PHYSICAL MEASURES FOR NON-UNIFORMLY EXPANDING MAPS

by

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1. Introduction

One of the main goals of Dynamical Systems is to describe the typical behavior of orbits, specially as time goes to infinity. Even in cases of very simple evolution laws the orbits may have a rather complicated behavior, specially because systems may display sensitivity on the initial conditions, i.e. a small variation on the initial state gives rise to a completely different behavior of its orbit. The approach to this kind of systems has been particularly well succeeded through *physical measures*, or *Sinai-Ruelle-Bowen (SRB) measures*, which characterize asymptotically, in time average, a large set of orbits in the phase space.

Systems displaying uniformly expanding/contracting behavior on Riemannian manifolds (uniformly expanding maps and Axiom A attractors for diffeomorphisms and flows) have been exhaustively studied in the last decades; see [**Bo70, Bo75, BR, KS, Ru, Si68, Si72**]. Systems exhibiting expansion only in asymptotic terms have been considered in [**Ja**], where the existence of physical measures for many quadratic transformations of the interval were established; see also [**BC85, BY92**]. Related to [**BC85**] is the work [**BC91**] for Hénon maps exhibiting strange attractors. Motivated by the results for multidimensional non-uniformly expanding systems in [**Vi2, Al00**], general conclusions for systems exhibiting non-uniformly expanding behavior were drawn in [**ABV**].

The aim of these notes is to present an introduction to physical measures for non-uniformly expanding maps in finite dimensional compact Riemannian manifolds. Here we follow the approach in [**ABV**]. To better illustrate it we first consider uniformly expanding maps, then we consider non-uniformly expanding local diffeomorphisms, and finally non-uniformly maps with critical sets.

Readers should be acquainted with concepts from Measure Theory such as measure spaces, integration, absolute continuity and weak* convergence of measures. For the sake of completeness we present all these concepts and major results in an Appendix. In the next subsection we make an introduction to some classical results in Ergodic Theory leading to the definition of physical measures. In a second subsection we depict a nice picture of the global situation for one-dimensional quadratic maps concerning the existence of physical measures.

1.1. Physical measures. — We start by presenting some classical results on ergodic theory. For a detailed explanation and complete proofs we recommend [**Ma87**] or [**Wa**].

Definition 1.1. — Let $(X, \mathcal{A}, \mu)$ be a measure space. We say that $f\colon X \to X$ is *measurable* if $f^{-1}(A) \in \mathcal{A}$ for all $A \in \mathcal{A}$. We say that f *preserves* μ, or μ is *invariant* by f, if $\mu(f^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{A}$.

It easily follows from this definition that if μ is an invariant measure, then the sets $\{x \in M\colon x \in A\}$ and $\{x \in M\colon f^n(x) \in A\}$ have the same μ measure for every $n \in \mathbb{N}$. This means that the probability of finding a point in a measurable set does not depend on the moment we consider.

Let X be a compact metric space. We denote by $\mathbb{P}(X)$ the space of probability measures defined on the Borel σ-algebra of X. We introduce the weak* topology on $\mathbb{P}(X)$ in the following way: a sequence $(\mu_n)_n$ in $\mathbb{P}(X)$ converges to $\mu \in \mathbb{P}(X)$ if and only if

$$\int \varphi d\mu_n \to \int \varphi d\mu, \quad \text{for each continuous } \varphi\colon X \to \mathbb{R}.$$

We associate to a measurable map $f\colon X \to X$ an operator $f_*\colon \mathbb{P}(X) \to \mathbb{P}(X)$, assigning to each $\mu \in \mathbb{P}(X)$ the *push-forward* $f_*\mu$, which is defined as

$$f_*\mu(A) = \mu(f^{-1}(A)), \quad \text{for each } A \in \mathcal{A}.$$

One can easily check that f_* is continuous whenever f is continuous. Note that μ is invariant by f if and only if $f_*\mu = \mu$. If f is continuous, then taking some measure $\mu \in \mathbb{P}(X)$, a Dirac measure for instance, we define a sequence of measures in $\mathbb{P}(X)$,

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j \mu.$$

A weak* accumulation point of this sequence is necessarily a fixed point for f_*.

Theorem 1.2. — *Let M be a compact metric space. If $f\colon M \to M$ is a continuous transformation, then f has some invariant Borel probability measure.*

Some of the first results on the probabilistic features of dynamical systems go back to the work of Poincaré for conservative systems, and can be translated to our context in the following way:

***Theorem 1.3* (Poincaré).** — *Let f preserve a probability measure μ. If A is a measurable set, then for almost every $x \in A$, there are infinitely many $n \in \mathbb{N}$ for which $f^n(x) \in A$.*

The previous result gives no information on the asymptotic *frequency* that typical orbits visit A, i.e.

$$\lim_{n\to\infty} \frac{\#\{0 \le j < n\colon f^j(x) \in A\}}{n}. \tag{1}$$

Does this limit exist? Does it depend on x? Birkhoff Ergodic Theorem gives answers to these questions and, in fact, more general conclusions. Before we state it, let us introduce some important concept on this subject.

Definition 1.4. — Assume that f preserves a measure μ. We say that μ is *ergodic* if $\mu(A) = 0$ or $\mu(M \setminus A) = 0$ whenever $A \in \mathcal{A}$ satisfies $f^{-1}(A) = A$.

Observing that $f^{-1}(A) = A$ implies that $f(A) \subset A$ and $f(M \setminus A) \subset M \setminus A$, this means that the space cannot be decomposed into two parts which are relevant (positive measure) that do not interact.

***Theorem 1.5* (Birkhoff).** — *Assume that f preserves a probability measure μ. If φ is integrable, then there is an integrable function φ^* such that for μ almost every $x \in M$*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(x)) = \varphi^*(x).$$

Moreover, $\varphi^*(x) = \int \varphi d\mu$ for μ almost every $x \in M$, provided μ is ergodic.

Taking φ as the characteristic function of a measurable set A, we easily deduce that the limit in (1) exists for μ almost every $x \in M$. Furthermore, if μ is ergodic, then that limit is equal $\mu(A)$. This means that the frequency of visits to A coincides with the proportion that A occupies in the phase space.

The results we have presented so far concern dynamics over a probability measure space with no additional structure on the underlying phase space M. Frequently M has a Riemannian manifold structure and a volume form which gives rise to a Lebesgue measure m on the Borel sets of M. Birkhoff Ergodic Theorem states that asymptotic time averages exist for almost every point, with respect to an invariant measure μ, and they coincide with the spatial average, provided μ is ergodic. However, an invariant measure can lack of physical meaning, in the sense that sets with full μ measure may have zero Lebesgue measure. This problem can be overcome by the notion that we present next.

***Definition 1.6*.** — An invariant probability measure μ is called a *physical measure* for $f\colon M \to M$ if, for a positive Lebesgue measure set of points $x \in M$,

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi\, d\mu, \tag{2}$$

for all continuous $\varphi : M \to \mathbb{R}$.

This means that the averages of Dirac measures over the orbit of x converge in the weak* topology to the measure μ. We define the *basin* of μ as the set of points $x \in M$ for which (2) holds for all continuous φ. The following useful result follows easily from Birkhoff Ergodic Theorem.

***Theorem 1.7*.** — *Every absolutely continuous (with respect to Lebesgue measure) ergodic probability measure is a physical measure.*

Indeed, if a measure μ is ergodic, then by Birkhoff Ergodic Theorem its basin has full μ measure. By absolute continuity, the basin of μ cannot have zero Lebesgue measure.

1.2. Low dimensional dynamics. — There is no need of big complexity in the formulation of evolution laws for which intricate dynamical behavior occurs. To illustrate this, the basic model is the family of quadratic maps $q_a(x) = 1-ax^2$, where $x \in [-1,1]$ and $a \in [0,2]$ is a parameter. Here is where the rich part of the dynamics lies. For parameters out of this range or points out of this domain the dynamics is essentially well understood. In spite of its simple appearance, the dynamics of these maps presents many remarkable phenomena. From the topological point of view, the situation is quite well understood in most situations.

Theorem 1.8 (**[Ly1]**, **[GS]**). — *There is an open and dense set of parameters $a \in [0,2]$ for which q_a has a periodic orbit attracting Lebesgue almost every point.*

In spite of its simple formulation, this remained as a long term conjecture in one dimensional dynamics. From a probabilistic point of view, the situation is completely different. Its richness first became apparent with the work of Jakobson, where it was shown that a positive measure set of parameters corresponds to quadratic maps with chaotic behavior.

Theorem 1.9 (**[Ja]**). — *There is a positive Lebesgue measure set of parameters $a \in [0,2]$ for which q_a has an absolutely continuous ergodic probability measure μ_a.*

By Theorem 1.7 we have that μ_a is a physical measure. Some extra knowledge on the properties of μ_a allows us to show that $\log|q_a'|$ is μ_a integrable and $\int \log|q_a'| d\mu_a > 0$. By Birkhoff Ergodic Theorem

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} \log|q_a'((q_a^j(x))| = \int \log|q_a'| d\mu,$$

and so, using the chain rule, we have a positive *Lyapunov exponent* at almost every x:

$$\lim_{n\to\infty} \frac{1}{n} \log|(q_a^n)'(x)| > 0.$$

The existence of this positive Lyapunov exponent gives one pervasive feature of chaos: *sensitivity to the initial conditions.*

As we have seen, at least two types of distinct behavior are present on the quadratic family, and they alternate in a complicate way. Besides these two types, different behaviors were shown to exist, including examples with bad statistics, like absence of a physical measure or a physical measure concentrated on a hyperbolic repeller. Finally, Lyubich depicted a nice picture of the global situation.

Theorem 1.10 (**[Ly2]**). — *For Lebesgue almost every $a \in [0,2]$ the map q_a has either a periodic attracting orbit or an absolutely continuous ergodic probability measure.*

Though we have used the absolutely continuous ergodic measure to obtain a positive Lyapunov exponent, the existence of this exponent can be deduced directly for a positive Lebesgue measure subset of parameters. The big difficulty in carrying this out is that quadratic maps combine regions of the phase space where the dynamics expands, together with a critical region where the derivative becomes arbitrarily small. In [**BC85**], Benedicks and Carleson implemented a strategy which enabled them to prove the existence of a positive Lyapunov exponent not only for quadratic maps, but also for the Hénon maps

$$\begin{array}{rccc} f_{a,b}: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (x,y) & \longmapsto & (1-ax^2+y, bx). \end{array}$$

In [**He**], Hénon proposed this two parameter family as a model for non-linear two dimensional dynamics. This can be thought as a simplified discrete-time version of the Lorenz flow and interpreted as an unfolding of the quadratic family. For $b=0$ orbits eventually lie on $\{y=0\}$ and dynamics can be thought as that of quadratic maps. Based on numerical experiments for $a=1.4$ and $b=0.3$, Hénon conjectured that this system should have a *strange attractor*. It was not at all *a priori* clear that the attractor detected experimentally by Hénon was not a long stable periodic orbit. Benedicks and Carleson managed to prove that Hénon's conjecture was true for small $b>0$. It remains an interesting open question to know if the chaotic attractor exists for Hénon's choice of parameters $a=1.4$ and $b=0.3$.

