{B_k} \phi d\lambda = 0$ we have that $\lim_{k \to \infty} \mu(B_k) = 0$, and so $\mu(A) = 0$, proving the absolute continuity of $\mu$ with respect to $\lambda$.

Before we give the definitions of the measures $X_n, Y_n, Z_n$ and $W_n$ we construct, inductively, a sequence of partitions $P_0 \prec P_1 \prec P_2 \prec \cdots$ of $U$ (modulus a zero Lebesgue measure set) into intervals. For each $n$ we also define a function

$$
r_n : \bigcup_{\omega \in P_n} \omega \longrightarrow \mathbb{Z}^+
$$

that is constant on the elements of $P_n$. The function $r_n$ is called the $n^{th}$ return time. For all $n \geq 0$, $P_n$ and $r_n$ will have the following two properties:

1. $\forall \omega \in P_n \ f^{r_n}_\omega$ is a diffeomorphism,

2. $\forall \omega \in P_n \ I_{m,i} \subset f^{r_n}(\omega) \subset I_{m,i}^+$ for some $|m| \geq \Delta$.

For $n = 0$ we define $P_0 = \{ I_{m,i} : |m| \geq \Delta, 1 \leq i \leq m^2 \}$ and $r_0 \equiv 0$. It is obvious from the definitions of $P_0$ and $r_0$ that (a) and (b) are verified for $n = 0$.

Assuming that $P_{n-1}$ and $r_{n-1}$ are defined and verify properties (a) and (b), we show how the inductive step works for defining $P_n$ and $r_n$. Consider $\omega \in P_{n-1}$ and let $k$ be the first iterate after $r_{n-1}$ for which $f^k(\omega)$ contains some interval $I_{m,i}$ with $|m| \geq \Delta$. Note that for every $j \in \mathbb{Z}^+$ and every interval $J$,

- $f^j_J$ diffeomorphism
- $0 \not\in f^j(J)$

\[ \Rightarrow f^{j+1}_J \text{ diffeomorphism.} \quad \text{(\star)} \]

Since property (b) is verified for $\omega \in P_{n-1}$ we have that $0 \not\in f^{r_{n-1}}(\omega)$. For $r_{n-1} + 1 \leq j < k$ we also have that $0 \not\in f^j(\omega)$; otherwise $f^j(\omega)$ would contain some $I_{m,i}$ with $|m| \geq \Delta$ in contradiction with the definition of $k$. Using the fact that $f^{r_{n-1}}_\omega$ is a diffeomorphism and applying (\star) repeatedly we conclude that $f^k_\omega$ is also a diffeomorphism. Now we distinguish the following two cases:
1. \( f^k(\omega) \subset U^+_\Delta \).

We write \( \omega = \bigcup \omega_{m,i} \pmod{0} \) with the \( \omega_{m,i} \) satisfying

\[
I_{m,i} \subset f^k(\omega_{m,i}) \subset I_{m,i}^+.
\]

This may be done by taking \( \omega_{m,i} = \omega \cap f^{-k}(I_{m,i}) \) except for each one of the two end intervals which, if necessary, is joined to the adjacent one. By definition, these \( \omega_{m,i} \) are the intervals of the partition \( P_n \) contained in \( \omega \) and \( r_n(\omega_{m,i}) = k \).

2. \( f^k(\omega) \not\subset U^+_\Delta \).

Consider \( \omega' = f^{-k}(U_\Delta) : \omega \setminus \omega' \) has either one or two connected components, which we denote by \( \omega_1 \) and \( \omega_2 \). If the image under \( f^k \) of one of the components, say \( \omega_1 \), is contained in \( U^+_\Delta \), we proceed as in the previous case for \( \omega' \cup \omega_1 \) in the place of \( \omega \). For a component \( \omega_l \) (\( l = 1 \) or \( 2 \)) whose image under \( f^k \) is not contained in \( U^+_\Delta \), we consider \( k_l \) to be the first integer after \( k \) for which \( f^{k_l}(\omega_l) \) contains some \( I_{m,i} \) with \( |m| \geq \Delta \), and repeat for \( \omega_l \) and \( k_l \) what we have done for \( \omega \) and \( k \).

**Definition 4.1** For each \( k \geq 0 \) and \( \omega \in P_k \) we define

\[
q_k(\omega) = \inf_{x \in \omega} \{r_{k+1}(x)\}.
\]

**Remark 4.2** Let \( \omega \in P_{n-1} \) and suppose \( f^{q_{n-1}}(\omega) \not\subset U^+_\Delta \). Take \( \omega_1 \) to be a connected component of \( \omega \setminus f^{-q_{n-1}}(U_\Delta) \) whose image under \( f^{q_{n-1}} \) is not contained in \( U^+_\Delta \). We obviously have that

\[
\lambda(f^{q_{n-1}}(\omega_1)) \geq \frac{e^{-\Delta+1} - e^{-\Delta}}{(\Delta - 1)^2}.
\]

Let \( k \) be the first time after \( q_{n-1} \) for which \( f^k(\omega_1) \cap U^+_{\Delta+1} \neq \emptyset \).