Theorem 1.11 **([BC91])**. — *There is a positive Lebesgue measure set $\mathcal{BC}$ of parameters such that for each $(a,b)\in\mathcal{BC}$ the map $f=f_{a,b}$ has the following properties:*

1. *there is an open set $U\subset\mathbb{R}^2$ such that $\overline{f(U)}\subset U$ and $\Lambda=\cap_{n=0}^{\infty} f^n(U)$ attracts the orbit of every $x\in U$;*
2. *there is $z_0\in\Lambda$ whose orbit is dense on Λ, and there is $c>0$ such that $\|Df^n(z_0)(0,1)\|\geq e^{cn}$ for all $n\geq 1$;*
3. *f has a unique physical measure supported on Λ.*

The physical measure was obtained by Benedicks and Young in [**BY92**]. The second item of the theorem gives the existence of a positive Lyapunov exponent in a dense orbit, thus showing that the attractor displays sensitive dependence to the initial conditions for the parameters in $\mathcal{BC}$.

2. Uniformly expanding maps

From here on we shall always consider maps $f\colon M\to M$ where M is a Riemannian manifold endowed with a normalized Lebesgue measure m. The absolute continuity of measures will be always meant with respect to the Lebesgue measure.

Definition 2.1. — A map $f\colon M \to M$ is called *uniformly expanding* if, for some choice of a Riemannian metric $\| \quad \|$, there is $0 < \sigma < 1$ such that $\|Df(x)^{-1}\| < \sigma$ for every $x \in M$.

This means that $Df(x)$ expands in *every direction*. Note that in dimension greater than one, saying that $\|A\| > 1$ for an invertible linear transformation A, is not necessarily equivalent to say that $\|A^{-1}\| < 1$. Indeed, considering

$$A = \begin{pmatrix} 1/2 & 0 \\ 0 & 3 \end{pmatrix}$$

with respect to the usual norm we have $\|A\| = 3$ but $\|A^{-1}\| = 2$. Actually, the condition $\|A^{-1}\| < 1$ ensures that A expands in every direction, whereas $\|A\| > 1$ means that A has some direction of expansion.

Example 2.2. — Let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear isomorphism of $\mathbb{R}^n$ for which $F(\mathbb{Z}^n) \subset \mathbb{Z}^n$. Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-dimensional torus and $\pi\colon \mathbb{R}^n \to \mathbb{T}^n$ the canonical projection. Then there is a unique transformation $f\colon \mathbb{T}^n \to \mathbb{T}^n$ such that $f \circ \pi = \pi \circ F$. If the eigenvalues $\lambda_1, \dots, \lambda_n$ of F are all distinct and bigger than one in absolute value, then f is uniformly expanding. Actually, we may take any σ such that $\max_i |\lambda_i^{-1}| < \sigma < 1$.

The goal of the next subsections is to prove the following result on the existence and uniqueness of physical measures for uniformly expanding maps.

Theorem 2.3. — *Let $f : M \to M$ be a C^1 uniformly expanding map on a compact connected manifold M. Assume that* $\det Df$ *is Hölder continuous. Then f has a unique ergodic absolutely continuous invariant probability measure whose support coincides with M and its basin has full Lebesgue measure in M.*

2.1. Preballs and distortion bounds. — Here we introduce hyperbolic preballs, which constitute a useful tool for proving the existence of absolutely continuous ergodic measures, also in the context of non-uniformly expanding maps. In this subsection we do not assume that f is uniformly expanding.

Definition 2.4. — Let $\delta_1 > 0$ be a small number and take $0 < \sigma < 1$. Given $n \geq 1$ and $x \in M$, we say that a neighborhood $V_n(x)$ of x is a (σ, δ_1)*-hyperbolic preball* if

1. f^n sends $V_n(x)$ diffeomorphically onto $B_{\delta_1}(f^n(x))$;
2. for every $y \in V_n(x)$ and $1 \leq k \leq n$ we have $\|Df^k(f^{n-k}(y))^{-1}\| \leq \sigma^k$.

For later use, let us observe that if $V_n(x)$ is a (σ, δ_1)-hyperbolic preball, then $f(V_n(x))$ is also a (σ, δ_1)-hyperbolic preball which is sent by f^{n-1} diffeomorphically onto $B_{\delta_1}(f^n(x))$. This can be obviously generalized for any $f^k(V_n(x))$ with $1 \leq k < n$.

Lemma 2.5. — *If* $V_n(x)$ *is a* (σ, δ_1)*-hyperbolic preball, then for every* $y \in V_n(x)$ *and* $1 \le k \le n$ *we have*

$$\operatorname{dist}(f^{n-k}(y), f^{n-k}(x)) \le \sigma^k \operatorname{dist}(f^n(y), f^n(x)).$$

Proof. — Let γ_n be a curve of minimal length connecting $f^n(x)$ to $f^n(y)$. This curve γ_n must obviously be contained in $B_{\delta_1}(f^n(x))$. For $1 \le k \le n$, let γ_{n-k} be the (unique) curve in $f^{n-k}(V_n(x))$ joining $f^{n-k}(x)$ to $f^{n-k}(y)$ such that $f^k(\gamma_{n-k}) = \gamma_n$. We have for every $n \ge 1$

$$\begin{aligned}
\operatorname{length}(\gamma_n) &= \int \|\gamma_n'(t)\| dt \\
&= \int \|Df^k(\gamma_{n-k}(t)) \cdot \gamma_{n-k}'(t)\| dt \\
&\ge \sigma^{-k} \int \|\gamma_{n-k}'(t)\| dt \\
&= \sigma^{-k} \operatorname{length}(\gamma_{n-k}).
\end{aligned}$$

As a consequence,

$$\operatorname{dist}(f^{n-k}(y), f^{n-k}(x)) \le \operatorname{length}(\gamma_{n-k}) \le \sigma^k \operatorname{length}(\gamma_n) = \sigma^k \operatorname{dist}(f^n(y), f^n(x)).$$

This gives the result. □

***Corollary 2.6* (Bounded distortion)**. — *Assume that* $\det Df$ *is Hölder continuous. Given* $\delta > 0$ *there is* $C_1 > 0$ *such that for each* (σ, δ_1)*-hyperbolic preball* $V_n(x)$ *and every* $z, w \in V_n(x)$

$$\frac{1}{C_1} \le \frac{|\det Df^n(z)|}{|\det Df^n(w)|} \le C_1.$$

Proof. — By the Hölder continuity, there are $C, \alpha > 0$ such that for all $z, w \in M$

$$|\log|\det Df(z)| - \log|\det Df(w)|| \le C \operatorname{dist}(z, w)^\alpha.$$

We have for all $z, w \in V_n(x)$

$$\begin{aligned}
\log \frac{|\det Df(f^n(z))|}{|\det Df(f^n(w))|} &= \sum_{j=0}^{n-1} \log \frac{|\det Df(f^j(z))|}{|\det Df(f^j(w))|} \\
&= \sum_{j=0}^{n-1} \log \frac{|\det Df(f^j(z))|}{|\det Df(f^j(x))|} + \sum_{j=0}^{n-1} \log \frac{|\det Df(f^j(x))|}{|\det Df(f^j(w))|} \\
&\le \sum_{j=0}^{n-1} C\sigma^{(n-j)\alpha} \left(\operatorname{dist}(f^n(z), f^n(x)) + \operatorname{dist}(f^n(x), f^n(w))\right).
\end{aligned}$$

It is enough to take $C_1 = \exp\left(\sum_{k=0}^{\infty} 2C\sigma^{k\alpha} \operatorname{diam}(M)\right)$. □

We will refer to the conclusion of Corollary 2.6 by saying that the the hyperbolic preballs have *uniform bounded distortion*. The following easy consequence of uniform bounded distortion will be used several times in the future.

Lemma 2.7. — *Given a family $\{V_n(x)\}$ of (σ,δ_1)-hyperbolic preballs with uniform bounded distortion, there is $C_2>0$ such that for any any $A_1,A_2\subset V_n(x)$*

$$\frac{1}{C_2}\frac{m(A_1)}{m(A_2)}\le\frac{m(f^n(A_1))}{m(f^n(A_2))}\le C_2\frac{m(A_1)}{m(A_2)}$$

Proof. — By a change of variables induced by f^n we may write

$$\begin{aligned}\frac{m(f^n(A_1))}{m(f^n(A_2))} &= \frac{\int_{A_1}|\det Df^n(z)|dm(z)}{\int_{A_2}|\det Df^n(z)|dm(z)}\\ &= \frac{|\det Df^n(z_1)|\int_{A_1}\left|\frac{\det Df^n(z)}{\det Df^n(z_1)}\right|dm(z)}{|\det Df^n(z_2)|\int_{A_2}\left|\frac{\det Df^n(z)}{\det Df^n(z_2)}\right|dm(z)},\end{aligned}$$

where z_1 and z_2 are points chosen arbitrarily in A_1 and A_2 respectively. Using the uniform bounded distortion property we easily find a uniform bound for this expression. □

2.2. Absolute continuity. — Consider the sequence of measures

$$\mu_n=\frac{1}{n}\sum_{j=0}^{n-1}f^j_*m,\quad n\ge 1. \tag{3}$$

As observed before, any weak* accumulation point of this sequence is an invariant measure for f. Our goal now is to prove that such weak* accumulation points are absolutely continuous with respect to m. We start by showing that every point in M has infinitely many hyperbolic preballs.

Lemma 2.8. — *Let $f\colon M\to M$ be uniformly expanding. There is $\delta_1>0$ such that for every $x\in M$ and $n\in\mathbb{N}$ there is a (σ,δ_1)-hyperbolic preball $V_n(x)$.*

Proof. — One just needs to show that there is $\delta_1>0$ such that f^n sends a neighborhood of x diffeomorphically onto $B_{\delta_1}(f^n(x))$, for any $x\in M$, since the second condition in Definition 2.4 obviously holds for uniformly expanding maps.

Since f is a local diffeomorphism, for each $x\in M$ there is a radius $\delta_x>0$ such that f sends a neighborhood of x in M onto $B_{\delta_x}(f(x))$. By compactness of M we may choose a uniform radius $\delta_1>0$.

Let us now prove the result by induction on $n\ge 1$. By uniform expansion and the choice of δ_1, the existence of (σ,δ_1)-hyperbolic preballs is obvious for $n=1$. Assume now that there is a (σ,δ_1)-hyperbolic preball $V_n(f(x))$. By Lemma 2.5 we have that $V_n(f(x))$ is contained in $B_{\delta_1}(f(x))$ and so, by the choice of δ_1, there is a neighborhood $V_{n+1}(x)$ which is sent diffeomorphically by f onto $V_n(f(x))$. Hence, f^{n+1} sends $V_{n+1}(x)$ diffeomorphically onto $B_{\delta_1}(f^{n+1}(x))$. □

Lemma 2.9. — *Let $f\colon M \to M$ be uniformly expanding. There is $C_3 > 0$ such that $f^n_* m(A) \le C_3 m(A)$ for every measurable set $A \subset M$ and all $n \ge 1$.*

Proof. — We may assume, with no loss of generality, that A is contained in some ball B of radius $\delta_1 > 0$ around some point $y \in M$. Let $x_1, \dots, x_k$ be the pre-images of y under f^n, and consider the respective preballs $V_n(x_1), \dots, V_n(x_k)$ given by Lemma 2.8. Since each of these balls is sent by f^n diffeomorphically onto B, then they must be all disjoint. For $1 \le i \le k$, let A_i be that part of $V_n(x_i)$ the is mapped by f^n onto $A \subset B$. We may write

$$m(f^{-n}(A)) = \sum_{i=1}^{k} m(A_i) \le C_2 \frac{m(A)}{m(B)} \sum_{i=1}^{k} m(V_n(x_i))$$

Since B is a ball of radius δ_1 and $V_n(x_1), \dots, V_n(x_k)$ are pairwise disjoint we get the desired conclusion. □

Lemma 2.10. — *Let $(\mu_n)_n$ be a sequence of probability measures converging in the weak* topology to a probability measure μ. If there is an integrable φ such that $d\mu_n/dm \le \varphi$ for every $n \ge 1$, then $\mu \ll m$.*

Proof. — Take $A \subset X$ with $m(A) = 0$. Assume, by contradiction, that $\mu(A) > 0$. By the regularity of μ there is some compact set $K \subset A$ such that $\mu(K) > 0$. On the other hand, since $m(A) = 0$, then also $\int_A \varphi dm = 0$. So, by regularity there is an open set $B \supset A$ such that $\int_B \varphi dm < \mu(K)$. Let $u : X \to [0, 1]$ be defined as

$$u(x) = \frac{\operatorname{dist}(x, M \setminus B)}{\operatorname{dist}(x, K) + \operatorname{dist}(x, M \setminus B)}$$

As K and $M \setminus B$ are disjoint compact sets, then φ is well defined. Moreover, φ is continuous and $\chi_K \le \varphi \le \chi_B$. We have

$$\begin{aligned} \mu(K) &= \int \chi_K d\mu \\ &\le \int u d\mu \\ &= \lim_{n\to\infty} \int u d\mu_n \\ &\le \limsup_{n\to\infty} \int \chi_B d\mu_n \\ &\le \limsup_{n\to\infty} \mu_n(B) \\ &\le \int_B \varphi dm \end{aligned}$$

This gives a contradiction. □

Using Lemma 2.9 and Lemma 2.10 we easily deduce that any weak* accumulation point of (3) is absolutely continuous with respect to m.

2.3. Ergodicity. — In this subsection we show that there is an ergodic absolutely continuous invariant measure which is in fact unique.

Lemma 2.11. — *Let $f\colon M \to M$ be uniformly expanding. Given $p \in M$ and $\epsilon > 0$ there is $N_\epsilon \geq 1$ such that $f^{N_\epsilon}(B_\epsilon(p)) = M$.*

Proof. — Assume by contradiction that $f^n(B_\epsilon(p)) \neq M$ for all $n \geq 1$. Then, for each $n \geq 1$ we may find a point $y_n \in M \setminus f^n(B_\epsilon(p))$. Let γ_n be a smooth curve joining $f^n(p)$ to y_n. We assume with no loss of generality that $\text{length}(\gamma_n) \leq \text{diam}(M)$. Since f is a local diffeomorphism, there is a unique curve $\hat\gamma_n$ joining p to some point $x \in M \setminus B_\epsilon(p)$ such that $f^n(\hat\gamma_n) = \gamma_n$. For each $n \geq 1$ we have

$$\begin{aligned}
\text{length}(\gamma_n) &= \int \|\gamma_n'(t)\| dt \\
&= \int \|Df^n(\hat\gamma_n(t)) \cdot \hat\gamma_n'(t)\| dt \\
&\geq \sigma^{-n} \int \|\hat\gamma_n'(t)\| dt \\
&= \sigma^{-n}\, \text{length}(\hat\gamma_n).
\end{aligned}$$

But since $\text{length}(\hat\gamma_n) \geq \epsilon$ for every $n \geq 1$ we arrive to a contradiction. □

The next result will be used later in a more general context where we can only assure that Lebesgue almost every point has infinitely many hyperbolic preballs. This is obviously true in the present situation of uniformly expanding maps.

Proposition 2.12. — *Assume that m almost every point has infinitely many (σ, δ_1)-hyperbolic preballs. Then given any forward invariant set $A \subset M$ with $m(A) > 0$, there is a ball B of radius $\delta_1/4$ such that $m(B \setminus A) = 0$.*

Proof. — It is enough to prove that there exist disks of radius $\delta_1/4$ where the relative measure of A is arbitrarily close to one.

Since the set of points with infinitely many (σ, δ_1)-hyperbolic preballs is positively invariant, we may assume, with no loss of generality, that every point in A has infinitely many (σ, δ_1)-hyperbolic preballs. Let $\epsilon > 0$ be some small number. By regularity of m, there is a compact set $A_c \subset A$ and an open set $A_o \supset A$ such that

$$m(A_o \setminus A_c) < \epsilon m(A). \tag{4}$$

Assume that n_0 is large enough so that for every $x \in A_c$, any (σ, δ_1)-hyperbolic preball $V_n(x)$ with $n \geq n_0$ is contained in A_o. Let $W_n(x)$ be that part of $V_n(x)$

that is sent diffeomorphically by f^n onto the ball $B_{\delta_1/4}(f^n(x))$. By compactness there are $x_1, ..., x_r \in A_c$ and $n(x_1), \dots, n(x_r) \geq n_0$ such that

$$A_c \subset W_{n(x_1)}(x_1) \cup ... \cup W_{n(x_r)}(x_r). \tag{5}$$

For the sake of notational simplicity we shall write for each $1 \leq i \leq r$

$$V_i = V_{n(x_i)}(x_i), \quad W_i = W_{n(x_i)}(x_i) \quad \text{and} \quad n_i = n(x_i).$$

Assume that

$$\{n_1, ..., n_r\} = \{n_1^*, ..., n_s^*\}, \quad \text{with } n_1^* < n_2^* < ... < n_s^*. \tag{6}$$

Let $I_1 \subset \mathbb{N}$ be a maximal subset of $\{1, ..., r\}$ such that for each $i \in I_1$ both $n_i = n_1^*$, and $W_i \cap W_j = \emptyset$ for every $j \in I_1$ with $j \neq i$. Inductively, we define I_k for $2 \leq k \leq s$ as follows: supposing that $I_1, \dots, I_{k-1}$ have already been defined, let I_k be a maximal set of $\{1, \dots, r\}$ such that for each $i \in I_k$ both $n_i = n_k^*$, and $W_i \cap W_j = \emptyset$ for every $j \in I_1 \cup ... \cup I_k$ with $i \neq j$.

Define $I = I_1 \cup \cdots \cup I_s$. By construction we have that $\{W_i\}_{i \in I}$ is a family of pairwise disjoint sets. We claim that $\{V_i\}_{i \in I}$ is a covering of A_c. To see this, recall that by construction, given any W_j with $1 \leq j \leq r$, there is some $i \in I$ with $n(x_i) \leq n(x_j)$ such that $W_{x_j} \cap W_{x_i} \neq \emptyset$. Taking images by $f^{n(x_i)}$ we have

$$f^{n(x_i)}(W_j) \cap B_{\delta_1/4}(f^{n(x_i)}(x_i)) \neq \emptyset.$$

It follows from Lemma 2.5 that

$$\operatorname{diam}(f^{n(x_i)}(W_j)) \leq \frac{\delta_1}{2} \sigma^{(n(x_j) - n(x_i))/2} \leq \frac{\delta_1}{2},$$

and so

$$f^{n(x_i)}(W_j) \subset B_{\delta_1}(f^{n(x_i)}(x_i)).$$

This gives that $W_j \subset V_i$. We have proved that given any W_j with $1 \leq j \leq r$, there is $i \in I$ so that $W_j \subset V_i$. Taking into account (5), this means that $\{V_i\}_{i \in I}$ is a covering of A_c.

By Lemma 2.7 one may find $\tau > 0$ such that

$$m(W_i) \geq \tau m(V_i), \quad \text{for all } i \in I.$$

Hence,

$$\begin{aligned} m\left(\bigcup_{i \in I} W_i\right) &= \sum_{i \in I} m(W_i) \\ &\geq \tau \sum_{i \in I} m(V_i) \\ &\geq \tau m\left(\bigcup_{i \in I} V_i\right) \\ &\geq \tau m(A_c). \end{aligned}$$

From (4) one easily deduces that $m(A_c) > (1-\epsilon)m(A)$. Noting that the constant τ does not depend on ϵ, choosing $\epsilon > 0$ small enough we may have

$$m\left(\bigcup_{i\in I} W_i\right) > \frac{\tau}{2}m(A). \tag{7}$$

We are going to prove that

$$\frac{m(W_i \setminus A)}{m(W_i)} < \frac{2\epsilon}{\tau}, \quad \text{for some } i \in I. \tag{8}$$

This is enough for our purpose, since taking $B = f^{n(x_i)}(W_i)$ we have by invariance of A and Lemma 2.7

$$\frac{m(B \setminus A)}{m(B)} \leq \frac{m(f^{n(x_i)}(W_i \setminus A))}{m(f^{n(x_i)}(W_i))} \leq C_2\frac{m(W_i \setminus A)}{m(W_i)} = \frac{2C_2\epsilon}{\tau},$$

which can obviously be made arbitrarily small. From this one easily deduces that there are disks of radius $\delta_1/4$ where the relative measure of A is arbitrarily close to one.