(i) If \( f^k(\omega_1) \not\subset U^+_\Delta \), then it must contain some \( I_{m,i} \) with \( |m| \geq \Delta \) and so, by definition, \( r_n(x) \leq k \) for the points \( x \in \omega_1 \) such that \( f^k(x) \in U_\Delta \).

(ii) If \( f^k(\omega_1) \subset U^+_\Delta \), then by Lemma 2.2

\[
\lambda(f^k(\omega_1)) = \lambda(f^{k-q_{n-1}}(f^{q_{n-1}}(\omega_1))) \geq 2 \frac{e^{-\Delta} - e^{-\Delta-1}}{\Delta^2}.
\]

Hence, it follows once more that \( f^k(\omega_1) \) must contain at least one \( I_{m,i} \) with \( |m| \geq \Delta \), and so \( r_n(x) \leq k \) for \( x \in \omega_1 \).
From (i) and (ii) we conclude that for the points $x \in \omega$ such that $r_n(x) > q_{n-1}(\omega)$ one has $f^j(x) \notin U_{\Delta+1}^r$, for $q_{n-1}(\omega) \leq j < r_n(x)$.

**Remark 4.3** It follows from Lemma 3.5 and Remark 4.2 that $r_n$ is defined for all the points in $\omega \in \mathcal{P}_{n-1}$, except, possibly, for a set of measure zero.

**Remark 4.4** For an interval $\omega \in \mathcal{P}_n$ such that $f^{r_n}(\omega) \subset I^+_{m_0,i_0}$ we have

$$q_n(\omega) \geq r_n(\omega) + p(m_0).$$

In order to prove this we only have to show that $f^{r_n+1}(\omega)$ contains no interval $I_{m,i}$ for $1 \leq j < p(m_0)$. Recall that by assumption on the parameter, $|\xi_j| \geq e^{-[\alpha j]}$. Note also that all $I_{m,i}$ with $|m| < [\alpha j]$ have lengths at least

$$e^{-[\alpha j]+1} - e^{-[\alpha j]} \left(\frac{[\alpha j]}{1}\right)^2.$$

For $x, y \in \omega$ we have

$$|f^{r_n+1}(x) - f^{r_n+1}(y)| \leq |f^j(f^{r_n}(x)) - \xi_j| + |f^j(f^{r_n}(y)) - \xi_j| \leq 2e^{-\beta j}$$

and so the length of $f^{r_n+1}(\omega)$ does not exceed $2e^{-\beta j}$. Since $\alpha \leq \beta$, we have

$$2e^{-\beta j} < \frac{e^{-[\alpha j]+1} - e^{-[\alpha j]}}{([\alpha j] - 1)^2}.$$

Therefore, $f^{r_n+1}(\omega)$ contains no interval $I_{m,i}$ with $|m| < [\alpha j]$. On the other hand, for every $z \in \omega$ we have that

$$|f^{r_n+1}(z)| \geq |\xi_j| - e^{-\beta j} \geq e^{-[\alpha j]} - e^{-\beta j}.$$ 

Since the length of the $I_{\pm[j]}$ is equal to

$$e^{-[\alpha j]} - e^{-[\alpha j]-1} \left(\frac{[\alpha j]}{1}\right)^2,$$

and our choice of $\alpha$ and $\beta$ gives

$$e^{-\beta j} < \frac{e^{-[\alpha j]} - e^{-[\alpha j]-1}}{([\alpha j] - 1)^2},$$

it follows that $f^{r_n+1}(\omega)$ contains no interval $I_{m,i}$ with $|m| \geq [\alpha j]$ either.
For the definition of the measures $X_n, Y_n, Z_n$ and $W_n$ it is convenient to distinguish the following types of iterates for each $\omega \in P_n$:

1. The $n^{th}$ return time $r_n$.

2. The *bound period* after the $n^{th}$ return, which consists of the iterates $j$ with $r_n + 1 \leq j < r_n + p_n$, where $p_n(\omega) = p(m)$ and $|m| \geq \Delta$ is the integer such that $f^{r_n}(\omega)$ contains some interval $I_{m,i}$.

3. The *free period* after the $n^{th}$ return, i.e., the set of iterates $j$ with $r_n + p_n \leq j < q_n$.

The measures $X_n, Y_n, Z_n$ and $W_n$ are defined in the following way:

\[
X_n = \sum_{\omega \in P_n} f^{r_n}_n(\lambda|\omega) \\
Y_n = \sum_{\omega \in P_n} \sum_{j=r_n+1}^{r_n+p_n-1} f^j_n(\lambda|\omega) \\
Z_n = \sum_{\omega \in P_n} \sum_{j=r_n+p_n}^{q_n-1} f^j_n(\lambda|\omega) \\
W_n = \sum_{\omega \in P_{n+1}} \sum_{j=q_n}^{r_{n+1}-1} f^j_n(\lambda|\omega)
\]

The first two measures correspond, respectively, to the contributions of the $n^{th}$ return time $r_n$ and the bound period following it. The measure $Z_n$ takes into account the contribution the free iterates following this bound period, up to the next return situation $q_n$. Finally, for the elements of $P_{n+1}$ which at time $q_n$ have not their $(n+1)^{st}$ return we must also consider the contribution of the subsequent iterates, up to the $(n+1)^{st}$ return, and this is done in $W_n$. With the previous definitions we have

\[
\mu_n \leq \frac{1}{n} \sum_{j=0}^{\infty} f^j_n(\lambda|\{x \in U_\Delta : r_n(x) > j\}) \leq \frac{1}{n} \sum_{k=0}^{n-1} (X_k + Y_k + Z_k + W_k)
\]

as we wanted.
5 Density estimates

Proposition 5.1 There is a constant $E_0 \in \mathbb{R}^+$ such that for all $n \geq 0$

$$\frac{d}{d\lambda}\lambda_n \leq E_0$$

Proof: We shall prove that if $E_0$ is sufficiently large, then

$$\frac{d}{d\lambda}\lambda_{n-1} \leq E_0 \Rightarrow \frac{d}{d\lambda}\lambda_n \leq E_0.$$ 

This implies the statement, since the density of $\lambda_0$ is, obviously, bounded by a constant. We are going to define $\mathcal{P}_{n-1}^*$, a refinement of $\mathcal{P}_{n-1}$, that will be written as the disjoint union of three collections of intervals: $\mathcal{P}_{n-1}^0$, $\mathcal{P}_{n-1}^1$ and $\mathcal{P}_{n-1}^2$. This refinement is done in such a way that every interval in $\mathcal{P}_{n-1}^*$ is the maximal subinterval of the interval in $\mathcal{P}_{n-1}$ containing it, on which $r_n$ is constant. Let $\omega$ be an interval of $\mathcal{P}_{n-1}$. We define the elements of $\mathcal{P}_{n-1}^0$, $\mathcal{P}_{n-1}^1$ and $\mathcal{P}_{n-1}^2$ contained in $\omega$ according to the following rules:

- If $f^{q_{n-1}}(\omega) \subset U_\Delta^+$ we say that $\omega \in \mathcal{P}_{n-1}^0$.
- If $f^{q_{n-1}}(\omega) \not\subset U_\Delta^+$ we take $\omega_1 = f^{-q_{n-1}}(U_\Delta) \cap \omega$ to be an element of $\mathcal{P}_{n-1}^1$.

For the other points, we say that $x, y \in \omega$ are in the same $\omega_2 \in \mathcal{P}_{n-1}^2$ if and only if $r_n(x) = r_n(y)$ (which is obviously greater than $q_{n-1}(\omega)$).

It follows from the definitions of $\mathcal{P}_{n-1}^0$, $\mathcal{P}_{n-1}^1$ and $\mathcal{P}_{n-1}^2$ that:

(a) If $\omega_0$ is an element of $\mathcal{P}_{n-1}^0$, then $\omega_0$ is also an element of $\mathcal{P}_{n-1}$. Let $I_{m,i}$ be the interval such that $f^{r_n}(\omega_0) \supset I_{m,i}$. It follows from Lemmas 2.2 and 3.2 that

$$\lambda(f^{r_n}(\omega_0)) = \lambda(f^{q_{n-1}}(\omega_0)) \geq e^{(1-\frac{4\beta}{m})|m|} \lambda(I_{m,i}) \geq \frac{e^{-\frac{4\beta}{m}|m|}}{2m^2}$$

(b) For the intervals $\omega_1 \in \mathcal{P}_{n-1}^1$

$$\lambda(f^{r_n}(\omega_1)) = \lambda(f^{q_{n-1}}(\omega_1)) \geq \frac{e^{-\Delta} - e^{-\Delta - 1}}{\Delta^2} \geq \frac{e^{-\Delta}}{2\Delta^2}$$

since $f^{q_{n-1}}(\omega_1)$ contains at least one $I_{\Delta,i}$.