Finally, let us prove (8). Assume, by contradiction, that it does not hold. Then, using (4) and (7)

$$\begin{aligned}
\epsilon m(A) &> m(A_o \setminus A_c) \\
&\geq m\left(\left(\bigcup_{i\in I} W_i\right) \setminus A\right) \\
&\geq \frac{2\epsilon}{\tau} m\left(\bigcup_{i\in I} W_i\right) \\
&> \epsilon m(A).
\end{aligned}$$

This gives a contradiction. □

Corollary 2.13. — *If $A \subset M$ is a forward invariant set with $m(A) > 0$, then $m(M \setminus A) = 0$.*

Proof. — Take a ball B of radius $\delta_1/4$ such that $m(B \setminus A) = 0$ given by Proposition 2.12. Then by Lemma 2.11 there is $N \in \mathbb{N}$ such that $f^N(B) = M$. Hence

$$m(M \setminus A) \leq m(f^N(B \setminus A)) = 0.$$

This last equality holds because f^N is a local diffeomorphism. □

Proposition 2.14. — *There is a unique absolutely continuous invariant measure. Moreover, this measure is ergodic, its support is the whole M and Lebesgue almost every point in M belongs to its basin.*

Proof. — As we have seen in Section 2.2, there is some absolutely continuous invariant probability measure μ. If μ is not ergodic, then there are two disjoint invariant sets A_1 and A_2 both with positive μ_0 measure, thus with positive Lebesgue measure. But this cannot happen, by Corollary 2.13.

Let μ be an ergodic absolutely continuous invariant measure. Since the basin $B(\mu)$ and the support $\text{supp}(\mu)$ of μ have positive μ measure, then they have positive Lebesgue measure, by absolute continuity. Since $B(\mu)$ and $\text{supp}(\mu)$ are also invariant, it follows from Corollary 2.13 that both $B(\mu)$ and $\text{supp}(\mu)$ have full Lebesgue measure in M. Additionally, noting that $\text{supp}(\mu)$ is a closed set, then $\text{supp}(\mu) = M$.

It remains to check uniqueness. Let μ_1 and μ_2 be ergodic absolutely continuous invariant measures. Then, as $B(\mu_1)$ and $B(\mu_2)$ have full Lebesgue measure, there is some $x \in B(\mu_1) \cap B(\mu_2)$. Given any continuous $\varphi\colon M \to \mathbb{R}$, we have $\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(x))$ converging to both $\int \varphi d\mu_1$ and $\int \varphi d\mu_1$. Thus we have $\int \varphi d\mu_1 = \int \varphi d\mu_2$. By Theorem A.5 one must have $\mu_1 = \mu_2$. □

3. Non-uniformly expanding local diffeomorphisms

Now we study the existence of physical measures for smooth maps with no critical sets for which expansion is only attained in time average for Lebesgue almost every point.

Definition 3.1. — Let $f\colon M \to M$ be a C^2 local diffeomorphism. We say that f is *non-uniformly expanding (NUE)* if there is $\lambda > 0$ such that for Lebesgue almost every $x \in M$

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log \|Df(f^i(x))^{-1}\| < -\lambda. \tag{9}$$

Note that in dimension one condition (9) is equivalent to the existence of a positive *Lyapunov exponent* at x. In dimension greater than one non-uniform expansion at a point x implies the existence of $\dim(M)$ positive Lyapunov exponents at $x \in M$, i.e.

$$\liminf_{n\to\infty} \frac{1}{n}\|Df^n(x)v\| > \lambda > 0, \qquad \forall v \in T_xM.$$

The converse does not necessarily hold, as the next example illustrates.

Example 3.2. — Consider a period 2 orbit $\{p, q\}$ for a local diffeomorphism f on a surface which, for a given choice of local basis at p and q, satisfies

$$Df(p) = \begin{pmatrix} 1/2 & 0 \\ 0 & 3 \end{pmatrix} \quad \text{and} \quad Df(q) = \begin{pmatrix} 3 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Then it is clear that both Lyapunov exponents at p or q are $\log(3/2)/2 > 0$ and the limit in (9) with $x = p$ or q equals $\log 2 > 0$.

However, the following interesting open question still remains.

Problem 3.3. — Does the existence of $\dim(M)$ positive Lyapunov exponents Lebesgue almost everywhere imply non-uniform expansion Lebesgue almost everywhere?

Example 3.4. — Let $f_0\colon M \to M$ be a uniformly expanding map. We may think of M as the d-dimensional torus $\mathbb{T}^d$ and f_0 as in Example 2.2. Let $V \subset M$ be some small compact domain, so that the restriction of f_0 to V is injective. Let f be any map in a sufficiently small C^1-neighborhood $\mathcal{N}$ of f_0 so that:

1. f *is expanding outside* V: there exists $\sigma_0 < 1$ such that
$$\|Df(x)^{-1}\| < \sigma_0, \quad \text{for every } x \in M \setminus V;$$
2. f *is volume expanding everywhere*: there exists $\sigma_1 > 1$ such that
$$|\det Df(x)| > \sigma_1, \qquad \text{for every } x \in M;$$
3. f *is not too contracting on* V: there is small $\delta > 0$ such that
$$\|Df(x)^{-1}\| < 1 + \delta, \qquad \text{for every } x \in V.$$

We are going to show that choosing appropriately the constants σ_0, σ_1 and δ, then every map f in such a C^1-neighborhood $\mathcal{N}$ of f_0 is non-uniformly expanding.

Lemma 3.5. — *Let $B_1, \dots, B_p, B_{p+1} = V$ be any partition of M into domains such that f is injective on B_j, for $1 \le j \le p+1$. There exists $\theta > 0$ such that the orbit of Lebesgue almost every point $x \in M$ spends a fraction θ of the time in $B_1 \cup \dots \cup B_p$, that is,*
$$\#\{0 \le j < n : f^j(x) \in B_1 \cup \dots \cup B_p\} \ge \theta\, n$$
for every large n.

Proof. — Let n be fixed. Given a sequence $\underline{i} = (i_0, i_1, \dots, i_{n-1})$ in $\{1, \dots, p+1\}$, we denote
$$[\underline{i}] = B_{i_0} \cap f^{-1}(B_{i_1}) \cap \dots \cap f^{-n+1}(B_{i_{n-1}}).$$
Moreover, we define $g(\underline{i})$ to be the number of values of $0 \le j \le n-1$ for which $i_j \le p$. We begin by noting that, given any $\theta > 0$, the total number of sequences $\underline{i}$ for which $g(\underline{i}) < \theta\, n$ is bounded by
$$\sum_{k<\theta\, n} \binom{n}{k} p^k \le \sum_{k \le \theta\, n} \binom{n}{k} p^{\theta\, n}$$
A standard application of Stirling's formula (see e.g. [**BoV**, Section 6.3]) gives that the last expression is bounded by $e^{\gamma n} p^{\theta\, n}$, where γ depends only on θ and goes to zero when θ goes to zero. On the other hand, since we are assuming that f is volume expanding everywhere, we have $m([\underline{i}]) \le m(M)\, \sigma_1^{-(1-\theta)n}$. Then the measure of the union I_n of all the sets $[\underline{i}]$ with $g(\underline{i}) < \theta\, n$ is less than $m(M)\sigma_1^{-(1-\theta)n} e^{\gamma n} p^{\theta\, n}$. Since $\sigma_1 > 1$, we may fix θ small so that $e^{\gamma} p^{\theta} < \sigma_1^{1-\theta}$.

This means that the Lebesgue measure of I_n goes to zero exponentially fast as $n \to \infty$. Thus, by the lemma of Borel-Cantelli, Lebesgue almost every point $x \in M$ belongs in only finitely many sets I_n. Clearly, any such point x satisfies the conclusion of the lemma. □

Let $\theta > 0$ be the constant given by Lemma 3.5, and fix $\delta > 0$ small enough so that $\sigma_0^\theta(1+\delta) \le e^{-\lambda}$ for some $\lambda > 0$. Let x be any point satisfying the conclusion of the lemma. Then

$$\prod_{j=0}^{n-1} \|Df(f^j(x))^{-1}\| \le \sigma_0^{\theta n}(1+\delta)^{(1-\theta)n} \le e^{-\lambda n}$$

for every large enough n. This implies that x satisfies

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \le -\lambda\,,$$

and since the conclusion of Lemma 3.5 holds Lebesgue almost everywhere we have that f is a non-uniformly expanding map.

Theorem 3.6. — *Let $f : M \to M$ be a non-uniformly expanding local diffeomorphism. Assume that* $\det Df$ *is Hölder continuous. Then there are ergodic absolutely continuous probability measures $\mu_1, \dots, \mu_p$ whose basins cover a full Lebesgue measure subset of M.*

The proof of this result will be given in the next two subsections. In Corollary 3.17 we show that if f has a dense orbit then it has a unique physical measure.

3.1. Hyperbolic times. — A powerful tool in the study of the ergodic properties of Viana maps was introduced in [**Al00**] through the notion of *hyperbolic times.* This concept has been extended to a more general setting in [**ABV**].

Definition 3.7. — Given $0 < \sigma < 1$, we say that n is a *σ-hyperbolic time* for a point $x \in M$ if

$$\prod_{j=n-k}^{n-1} \|Df(f^j(x))^{-1}\| \le \sigma^k.$$

for all $1 \le k \le n$.

Since f is a local diffeomorphism, for each $x \in M$ there is a radius $\delta_x > 0$ such that f sends a neighborhood of x diffeomorphically onto $B_{\delta_x}(f(x))$, the ball of radius δ_x around $f(x)$. By compactness of M we may choose a uniform radius $\delta_1 > 0$. We choose $\delta_1 > 0$ small enough so that also

$$\|Df(y)^{-1}\| \le \sigma^{-1/2}\|Df(x)^{-1}\|, \quad \text{whenever} \quad \text{dist}(f(y), f(x)) < \delta_1. \tag{10}$$

Proposition 3.8. — *If n is a σ-hyperbolic time for $x \in M$, then there is a $(\sqrt{\sigma}, \delta_1)$-hyperbolic preball $V_n(x)$.*

Proof. — We shall prove the result by induction on $n \geq 1$. Starting the induction argument, let $n = 1$ be a σ-hyperbolic time for $x \in M$. By the choice of δ_1 in (10) there is a neighborhood $V_1(x)$ of x that is sent diffeomorphically by f onto $B_{\delta_1}(f(x))$. Moreover, using (10) and the definition of σ-hyperbolic time, we have for $y \in V_1(x)$

$$\|Df(y)^{-1}\| \leq \sigma^{-1/2}\|Df(x)^{-1}\| \leq \sigma^{1/2}.$$

This means that, $V_1(x)$ is a $(\sqrt{\sigma}, \delta_1)$-hyperbolic preball.

Assume now that if $n \geq 1$ is a σ-hyperbolic time for a point x, then there is $(\sqrt{\sigma}, \delta_1)$-hyperbolic preball $V_n(x)$ containing x. Let $n+1$ be a σ-hyperbolic time for x. By the definition of σ-hyperbolic time we easily deduce that n is a σ-hyperbolic time for $f(x)$. Hence, by the induction hypothesis, there is a $(\sqrt{\sigma}, \delta_1)$-hyperbolic preball $V_n(f(x))$. By Lemma 2.5 we have that $V_n(f(x))$ is contained in $B_{\delta_1}(f(x))$ and so, by the choice of δ_1, there is a neighborhood $V_{n+1}(x)$ which is sent diffeomorphically by f onto $V_n(f(x))$.

For concluding that $V_{n+1}(x)$ is a $(\sqrt{\sigma}, \delta_1)$-hyperbolic preball, we just have to check that for every $y \in V_{n+1}(x)$ we have $\|Df^{n+1}(y)^{-1}\| \leq \sigma^{(n+1)/2}$. Indeed, using the fact that $n+1$ is a hyperbolic time for x and (10) we have

$$\|Df^{n+1}(y)^{-1}\| \leq \prod_{i=0}^{n} \|Df(f^i(y))^{-1}\| \leq \prod_{i=0}^{n} \sigma^{-1/2}\|Df(f^i(x))^{-1}\| \leq \sigma^{(n+1)/2}$$

Note that by Lemma 2.5 we have $f^i(V_{n+1}(x)) \subset B_{\delta_1}(f^i(x))$ for $0 \leq i \leq n$. □

***Lemma 3.9* (Pliss)**. — *Given $0 < c \leq A$ let $\theta = c/A$. Assume that $a_1, \ldots, a_N$ are real numbers satisfying $a_j \leq A$ for every $1 \leq j \leq N$ and*

$$\sum_{j=1}^{N} a_j \geq cN.$$

Then there are $\ell \geq \theta N$ and $1 \leq n_1 < \cdots < n_\ell \leq N$ so that

$$\sum_{j=n}^{n_i} a_j \geq 0$$

for every $1 \leq n \leq n_i$ and $1 \leq i \leq \ell$.

Proof. — Define for each $1 \leq n \leq N$,

$$S_n = \sum_{j=1}^{n} a_j, \quad \text{and also} \quad S_0 = 0.$$

Then let $1 \leq n_1 < \cdots < n_\ell \leq N$ be the maximal sequence such that $S_{n_i} \geq S_n$ for every $0 \leq n \leq n_i$ and $1 \leq i \leq \ell$. Note that $\ell \geq 1$, since $S_N > S_0$. Moreover, by the choice of S_{n_i}, for each $1 \leq i \leq \ell$ we have

$$\sum_{j=n}^{n_i} a_j \geq 0, \quad \text{for } 1 \leq n \leq n_i.$$

We are left to verify that $\ell > \theta N$. Defining for convenience $n_0 = 0$, we have by definition of n_i that for each $1 \leq i \leq \ell$

$$S_{n_i - 1} \leq S_{n_{i-1}}.$$

Adding a_{n_i} in both sides and using that $a_{n_i} \leq A$, we easily deduce that

$$S_{n_i} - S_{n_{i-1}} \leq A.$$

Observing that $S_{n_\ell} \geq S_N \geq cN$, we finally have

$$cN \leq S_{n_\ell} = \sum_{i=1}^{\ell} \left(S_{n_i} - S_{n_{i-1}} \right) \leq \ell A,$$

which completes the proof. □

We say that the *frequency* of σ-hyperbolic times for $x \in M$ is bigger than θ if for large $n \in \mathbb{N}$ there are $\ell \geq \theta n$ and integers $1 \leq n_1 < n_2 \cdots < n_\ell \leq n$ which are σ-hyperbolic times for x. The frequency of (σ, δ_1)-hyperbolic preballs is defined similarly.

Corollary 3.10. — *There are $0 < \sigma < 1$ and $\theta > 0$ such that the frequency of σ-hyperbolic times for Lebesgue almost every point is greater than θ.*

Proof. — By NUE we know that for Lebesgue almost every $x \in M$ and N sufficiently large we have

$$\sum_{j=1}^{N} -\log \|Df(f^{j-1}(x))^{-1}\| \geq \lambda N.$$

Using Lemma 3.9 for the sequence

$$a_j = -\log \|Df(f^{j-1}(x))^{-1}\| + \frac{\lambda}{2},$$

with $c = \lambda/2$ and $A = \max_{x \in M}\{-\log \|Df(x)^{-1}\| + \lambda/2\}$, we obtain the result for $\theta = \lambda/(2A)$ and $\sigma = e^{-\lambda/2}$. □

It follows from Lemma 2.7, Proposition 3.8 and Corollary 3.10 that Lebesgue almost every point has (σ, δ_1)-hyperbolic preballs with uniform bounded distortion and frequency bigger than $\theta > 0$ given by Corollary 3.10. Theorem 3.6 is then a consequence of Theorem 3.12 below whose proof we give in Subsections 3.2 and 3.3.

Definition 3.11. — We say that a map $f\colon M \to M$ is *nonsingular* if both $m(f(A)) = 0$ and $m(f^{-1}(A)) = 0$, whenever $m(A) = 0$.

Observe that a local diffeomorphism is obviously a nonsingular map.

Theorem 3.12. — *Let $f : M \to M$ be a nonsingular map. Assume that there is $\theta > 0$ such that Lebesgue almost every point has (σ, δ_1)-hyperbolic preballs with uniform bounded distortion and frequency bigger than $\theta > 0$. Then there are ergodic absolutely continuous probability measures $\mu_1, \dots, \mu_p$ whose basins cover a full Lebesgue measure set in M.*

3.2. Absolute continuity. — Let $f : M \to M$ be a nonsingular map. Assume that, for some $\theta > 0$, Lebesgue almost every point in M has (σ, δ_1)-hyperbolic preballs with frequency bigger than θ. Define H_n as the set of points x that have a (σ, δ_1)-hyperbolic preball $V_n(x)$.

Lemma 3.13. — *There is $C_3 > 0$ such that for every $n \geq 0$*

$$\frac{d}{dm} f_*^n\big(m \mid H_n\big) \leq C_3.$$

Proof. — It is enough to show that there is some uniform constant $C > 0$ such that for any Borel set $A \subset M$ with diameter smaller than $\delta_1/2$ we have

$$m\big(f^{-n}(A) \cap H_n\big) \leq C m(A).$$

Let B be an open ball of radius $\delta_1/2$ containing A. Taking the connected components of $f^{-n}(B)$ we may write

$$f^{-n}(B) = \bigcup_{k\geq 1} B_k,$$

where $(B_k)_{k\geq 1}$ is a (possibly finite) family of pairwise disjoint sets in M. Considering only those B_k that intersect H_n, we choose, for each $k \geq 1$, a point $x_k \in H_n \cap B_k$. Since B is contained in $B_{\delta_1}\big(f^n(x_k)\big)$ and f^n is a diffeomorphism from the (σ, δ_1)-hyperbolic preball $V_n(x_k)$ onto $B_{\delta_1}\big(f^n(x_k)\big)$, one must have $B_k \subset V_n(x_k)$. It follows from Corollary 2.6 and Lemma 2.7 that there is $C_2 > 0$ such that for each $n \geq 1$ and $k \geq 1$

$$\begin{aligned} m\big(f^{-n}(A) \cap H_n\big) &\leq \sum_{k\geq 1} m\big(f^{-n}(A \cap B) \cap B_k\big) \\ &\leq \sum_{k\geq 1} C_2 \frac{m(A \cap B)}{m(B)} m(B_k) \\ &\leq C_3 m(A), \end{aligned}$$

for some uniform constant $C_3 > 0$ only depending on C_2 and on the volume of the balls of radius $\delta_1/2$. □

Lemma 3.14. — *For large n we have*

$$\frac{1}{n}\sum_{j=1}^{n} m(H_j) \geq \theta.$$

Proof. — Let ξ_n be the measure in $\{1,\dots,n\}$ defined by $\xi_n(J) = \#J/n$, for each subset J. Then, using Fubini Theorem

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n} m(B\cap H_j) &= \int\left(\int_B \mathbf{1}(x,i)\, dm(x)\right) d\xi_n(i)\\ &= \int_B\left(\int \mathbf{1}(x,i)\, d\xi_n(i)\right) dm(x),\end{aligned}$$

where $\mathbf{1}(x,i) = 1$ if $x \in H_i$, and $\mathbf{1}(x,i) = 0$ otherwise. Since almost every point has (σ,δ_1)-hyperbolic preballs with frequency bigger than θ, the conclusion follows. □

Consider, for each $n\in\mathbb{N}$, the measure

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j m.$$

and its submeasure

$$\nu_n = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j (m \mid H_j).$$

By Lemma 3.14 we have $\theta > 0$ such that for large n

$$\nu_n(M) \geq \frac{1}{n}\sum_{i=0}^{n-1} m(H_i) \geq \frac{1}{n}\sum_{i=0}^{n-1} m(H_i) \geq \theta. \tag{11}$$

Moreover, by Corollary 3.13, every $f_*^j(m|H_j)$ is absolutely continuous with respect to Lebesgue measure, with density uniformly bounded from above, and so the same is true for every ν_n.

Since we are working with a continuous map in the compact space M, we know that sequences of probability measures in M have weak* accumulation points. Take $n_k\to\infty$ such that both μ_{n_k} and ν_{n_k} converge in the weak* sense to measures μ and ν, respectively. Then μ is an invariant probability measure, $\mu = \nu+\eta$ for some measure η, ν is absolutely continuous with respect to Lebesgue measure, and $\nu(H) > 0$ by (11).

Now, if $\eta = \eta_{ac} + \eta_s$ denotes the decomposition of η with respect to Lebesgue measure given by Lebesgue Decomposition Theorem, then $\mu_{ac} = \nu + \eta_{ac}$ gives the absolutely continuous component in the corresponding decomposition of μ. By uniqueness of the decomposition, and since f is nonsingular, we may conclude that μ_{ac} is an invariant measure. Clearly, $\mu_{ac}(M) \geq \nu(M) > 0$. Normalizing μ_{ac} we obtain an absolutely continuous f-invariant probability measure.

3.3. Ergodic components. — Let us now show that under the assumptions of Theorem 3.12 there cannot be an infinite number of ergodic absolutely continuous invariant measures.