(c) For an interval $\omega_2 \in \mathcal{P}_{n-1}^2$ let $k$ be the last time before $r_n = r_n(\omega_2)$ such that $\omega$ gave rise to another element of $\mathcal{P}_{n-1}^*$. So, there must be an interval $\omega'$, $\omega_2 \subset \omega' \subset \omega$, such that

$$f^k(\omega') \cap U_\Delta = \emptyset \text{ and } \lambda(f^k(\omega')) \geq \frac{e^{-\Delta+1} - e^{-\Delta}}{(\Delta - 1)^2}.$$
(i) If $f r(n) \omega' \subset U_{\Delta}$, then $\omega' = \omega_2 \in P^2_{n-1}$ and from Lemma 2.2 and Remark 4.2 we have
$$\lambda(f r(n) \omega') \geq \lambda(f k(\omega')) \geq \frac{e^{-\Delta+1}}{2(\Delta - 1)^2}.$$

(ii) If $f r(n) \omega' \not\subset U_{\Delta}$, then
$$\lambda(f r(n) \omega_2) \geq \frac{e^{-\Delta} - e^{-\Delta-1}}{\Delta^2},$$

since $f r(n) \omega'$ must contain at least one $I_{\Delta,i}$ and $\omega_2$ is precisely the part of $\omega'$ that falls into $U_{\Delta}$.

In both cases we have that $\lambda(f r(n) \omega_2) \geq e^{-\Delta}/2\Delta^2$.

Now, let $J$ be an interval in $[-1, 1]$ and assume that $d\mathcal{X}_{n-1}/d\lambda \leq E_0$. We write
$$\mathcal{X}_n(J) = \sum_{i=0}^{2} \sum_{\omega_i \in P^i_{n-1}} \lambda(f^{-r_n}(J) \cap \omega_i).$$

For $i = 0, 1, 2$ we have
$$\lambda(f^{-r_n}(J) \cap \omega_i) = \lambda(f^{-r_n}(J \cap f^{r_n}(\omega_i))) = \frac{1}{||(f^{r_n})'(x)||} \cdot \lambda(J \cap f^{r_n}(\omega_i)), \text{ for some } x \in \omega_i$$
$$\leq \frac{1}{||(f^{r_n})'(x)||} \cdot \frac{\lambda(f^{r_n}(\omega_i))}{\lambda(f^{r_n}(\omega_i))} \cdot \lambda(J)$$
$$\leq \frac{1}{||(f^{r_n})'(y)||} \cdot \frac{\lambda(\omega_i)}{\lambda(f^{r_n}(\omega_i))} \cdot \lambda(J), \text{ for some } y \in \omega_i$$
$$\leq C_2 \cdot \frac{\lambda(\omega_i)}{\lambda(f^{r_n}(\omega_i))} \cdot \lambda(J) \text{ by Lemma 3.4.}$$

Thus, we have
$$\sum_{i=1}^{2} \sum_{\omega_i \in P^i_{n-1}} \lambda(f^{-r_n}(J) \cap \omega_i) \leq \sum_{i=1}^{2} \sum_{\omega_i \in P^i_{n-1}} C_2 \cdot \frac{\lambda(\omega_i)}{\lambda(f^{r_n}(\omega_i))} \cdot \lambda(J)$$
$$\leq C_2 \sum_{i=1}^{2} \sum_{\omega_i \in P^i_{n-1}} \lambda(\omega_i) \cdot 2\Delta^2 e^\Delta \cdot \lambda(J)$$
$$\leq 2C_2 \Delta^2 e^\Delta \lambda(J) = K(\Delta) \lambda(J)$$