Proposition 3.15. — *There are ergodic absolutely continuous probability measures $\mu_1, \dots, \mu_p$ whose basins cover a full Lebesgue measure set in M.*

Proof. — Let μ_0 be any absolutely continuous invariant probability measure. If μ_0 is not ergodic, then we may decompose M into two disjoint invariant sets H_1 and H_2 both with positive μ_0-measure. In particular, both H_1 and H_2 have positive Lebesgue measure. Let μ_1 and μ_2 be the normalized restrictions of μ_0 to H_1 and H_2, respectively. Clearly, they are also absolutely continuous invariant measures. If they are not ergodic, we continue decomposing them, in the same way as we did for μ_0.

On the other hand, by Proposition 2.12, each one of the invariant sets we find in this decomposition has full Lebesgue measure in some disk with fixed radius. Since these disks must be disjoint, and the ambient manifold is compact, there can only be finitely many of them. So, the decomposition must stop after a finite number of steps, giving that μ_0 can be written $\mu_0 = \sum_{i=1}^{p} \mu_0(H_i)\mu_i$ where $H_1, \dots, H_p$ is a partition of M into invariant sets with positive measure and each $\mu_i = (\mu_0|H_i)/\mu_0(H_i)$ is an ergodic probability measure. □

Remark 3.16. — We have proved that if μ is an absolutely continuous invariant probability measure, then μ is a convex linear combination of the finitely many physical measures given by Theorem 3.12: there are $\alpha_1 \geq 0, \dots, \alpha_p \geq 0$ with $\alpha_1 + \cdots + \alpha_p = 1$ such that $\alpha_1\mu_1 + \cdots + \alpha_p\mu_p = \mu$.

Corollary 3.17. — *If f is transitive, then M is covered (Lebesgue mod 0) by the basin of a unique physical measure, which is ergodic and absolutely continuous.*

Proof. — Assume, by contradiction, that there two distinct ergodic absolutely continuous invariant measures μ_1 and μ_2. Since $B(\mu_1)$ and $B(\mu_2)$ are positively invariant sets, then by Proposition 2.12 there are disks Δ_1 and Δ_2 such that $m(\Delta_i \setminus B(\mu_i)) = 0$ for $i = 1, 2$. The transitivity of f and the invariance of $B(\mu_1)$ and $B(\mu_2)$ imply that $m(B(\mu_1) \cap B(\mu_2)) > 0$. Since distinct SRB measures have disjoint basins we have a contradiction. □

4. Non-uniformly expanding maps with critical sets

Let $f\colon M \to M$ be a continuous map which is a local diffeomorphism in the whole manifold but in a critical set $\mathcal{C} \subset M$ with zero Lebesgue measure. This may be thought as a set of points where the derivative of f fails to be invertible or simply does not exist. In particular, f is nonsingular.

Definition 4.1. — We say that a critical set $\mathcal{C} \subset M$ as above is *non-degenerate* if there are constants $B > 1$ and $\beta > 0$ such that

1. for every $x \in M \setminus \mathcal{C}$ and $v \in T_x M$

$$(\mathrm{c}_1) \qquad \frac{1}{B}\operatorname{dist}(x,\mathcal{C})^{\beta} \le \frac{\|Df(x)v\|}{\|v\|} \le B\operatorname{dist}(x,\mathcal{C})^{-\beta};$$

2. for every $x, y \in M \setminus \mathcal{C}$ with $\operatorname{dist}(x,y) < \operatorname{dist}(x,\mathcal{C})/2$ we have

$$(\mathrm{c}_2) \qquad \big|\log\|Df(x)^{-1}\| - \log\|Df(y)^{-1}\|\,\big| \le \frac{B}{\operatorname{dist}(x,\mathcal{C})^{\beta}}\operatorname{dist}(x,y);$$

$$(\mathrm{c}_3) \qquad \big|\log|\det Df(x)| - \log|\det Df(y)|\,\big| \le \frac{B}{\operatorname{dist}(x,\mathcal{C})^{\beta}}\operatorname{dist}(x,y).$$

The first condition says that f *behaves like a power of the distance* to $\mathcal{C}$ whilst the last two conditions say that the functions $\log|\det Df|$ and $\log\|Df^{-1}\|$ are *locally Lipschitz* at points $x \in M \setminus \mathcal{C}$, with Lipschitz constant depending on $\operatorname{dist}(x,\mathcal{C})$. Given $\delta > 0$ and $x \in M \setminus \mathcal{C}$ we define the *δ-truncated distance* from x to $\mathcal{C}$ as

$$\operatorname{dist}_{\delta}(x,\mathcal{C}) = \begin{cases} 1, & \text{if } \operatorname{dist}(x,\mathcal{C}) \ge \delta; \\ \operatorname{dist}(x,\mathcal{C}), & \text{otherwise.} \end{cases}$$

Note that it makes sense to consider non-uniform expansion condition (9) for points whose orbits remain in $M\setminus\mathcal{C}$. This set of points has full Lebesgue measure as long as we take $\mathcal{C}$ zero Lebesgue measure. For proving our main result on the existence and finiteness of physical measures for maps with critical sets, we need an extra assumption on the behavior of orbits, essentially saying that a typical orbit does not come too often to close to the critical set.

Definition 4.2. — We say that f has *slow recurrence to $\mathcal{C}$* if for every $\epsilon > 0$ there exists $\delta > 0$ such that for Lebesgue almost every $x \in M \setminus \mathcal{C}$

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} -\log\operatorname{dist}_{\delta}(f^j(x),\mathcal{C}) < \epsilon. \tag{12}$$

In practice we do not need this condition for *every* $\varepsilon > 0$. As it will become clear in the proof of Proposition 4.9 it is enough that it holds for some sufficiently small $\varepsilon > 0$ which essentially depends on the of non-uniform expansion rate and on the constants that appear in the definition of non-degenerate critical set.

***Example 4.3* (Viana maps)**. — Let $a_0 \in (1,2)$ be such that the critical point $x = 0$ is pre-periodic for the quadratic map $Q(x) = a_0 - x^2$. Let $S^1 = \mathbb{R}/\mathbb{Z}$ and $b : S^1 \to \mathbb{R}$ be a Morse function, for instance, $b(s) = \sin(2\pi s)$. For fixed small $\alpha > 0$, consider the map

$$\begin{array}{rccc} \hat f: & S^1\times\mathbb{R} & \longrightarrow & S^1\times\mathbb{R} \\ & (s,x) & \longmapsto & \big(\hat g(s), \hat q(s,x)\big) \end{array}$$

where $\hat g$ is the uniformly expanding map of the circle defined by $\hat g(s) = ds$ (mod $\mathbb{Z}$) for some $d \ge 16$, and $\hat q(s,x) = a(s) - x^2$ with $a(s) = a_0 + \alpha b(s)$. It is easy to check that for $\alpha > 0$ small enough there is an interval $I \subset (-2,2)$ for which $\hat f(S^1 \times I)$ is contained in the interior of $S^1 \times I$. Thus, any map f sufficiently close to $\hat{f}$ in the C^0 topology has $S^1 \times I$ as a forward invariant region. We consider from here on these maps f close to $\hat{f}$ restricted to $S^1 \times I$. Taking into account the expression of $\hat{f}$ it is not difficult to check that for $\hat{f}$ (and any map f close to $\hat{f}$ in the C^2 topology) the critical set is non-degenerate. Using the results in [**Vi2**] it is possible to show that if the map f is sufficiently close to $\hat{f}$ in the C^3 topology then there is a set Γ_n with

$$m(\Gamma_n) \leq \text{const}\, e^{-\gamma\sqrt{n}}, \quad \text{for all } n \geq 1. \tag{13}$$

such that for each $(s, x) \notin \Gamma_n$

$$\frac{1}{k}\sum_{j=0}^{k-1} -\log \operatorname{dist}_\delta(f^j(s,x), \mathcal{C}) \leq \varepsilon \quad \text{for all } k \geq n,$$

and

$$\frac{1}{k}\sum_{j=0}^{k-1} \log \|Df(s_j, x_j))^{-1}\| \leq -\lambda \quad \text{for all } k \geq n.$$

This shows that f is non-uniformly expanding.

Theorem 4.4. — *Let $f\colon M \to M$ be a continuous map which is a local diffeomorphism everywhere but in a critical set $\mathcal{C} \subset M$ with zero Lebesgue measure. Assume that f is non-uniformly expanding and has slow recurrence to $\mathcal{C}$ Lebesgue almost everywhere. Then there are ergodic absolutely continuous probability measures $\mu_1, \ldots, \mu_p$ whose basins cover a full Lebesgue measure set in M.*

By Theorem 3.12 it is enough to show that there exists some $\theta > 0$ such that Lebesgue almost every point has hyperbolic preballs with uniform bounded distortion and frequency bigger than θ. That is the aim of the next subsection.

4.1. Hyperbolic times. — We fix once and for all $B > 1$ and $\beta > 0$ as in Definition 4.1, and take a constant $b > 0$ such that $2b < \min\{1, \beta^{-1}\}$. For maps with critical sets we impose an extra assumption in the definition of hyperbolic times bounding the approaches to the critical set.

Definition 4.5. — Given $0 < \sigma < 1$ and $\delta > 0$, we say that n is a (σ, δ)-*hyperbolic time* for $x \in M$ if, for all $1 \leq k \leq n$,

$$\prod_{j=n-k}^{n-1} \|Df(f^j(x))^{-1}\| \leq \sigma^k \quad \text{and} \quad \operatorname{dist}_\delta(f^{n-k}(x), \mathcal{C}) \geq \sigma^{bk}.$$

For local diffeomorphisms, the choice of $\delta_1 > 0$ has been made in (10) and it was actually quite simple. Here, due to the presence of a critical set, we need some extra work.

Lemma 4.6. — *Given* $0 < \sigma < 1$ *and* $\delta > 0$, *there is* $\delta_1 > 0$ *such that if* n *is a* (σ, δ)*-hyperbolic time for* x, *then*

$$\|Df(y)^{-1}\| \leq \sigma^{-1/2}\|Df(x)^{-1}\|,$$

for any point y *in the ball of radius* $\delta_1\sigma^{n/2}$ *around* x.

Proof. — Since n is a (σ, δ)-hyperbolic time for x, we have

$$\mathrm{dist}_\delta(x, \mathcal{C}) \geq \sigma^{bn}.$$

According to the definition of the truncated distance, this means that

$$\mathrm{dist}(x, \mathcal{C}) = \mathrm{dist}_\delta(x, \mathcal{C}) \geq \sigma^{bn}, \quad \text{or else} \quad \mathrm{dist}(x, \mathcal{C}) \geq \delta.$$

In either case, taking $\delta_1 < \delta/2 < 1/2$, and since we haven chosen $b < 1/2$, for any point y in the ball of radius $\delta_1\sigma^{n/2}$ around x we have

$$\mathrm{dist}(y, x) < \frac{1}{2}\,\mathrm{dist}(x, \mathcal{C}).$$

Therefore, we may use (c_2) to conclude that

$$\log \frac{\|Df(y)^{-1}\|}{\|Df(x)^{-1}\|} \leq \frac{B}{\mathrm{dist}(x, \mathcal{C})^\beta}\,\mathrm{dist}(y, x) \leq B\frac{\delta_1\sigma^{n/2}}{\min\{\sigma^{b\beta n}, \delta^\beta\}}.$$

Since $0 < \delta, \sigma < 1$ and we have taken $b\beta < 1/2$, the term on the right hand side is bounded by $B\delta_1\delta^{-\beta}$. Choosing $\delta_1 > 0$ small so that $B\delta_1\delta^{-\beta} < \log\sigma^{-1/2}$ we get the conclusion. □

From here one we assume that, given (σ, δ) as before, we take δ_1 small so that Lemma 4.6 holds. We further require that δ_1 is sufficiently small so that the exponential map in the ball of radius δ_1 is an isometry onto its image. This in particular implies that any point in the boundary of a ball of radius δ_1 can be joined to the center of that ball through a smooth curve of minimal length (a geodesic arc) which lies completely inside the ball.

Proposition 4.7. — *If* n *is a* (σ, δ)*-hyperbolic time for* $x \in M$, *then there is a* $(\sqrt{\sigma}, \delta_1)$*-hyperbolic preball* $V_n(x)$.

Proof. — We shall prove the result by induction on n. Let $n = 1$ be a (σ, δ)-hyperbolic time for $x \in M$. It follows from Lemma 4.6 that for any y in the ball $B_{\delta_1\sigma^{1/2}}(x)$ of radius $\delta_1\sigma^{1/2}$ around x

$$\|Df(y)^{-1}\| \leq \sigma^{-1/2}\|Df(x)^{-1}\| \leq \sigma^{1/2}. \tag{14}$$

This means that f is a $\sigma^{-1/2}$-dilation in the ball $B_{\delta_1\sigma^{1/2}}(x)$. Then, there exists some neighborhood $V_1(x)$ of x contained in $B_{\delta_1\sigma^{1/2}}(x)$ which is mapped diffeomorphically onto the ball $B_{\delta_1}(f(x))$. It follows from (14) that $V_1(x)$ is a $(\sqrt{\sigma}, \delta_1)$-hyperbolic preball.

Assume now that the conclusion holds for $n \geq 1$. Let $n+1$ be a (σ, δ)-hyperbolic time for $x \in M$. Take any $z \in \partial B_{\delta_1}(f^{n+1}(x))$, and let $\gamma \colon [0, 1] \to M$ be a smooth curve of minimal length joining z to $f^{n+1}(x)$. The curve γ necessarily lies inside $B_{\delta_1}(f^{n+1}(x))$ by the choice of δ_1. Consider γ_n and γ_{n+1} smooth curves which are lifts of γ starting at $f(x)$ and x, respectively. This means that

$$\gamma = f^n \circ \gamma_n \quad \text{and} \quad \gamma = f^{n+1} \circ \gamma_{n+1},$$

at least in the domains where the lifts make sense. Since n is a (σ, δ)-hyperbolic time for $f(x)$, by induction hypothesis there exists a $(\sqrt{\sigma}, \delta_1)$-hyperbolic preball $V_n(f(x))$ which is sent diffeomorphically by f^n onto the ball of radius δ_1 around $f^{n+1}(x)$. One obviously has that γ_n lies inside $V_n(f(x))$.

Claim. *The curve γ_{n+1} remains inside the ball of radius $\delta_1\sigma^{(n+1)/2}$ around x.*

Assume, by contradiction, that γ_{n+1} hits the boundary of $B_{\delta_1\sigma^{(n+1)/2}}(x)$ before the end time. Let $0 < t_0 < 1$ be the first moment in such conditions. One necessarily has that $\gamma_{n+1}|[0, t_0]$ is a curve inside the ball $B_{\delta_1\sigma^{(n+1)/2}}(x)$ joining x to a point in the boundary of that ball. Moreover, $\gamma_n([0, t_0]) \subset V_n(f(x))$. This yields for each $0 \le t \le t_0$

$$\begin{aligned} \|Df^{n+1}(\gamma_{n+1}(t))^{-1}\| &\le \prod_{j=0}^{n} \|Df(\gamma_{j+1}(t))^{-1}\| \\ &= \|Df(\gamma_{n+1}(t))^{-1}\| \cdot \prod_{j=0}^{n-1} \|Df(\gamma_{j+1}(t))^{-1}\| \\ &\le \sigma^{-1/2}\|Df(x)^{-1}\| \cdot \prod_{j=0}^{n-1} \sigma^{-1/2}\|Df(f^{n-j}(x))^{-1}\| \\ &\le \sigma^{(n+1)/2}. \end{aligned} \tag{15}$$

In the last inequality we have used that $n+1$ is a (σ, δ)-hyperbolic time for x. Hence

$$\begin{aligned} \int_0^{t_0} \|\gamma'(t)\| dt &= \int_0^{t_0} \|Df^{n+1}(\gamma_{n+1}(t)) \cdot \gamma'_{n+1}(t)\| dt \\ &\ge \int_0^{t_0} \sigma^{-(n+1)/2} \|\gamma'_{n+1}(t)\| dt \\ &= \delta_1 \end{aligned}$$

This gives a contradiction, since $t_0 < 1$. Thus we have proved the claim.

Let us now finish the proof of the lemma. We simply consider the lifts by f^{n+1} of the geodesics joining $f^{n+1}(x)$ to the points in the boundary of $B_{\delta_1}(f^{n+1}(x))$. This defines a neighborhood $V_{n+1}(x)$ of x which is necessarily a $(\sqrt{\sigma}, \delta_1)$-hyperbolic preball, by (15). □

***Lemma 4.8* (Bounded distortion).** — *The family of $(\sqrt{\sigma}, \delta_1)$-hyperbolic preballs $V_n(x)$ has uniform bounded distortion.*

Proof. — Let $x \in M$ have n as a (σ, δ)-hyperbolic time and let $V_n(x)$ be the respective $(\sqrt{\sigma}, \delta_1)$-hyperbolic preball. It follows from Lemma 2.5 that for all $y, z \in V_n(x)$ and all $0 \le k \le n$

$$\operatorname{dist}(f^k(y), f^k(z)) \le 2\delta_1 \sigma^{(n-k)/2}. \tag{16}$$

On the other hand, since n is a hyperbolic time for x, choosing $\delta_1 < 1/8$ and recalling that we have taken $b < 1/2$, then

$$\begin{aligned} \operatorname{dist}(f^k(y), \mathcal{C}) &\ge \operatorname{dist}(f^k(x), \mathcal{C}) - \operatorname{dist}(f^k(x), f^k(y)) \\ &\ge \sigma^{b(n-k)} - 2\delta_1 \sigma^{(n-k)/2} \\ &\ge \frac{1}{2}\sigma^{b(n-k)} \\ &\ge 4\delta_1 \sigma^{(n-k)/2}. \\ &\ge 2 \operatorname{dist}(f^k(y), f^k(z)) \end{aligned} \tag{17}$$

Thus, we may apply (c_3) to obtain

$$\log \frac{|\det Df(f^k(y))|}{|\det Df(f^k(z))|} \le \frac{B}{\operatorname{dist}(f^k(y), \mathcal{C})^\beta} \operatorname{dist}(f^k(y), f^k(z)).$$

Using (16) and (17) we get

$$\begin{aligned} \log \frac{|\det Df^n(y)|}{|\det Df^n(z)|} &= \sum_{k=0}^{n-1} \log \frac{|\det Df(f^k(y))|}{|\det Df(f^k(z))|} \\ &\le \sum_{k=0}^{n-1} 2^{\beta+1} B \delta_1 \frac{\sigma^{(n-k)/2}}{\sigma^{b\beta(n-k)}}. \end{aligned}$$

Recall that $b\beta < 1/2$. □

We are left to prove that, under the assumptions of Theorem 4.4, Lebesgue almost every point has positive frequency of hyperbolic times.

Proposition 4.9. — *There are $0 < \sigma < 1$, $\delta > 0$ and $\theta > 0$ for which the frequency of (σ, δ)-hyperbolic times for Lebesgue almost every point is greater than θ.*

Proof. — The strategy is to use Lemma 3.9 twice, first for the sequence given by $a_j = -\log \|Df(f^{j-1}(x))^{-1}\|$ (up to a cut off that makes it bounded from above), and then with $a_j = \log \operatorname{dist}_\delta(f^{j-1}(x), \mathcal{C})$ for a convenient choice of $\delta > 0$. We prove that there exist many times n_i for which the conclusion of Lemma 3.9 holds, simultaneously, for both sequences. Then we check that any such n_i is a (σ, δ)-hyperbolic time for x.

Assuming that (9) holds for x, then for large N we have

$$\sum_{j=1}^{N} -\log \|Df(f^{j-1}(x))^{-1}\| \ge \lambda N\,.$$

Take $B, \beta > 0$ given by Definition 4.1. Then (c_1) implies that for large $\rho > 0$

$$\big| \log \|Df(x)^{-1}\| \big| \le \rho \, |\log \operatorname{dist}(x, \mathcal{C})| \tag{18}$$

for every $x \in M \setminus \mathcal{C}$. Fix $\varepsilon_1 > 0$ so that $\rho\varepsilon_1 \le \lambda/2$. Slow recurrence to $\mathcal{C}$ ensures that we may choose $r_1 > 0$ so that for large N

$$\sum_{j=1}^{N} \log \operatorname{dist}_{r_1}(f^{j-1}(x), \mathcal{C}) \ge -\varepsilon_1 N\,. \tag{19}$$

Fix any open neighborhood V of $\mathcal{C}$ and take $K_1 \ge \rho \, |\log r_1|$ large enough so that it is also an upper bound for $-\log \|Df^{-1}\|$ on $M \setminus V$. Then let

$$J = \big\{ 1 \le j \le N : \; -\log \|Df(f^{j-1}(x))^{-1}\| > K_1 \big\}.$$

Note that if $j \in J$, then $f^{j-1}(x) \in V$. Moreover, for each $j \in J$

$$\rho \, |\log r_1| \le K_1 < -\log \|Df(f^{j-1}(x))^{-1}\| < \rho \, |\log \operatorname{dist}(f^{j-1}(x), \mathcal{C})|,$$

which shows that $\operatorname{dist}(f^{j-1}(x), \mathcal{C}) < r_1$ for every $j \in J$. In particular,

$$\operatorname{dist}_{r_1}(f^{j-1}(x), \mathcal{C}) = \operatorname{dist}(f^{j-1}(x), \mathcal{C}) < r_1, \quad \forall j \in J.$$