26
where \( K(\Delta) = 2C_2\Delta^2 e^\Delta \). For \( i = 0 \) we have
\[
\sum_{\omega_0 \in P_{n-1}^0} \lambda (f^{-r_n}(J) \cap \omega_0) = \sum_{|m| \geq \Delta} \sum_{1 \leq i \leq m^2} \sum_{\omega_0 \in P_{n-1}^0} \lambda (f^{-r_n}(J) \cap \omega_0)
\]
\[
= \sum_{|m| > q} \sum_{1 \leq i \leq m^2} \sum_{\omega_0 \in P_{n-1}^0} \lambda (f^{-r_n}(J) \cap \omega_0) +
\]
\[
+ \sum_{|m| \leq q} \sum_{1 \leq i \leq m^2} \sum_{\omega_0 \in P_{n-1}^0} \lambda (f^{-r_n}(J) \cap \omega_0)
\]
where \( q \) is to be determined in such a way that
\[
S_1 = \sum_{|m| > q} \sum_{1 \leq i \leq m^2} \sum_{\omega_0 \in P_{n-1}^0} \lambda (f^{-r_n}(J) \cap \omega_0) \leq \frac{1}{2} E_0 \lambda (J) \tag{1}
\]
and
\[
S_2 = \sum_{|m| \leq q} \sum_{1 \leq i \leq m^2} \sum_{\omega_0 \in P_{n-1}^0} \lambda (f^{-r_n}(J) \cap \omega_0) \leq \hat{K}(q) \lambda (J) \tag{2}
\]
where \( \hat{K}(q) \) is a constant not depending on \( n \) or \( J \). If this happens, we have
\[
\mathcal{X}_n(J) \leq (K(\Delta) + \hat{K}(q) + \frac{1}{2} E_0) \lambda (J),
\]
and so, choosing \( E_0 \) such that
\[
\frac{1}{2} E_0 \geq K(\Delta) + \hat{K}(q),
\]
we have
\[
\frac{d}{d\lambda} \mathcal{X}_n \leq E_0.
\]
Now, we only have to prove that it is possible to choose \( q \) so that \( S_1 \) and \( S_2 \) satisfy (1) and (2). For \( S_1 \) we have
\[
S_1 \leq \sum_{|m| > q} \sum_{1 \leq i \leq m^2} C_2 \cdot \frac{\lambda (\omega_0)}{\lambda (f^{-r_n}(\omega_0))} \cdot \lambda (J)
\]
\[
\sum_{|m| > q} \sum_{1 \leq i \leq m^2} C_2 \cdot 2m^2 e^{\frac{4|m|}{e}} \cdot \lambda(|\omega_0|) \cdot \lambda(J) 
\]
\[
\leq \sum_{|m| > q} C_2 \cdot 2m^2 e^{\frac{4|m|}{e}} \cdot \lambda(J) \sum_{\omega_0 \in P_{n-1}^0} \lambda(f^{-r_{n-1}}(I_{m,i}^n) \cap \omega_0) 
\]
\[
\leq \sum_{|m| > q} C_2 \cdot 2m^2 e^{\frac{4|m|}{e}} \cdot \lambda(J) \cdot \lambda_0(I_{m,i}^n) 
\]
\[
\leq \sum_{|m| > q} C_2 \cdot 2m^2 e^{\frac{4|m|}{e}} \cdot \lambda(J) \cdot E_0 \lambda(I_{m,i}^n), \text{ by induction.} 
\]

Attending to the fact that
\[
\lambda(I_{m,i}^n) \leq 10 e^{-|m| \frac{m}{m^2}}, 
\]
and summing on \(1 \leq i \leq m^2\), we have
\[
S_1 \leq 2E_0 C_2 \sum_{|m| > q} m^2 e^{\frac{4|m|}{e}} \cdot 10 e^{-|m| \frac{m}{m^2}} \cdot \lambda(J) 
\]
\[
\leq 2E_0 C_2 \sum_{|m| > q} 10m^2 e^{(-1+\frac{4|m|}{e})|m|} \lambda(J). 
\]

Thus, we only have to choose \(q\) such that
\[
2C_2 \sum_{|m| > q} 10m^2 e^{(-1+\frac{4|m|}{e})|m|} \leq \frac{1}{2}. 
\]

For \(S_2\) we have
\[
S_2 \leq \sum_{|m| \leq q} \sum_{1 \leq i \leq m^2} C_2 \cdot \frac{\lambda(|\omega_0|)}{\lambda(f^{-r_{n-1}}(\omega_0))} \cdot \lambda(J) 
\]
\[
\leq \sum_{|m| \leq q} \sum_{\omega_0 \in P_{n-1}^0} C_2 \cdot 2m^2 e^{\frac{4|m|}{e}} \cdot \lambda(|\omega_0|) \cdot \lambda(J) 
\]
\[
\leq \sum_{|m| \leq q} 2C_2 m^4 e^{\frac{4|m|}{e}} \cdot \lambda(J) 
\]
\[
\leq 2q \cdot 2C_2 q^4 e^{\frac{4|q|}{e}} \cdot \lambda(J) 
\]
\[
28 
\]
So, we only have to take 

\[ \hat{K}(q) = 4C_2 q^5 e^{\frac{4q}{|q|}}. \]

**Proposition 5.2** There is \( \phi_0 \in L^1(\lambda) \) such that for all \( n \geq 1 \)

\[ \frac{d}{d\lambda} Y_n \leq \phi_0 \]

**Proof:** For every interval \( J \) in \([-1,1]\) we have

\[ Y_n(J) = \sum_{\omega \in P_n} \sum_{j=1}^{p_n-1} \lambda \left( f^{-r_n} \left( f^{-j}(J) \right) \cap \omega \right). \]

Since \( f^{r_n}(\omega) \subset I_{m,i}^+ \subset I_m^+ \) and \( p_n(\omega) = p(m) \) for some \( |m| \geq \Delta \), it follows that

\[ Y_n(J) \leq \sum_{|m| \geq \Delta} \sum_{j=1}^{p(m)-1} \lambda \left( f^{-r_n} \left( f^{-j}(J) \cap I_m^+ \right) \cap \omega \right) \]

\[ = \sum_{|m| \geq \Delta} \sum_{j=1}^{p(m)-1} \lambda \left( f^{-r_n} \left( f^{-j}(J) \cap I_m^+ \right) \cap \omega \right) \]

\[ = \sum_{|m| \geq \Delta} \sum_{j=1}^{p(m)-1} \lambda \left( f^{-j}(J) \cap I_m^+ \right) \]

\[ \leq \sum_{|m| \geq \Delta} \sum_{j=1}^{p(m)-1} E_0 \lambda \left( f^{-j}(J) \cap I_m^+ \right), \] by Proposition 5.1.

Considering the measure

\[ \mathcal{Y} = E_0 \sum_{|m| \geq \Delta} \sum_{j=1}^{p(m)-1} f_j^2(\lambda|I_m^+) \]

we get from the previous computations that \( \mathcal{Y}_n \leq \mathcal{Y} \) for all \( n \geq 1 \). We have that

\[ \mathcal{Y}([-1,1]) = E_0 \sum_{|m| \geq \Delta} \sum_{j=1}^{p(m)-1} \lambda \left( f^{-j}([-1,1] \cap I_m^+) \right) \]

\[ \leq E_0 \sum_{|m| \geq \Delta} \sum_{j=1}^{p(m)-1} \lambda \left( I_m^+ \right) \]
For each \( |m| \geq \Delta \), we have

\[
\sum_{j=1}^{p(m)-1} (e - e^{-2}) e^{-|m|} \leq (e - e^{-2}) E_0 \sum_{|m| \geq \Delta} \frac{3}{c} |m| \cdot e^{-|m|}, \quad \text{by Lemma 3.2.}
\]

Hence, \( \mathcal{Y} \) is a finite measure. Since it is also absolutely continuous with respect to Lebesgue measure, we take

\[
\phi_0 = \frac{d}{d\lambda} \mathcal{Y}
\]

and the result is proved, attending to the fact that \( \mathcal{Y}_n \leq \mathcal{Y} \) for all \( n \geq 1 \).

**Definition 5.3** For each \( |m| \geq \Delta \) and \( 1 \leq i \leq m^2 \) let \( q(m,i) \) be the smallest \( j > 0 \) for which \( f^j(I_{m,i}) \) contains some interval \( I_{m_0,i_0} \) with \( |m_0| \geq \Delta \).

**Lemma 5.4** There is a constant \( C_4 \) such that \( q(m,i) \leq C_4 |m| \) for all \( |m| \geq \Delta \) and \( 1 \leq i \leq m^2 \).

**Proof:** For each interval \( I_{m,i} \) we define \( p_0 = p(m) \) and let \( t_1 \) be the first \( j \geq p_0 \) such that \( f^j(I_{m,i}) \cap U_{\Delta} \neq \emptyset \). If \( t_1 < q(m,i) \) and \( f^{t_1}(I_{m,i}) \subset I_+^{m_1,i_1} \) we define \( p_1 = p(m_1) \) and let \( t_2 \) be the smallest \( j \geq t_1 + p_1 \) such that \( f^j(I_{m,i}) \cap U_{\Delta} \neq \emptyset \). We proceed this way until \( t_s = q(m,i) \). For \( k = 0, \ldots, s-1 \) we have

\[
\lambda(f^{t_{k+1}}(I_{m,i})) = \lambda(f^{t_{k+1}-t_k}(f^{t_k}(I_{m,i}))) = |(f^{t_{k+1}-t_k})'(x)| \cdot \lambda(f^{t_k}(I_{m,i})), \quad \text{for some } x \in f^{t_k}(I_{m,i})
\]

\[
= |(f^{t_{k+1}-t_k-p_k})'(f^{p_k}(x))| \cdot |(f^{p_k})'(x)| \cdot \lambda(f^{t_k}(I_{m,i})).
\]

From Lemma 2.2, Lemma 3.2, and since \( x \in f^{t_k}(I_{m,i}) \subset I_+^{m_1,i_1} \), we get

\[
\lambda(f^{t_{k+1}}(I_{m,i})) \geq e^{c_0(t_{k+1}-t_k-p_k)} \cdot e^{(1-\frac{4\beta}{e})|m_k|} \cdot \lambda(f^{t_k}(I_{m,i})).
\]

We also have from Lemma 3.2 that \( p_k \leq 3|m_k|/c \). So,

\[
\lambda(f^{t_{k+1}}(I_{m,i})) \geq e^{c_0(t_{k+1}-t_k-p_k)} \cdot e^{(1-\frac{4\beta}{e})|p_k|} \cdot \lambda(f^{t_k}(I_{m,i})) \geq e^{(1-\frac{4\beta}{e}) (t_{k+1}-t_k)} \cdot \lambda(f^{t_k}(I_{m,i})).
\]

Repeating the argument we get

\[
\lambda(f^{t_s}(I_{m,i})) \geq e^{(1-\frac{4\beta}{e}) (t_s-t_0)} \cdot \lambda(f^{t_0}(I_{m,i})).
\]
Taking into account that $t_s = q(m, i)$, $t_0 = 0$ and $\lambda(f^{t_s}(I_{m,i})) \leq 2$, we have
\[
2 \geq e^{\left(\frac{c-4\beta}{3}\right)q(m,i)} \cdot \lambda(I_{m,i}) \\
\geq e^{\left(\frac{c-4\beta}{3}\right)q(m,i)} \cdot \frac{1 - e^{-1}}{m^2} e^{-|m|}
\]
and so
\[
q(m, i) \leq \frac{3}{c - 4\beta} \cdot \left(|m| \log m^2 - \log(1 - e^{-1}) + \log 2 \right) \\
\leq \frac{6|m|}{c - 4\beta}.
\]
The result follows by choosing $C_4 = 6/(c - 4\beta)$.

**Proposition 5.5** There is $\phi_1 \in L^1(\lambda)$ such that for all $n \geq 0$
\[
\frac{d}{d\lambda} Z_n \leq \phi_1
\]

**Proof:** For every interval $J$ in $[-1, 1]$ we have
\[
Z_n(J) = \sum_{\omega \in \mathcal{P}_n} \sum_{j=p_n}^{q_n-r_n-1} \lambda \left( f^{-r_n-j}(J) \cap \omega \right) \\
\leq \sum_{|m| \geq \Delta} \sum_{1 \leq i \leq m^2} \sum_{\omega \in \mathcal{P}_n} \sum_{j=p_n}^{q_n-r_n-1} \lambda \left( f^{-r_n} \left( f^{-j}(J) \cap I^+_{m,i} \right) \cap \omega \right).
\]

For $\omega \in \mathcal{P}_n$ let $I_{m,i}$ be the interval such that $f^{r_n}(\omega) \supset I_{m,i}$. It easily follows from the definitions that $q_n(\omega) - r_n(\omega) \leq q(m, i)$. Thus,
\[
Z_n(J) \leq \sum_{|m| \geq \Delta} \sum_{j=p(m)}^{q(m,i)-1} \sum_{1 \leq i \leq m^2} \lambda \left( f^{-r_n} \left( f^{-j}(J) \cap I^+_{m,i} \right) \cap \omega \right) \\
\leq E_0 \sum_{|m| \geq \Delta} \sum_{j=p(m)}^{q(m,i)-1} \lambda \left( f^{-j}(J) \cap I^+_{m,i} \right), \text{ by Proposition 5.1.}
\]
Defining the measure
\[
Z = \sum_{|m| \geq \Delta} \sum_{1 \leq i \leq m^2} q(m,i) - 1 \sum_{j=p(m)} f_j^i(\lambda | I_{m,i}^+ )
\]
we have that
\[
Z([-1,1]) = \sum_{|m| \geq \Delta} \sum_{1 \leq i \leq m^2} q(m,i) - 1 \sum_{j=p(m)} \lambda(I_{m,i}^+ ) \cap I_{m,i}^+
\]
\[
\leq \sum_{|m| \geq \Delta} \sum_{1 \leq i \leq m^2} \lambda(I_{m,i}^+ )
\]
\[
\leq \sum_{|m| \geq \Delta} C_4 |m| \lambda(I_{m,i}^+ ) \text{, by Lemma 5.4}
\]
\[
\leq \sum_{|m| \geq \Delta} C_4 |m| \cdot \frac{10 e^{-|m|}}{m^2}
\]
\[
\leq \sum_{|m| \geq \Delta} 10 C_4 |m| e^{-|m|}.
\]
Hence, \( Z \) is a finite measure. Since \( Z \) is also absolutely continuous with respect to Lebesgue measure and \( Z_n \leq E_0 Z \) for all \( n \geq 0 \), taking
\[
\phi_1 = E_0 \frac{d}{d\lambda} Z
\]
we have proved the result.

**Proposition 5.6** There is \( \phi_2 \in L^1(\lambda) \) such that for all \( n \geq 1 \)
\[
\frac{d}{d\lambda} W_n \leq \phi_2
\]

**Proof:** For every interval \( J \) in \([-1,1]\) we have
\[
W_n(J) = \sum_{\omega \in P_{n+1}} \sum_{j=q_n}^{r_{n+1} - 1} \lambda(f^{-j}(J) \cap \omega).
\]
So, we may write
\[
W_n(J) = \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \sum_{\omega \in P_{n+1}^{\omega} \cap \omega} \lambda(f^{-q_n-j}(J) \cap \omega).
\]
Let \( \omega \in P_{n+1} \) be such that \( r_n + 1 = q_n + k \), and consider \( x \in \omega \). It follows from Remark 4.2 that
\[
f^j(x) \not\in U_{\Delta+1}^+ \text{ for } q_n \leq j < r_n + 1 = q_n + k,
\]
and so
\[
f^{q_n}(x) \in R_k.
\]
Hence, we have
\[
W_n(J) \leq \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \sum_{\omega \in P_{n+1}^{\omega} \cap \omega} \lambda(f^{-q_n}(f^{-j}(J) \cap R_k) \cap \omega)
\]
\[
\leq \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \sum_{\omega \in P_n^{\omega} \cap \omega} \lambda(f^{-q_n}(f^{-j}(J) \cap R_k) \cap \omega)
\]
\[
\leq \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \sum_{\omega \in P_n^{\omega} \cap \omega} \frac{1}{|(f^{q_n})'(x)|} \cdot \lambda(f^{-j}(J) \cap R_k), \text{ for some } x \in \omega
\]
\[
= \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \sum_{\omega \in P_n^{\omega} \cap \omega} \frac{1}{|(f^{q_n})'(x)|} \cdot \lambda(f^{q_n}(\omega)) \cdot \lambda(f^{-j}(J) \cap R_k)
\]
\[
\leq \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \sum_{\omega \in P_n^{\omega} \cap \omega} \frac{|(f^{q_n})'(y)|}{|(f^{q_n})'(x)|} \cdot \lambda(\omega) \cdot \lambda(f^{-j}(J) \cap R_k).
\]
For the intervals \( \omega \in P_n \) such that \( f^{q_n}(\omega) \not\in U_{\Delta} \), we have
\[
\lambda(f^{q_n}(\omega)) \geq \frac{e^{-\Delta}}{2\Delta^2},
\]
and so
\[
W_n(J) \leq \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \sum_{\omega \in P_n^{\omega} \cap \omega} \lambda(f^{q_n}(\omega)) \cdot \lambda(f^{-j}(J) \cap R_k)
\]
\[
\leq \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} C_2 \cdot \frac{|f^{q_n-1}(y)|}{f^{q_n-1}(x)} \cdot 2e^2 \Delta^2 \lambda(\omega) \cdot \lambda(f^{-j}(J) \cap R_k), \text{ by Lemma 3.4.}
\]
We remark that for $\omega \in \mathcal{P}_n$ we have $f^{q_n-1}(\omega) \cap U_{\Delta+1} = \emptyset$. Indeed, if $f^{q_n-1}(\omega)$ intersected $U_{\Delta+1}$, and since $f^{q_n-1}(\omega)$ contains no interval $I_{m,i}$ with $|m| \geq \Delta$, then, $f^{q_n-1}(\omega)$ would be contained in $U_{\Delta}$. Hence, we would have $f(f^{q_n-1}(\omega)) = f^{q_n}(\omega)$ not intersecting $U_{\Delta}$, in contradiction with the definition of $q_n$. Thus,

$$W_n(J) \leq \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} C_2 \cdot \frac{1}{e^{-\Delta-1}} \cdot 2e^2 \Delta^2 \cdot \lambda(f^{-j}(J) \cap R_k)$$

$$= E_1 \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \lambda(f^{-j}(J) \cap R_k)$$

where $E_1 = 2C_2 e^{2\Delta+1} \Delta^2$. Let us define the measure

$$\mathcal{W} = E_1 \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} f_j^* (\lambda|R_k).$$

We have proved before that $W_n \leq \mathcal{W}$ for all $n \geq 1$. The measure $\mathcal{W}$ satisfies

$$\mathcal{W}([-1, 1]) = E_1 \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \lambda(f^{-j}([-1, 1]) \cap R_k)$$

$$\leq E_1 \sum_{k=1}^{\infty} kC_3 \eta^k, \text{ by Lemma 3.5.}$$

Hence, $\mathcal{W}$ is a finite measure. Since $\mathcal{W}$ is also absolutely continuous with respect to Lebesgue measure, we take

$$\phi_2 = \frac{d}{d\lambda} \mathcal{W}$$

and the statement is proved.

References