Therefore, by (18) and (19),

$$\sum_{j \in J} -\log \|Df(f^{j-1}(x))^{-1}\| \le \rho \sum_{j \in J} |\log \operatorname{dist}(f^{j-1}(x), \mathcal{C})| \le \rho \, \varepsilon_1 N \le \frac{\lambda}{2} N.$$

Define

$$b_j = \begin{cases} -\log \|Df(f^{j-1}(x))^{-1}\|, & \text{if } j \notin J \\ 0 & \text{if } j \in J. \end{cases}$$

By definition, $b_j \le K_1$ for each $1 \le j \le N$. As a consequence,

$$\sum_{j=1}^{N} b_j = \sum_{j=1}^{N} -\log \|Df(f^{j-1}(x))^{-1}\| - \sum_{j \in J} -\log \|Df(f^{j-1}(x))^{-1}\| \ge \frac{\lambda}{2} N\,.$$

Defining $a_j = b_j - \lambda/4$, we have

$$\sum_{j=1}^{N} a_j \ge \frac{\lambda}{4} N\,.$$

Thus, we may apply Lemma 3.9 to $a_1, \ldots, a_N$, with $c = \lambda/4$ and $A = K_1$. The lemma provides $\theta_1 > 0$ and $\ell_1 \ge \theta_1 N$ times $1 \le p_1 < \cdots < p_{\ell_1} \le N$ such that

$$\sum_{j=n+1}^{p_i} -\log \|Df(f^{j-1}(x))^{-1}\| \ge \sum_{j=n+1}^{p_i} b_j = \sum_{j=n+1}^{p_i} \left(a_j + \frac{\lambda}{4} \right) \ge \frac{\lambda}{4}(p_i - n) \tag{20}$$

for every $0 \le n \le p_i - 1$ and $1 \le i \le \ell_1$.

Now fix $\varepsilon_2 > 0$ small enough so that $\varepsilon_2 < \theta_1 b\lambda/4$, and let $r_2 > 0$ be such that

$$\sum_{j=1}^{N} \log \mathrm{dist}_{r_2}(f^{j-1}(x), \mathcal{C}) \geq -\varepsilon_2 N\,. \tag{21}$$

Defining $a_j = \log \mathrm{dist}_{r_2}(f^{j-1}(x), \mathcal{C}) + b\lambda/4$ we have

$$\sum_{j=1}^{N} a_j \geq \left(\frac{b\lambda}{4} - \varepsilon_2\right) N\,.$$

Applying now Lemma 3.9 to $a_1, \dots, a_N$ with $c = b\lambda/4 - \varepsilon_2$ and $A = b\lambda/4$, we conclude that there are $l_2 \geq \theta_2 N$ times $1 \leq q_1 < \cdots < q_{\ell_2} \leq N$ such that

$$\sum_{j=n+1}^{q_i} \log \mathrm{dist}_{r_2}(f^{j-1}(x), \mathcal{C}) \geq -\frac{b\lambda}{4}\,(q_i - n) \tag{22}$$

for every $0 \leq n \leq q_i - 1$ and $1 \leq i \leq \ell_2$. Moreover,

$$\theta_2 = \frac{c}{A} = 1 - \frac{4\varepsilon_2}{b\,\lambda}\,.$$

Finally, our condition on ε_2 implies that $\theta = \theta_1 + \theta_2 - 1 > 0$. Then there exist $\ell \geq \theta N$ times $1 \leq n_1 < \cdots < n_\ell \leq N$ at which (20) and (22) occur simultaneously:

$$\sum_{j=n}^{n_i-1} -\log \|Df(f^j(x))^{-1}\| \geq \frac{\lambda}{4}(n_i - n)$$

and

$$\sum_{j=n}^{n_i-1} \log \mathrm{dist}_{r_2}(f^j(x), \mathcal{C}) \geq -\frac{b\lambda}{4}(n_i - n),$$

for every $0 \leq n \leq n_i - 1$ and $1 \leq i \leq \ell$. Letting $\sigma = e^{-\lambda/4}$ we easily obtain from the inequalities above

$$\prod_{j=n_i-k}^{n_i-1} \|Df(f^j(x))^{-1}\| \leq \sigma^k \quad \text{and} \quad \mathrm{dist}_{r_2}(f^{n_i-k}(x), \mathcal{C}) \geq \sigma^{bk},$$

for every $1 \leq i \leq \ell$ and $1 \leq k \leq n_i$. In other words, all those n_i are (σ, δ)-hyperbolic times for x, with $\delta = r_2$. □

A
Appendix: Measure and integration

A.1. Measure spaces. — Let X be set and $\mathcal{A}$ be a collection of subsets of X. We say that $\mathcal{A}$ is a *σ-algebra* if the following conditions hold:

1. $X \in \mathcal{A}$;
2. if $A \in \mathcal{A}$ then $X \backslash A \in \mathcal{A}$;

3. if $A_1, A_2, A_3 \dots \in \mathcal{A}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

We will refer to a pair $(X, \mathcal{A})$ where $\mathcal{A}$ is a σ-algebra on X as a *measurable space*. Let $\mu\colon \mathcal{A} \to [0, +\infty]$ be a function defined on a σ-algebra $\mathcal{A}$ of X.We say that μ is a *measure* if the following conditions hold:

1. $\mu(\emptyset) = 0$.
2. If $A_1, A_2, \dots \in \mathcal{A}$ are pairwise disjoint, then $\mu\big(\cup_{i=1}^{\infty} A_i\big) = \sum_{i=1}^{\infty} \mu(A_i)$.

We say $(X, \mathcal{A}, \mu)$ is a *measure space* if $(X, \mathcal{A})$ is a measurable space and μ is a measure on $\mathcal{A}$. If $\mu(X) = 1$ then μ is called a *probability measure* and $(X, \mathcal{A}, \mu)$ a *probability space*. We say that $A \subset X$ has *null measure* if there is $B \in \mathcal{A}$ such that $A \subset B$ and $\mu(B) = 0$. We say that some property on the elements of X holds *almost everywhere* (*a.e.* for short), if the set of points for which that property does not hold has null measure.

***Example A.1* (Counting measure).** — Let X be a set. We define a function $\#$ in the σ-algebra $\mathcal{P}(X)$ (the collection of all subsets of X) taking $\#(A)$ as the number of elements of A ($+\infty$ if A infinite) for each $A \subset X$. $\#$ defines a measure on $\mathcal{P}(X)$ that will be called the *counting measure* on X.

***Example A.2* (Dirac measure).** — Let X be a set and fix a point $x \in X$. Given $A \subset X$ we define δ_x in $\mathcal{P}(X)$ as $\delta_x(A) = 1$ if $x \in A$, and $\delta_x(A) = 0$ otherwise. δ_x defines a probability measure on X that will be called the *Dirac measure* supported on x.

Now we assume that X is a metric space. We define $\mathcal{B}(X)$, the *Borel σ-algebra* on X, as the σ-algebra generated by the open sets of X, i.e. the smallest (in terms of inclusion) σ-algebra that contains the open sets of X. This smallest σ-algebra always exists, since $\mathcal{P}(X)$ is a σ-algebra containing the open subsets of X and the intersection of σ-algebras is still a σ-algebra.

***Example A.3* (Lebesgue measure).** — Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}^d$. There is a unique measure m defined on $\mathcal{B}$ such that for intervals $I_1, \dots, I_d \subset \mathbb{R}$ one has

$$m\bigg(\prod_{i=1}^{n} I_i\bigg) = |I_1| \times \cdots \times |I_n|,$$

where each $|I_i|$ denotes the length of I_i. This is called *Lebesgue measure* on $\mathbb{R}^d$. Using a volume form we introduce the Lebesgue measure on Riemnannian manifolds in a similar way.

The *support* of a measure μ, which is denoted by $\mathrm{supp}(\mu)$, is defined as

$$\mathrm{supp}(\mu) = \{x \in X\colon \mu(U) > 0 \text{ for each neighborhood } U \text{ of } x\}.$$

A Borel probability measure μ on a metric space X is called *regular* if for all $A \in \mathcal{B}(X)$ and $\epsilon > 0$ there are a compact set $F_\epsilon \subset A$ and an open set $U_\epsilon \supset A$ such that $\mu(U_\epsilon \backslash F_\epsilon) < \epsilon$.

Theorem A.4. — *Every Borel probability measure on a compact metric space is regular.*

A.2. Integration. — Let $(X, \mathcal{A}, \mu)$ be a measure space. We say that $\varphi\colon X \to \mathbb{R}$ is *measurable function*, if $\varphi^{-1}(B) \in \mathcal{A}$ for every Borel set $B \subset \mathbb{R}$. A function $\varphi\colon X \to \mathbb{C}$ is said to be measurable if both its real part and its imaginary part are measurable functions. We say that $\varphi\colon X \to \mathbb{C}$ is a *simple function* if there are $A_1, \dots, A_n \in \mathcal{A}$ and $a_1, \dots, a_n \in \mathbb{C}$ such that

$$\varphi = \sum_{i=1}^{n} a_i \mathbf{1}_{A_i},$$

where $\mathbf{1}_A$ denotes the characteristic function of $A \subset M$. A simple function φ is said to be an *integrable function* if $\sum_{i=1}^{n} a_i \mu(A_i) < \infty$ (we assume that $0 \cdot \infty = 0$). In such case we define the *integral* of φ with respect to μ as

$$\int_X \varphi d\mu = \sum_{i=1}^{n} a_i \mu(A_i).$$

This value does not depend on the way we write φ as a combination of characteristic functions. We say that a measurable $\varphi\colon X \to \mathbb{C}$ is an *integrable function* if there is a sequence of simple functions $\varphi_n\colon X \to \mathbb{C}$ such that

$$\lim_{n\to\infty} \varphi_n(x) = \varphi(x), \quad \text{for almost every } x \in X,$$

and

$$\lim_{n\to\infty} \int_X |\varphi_n - \varphi_m| d\mu = 0.$$

In this case we define the *integral* of φ with respect to μ as

$$\int_X \varphi d\mu = \lim_{n\to\infty} \int_X \varphi_n d\mu.$$

One can prove that this limit exists and is independent of the sequence we take. Moreover, a function φ is integrable if and only if $|\varphi|$ is integrable. Given $A \in \mathcal{A}$ we say that φ is integrable on A if $\varphi \mathbf{1}_A$ is integrable. In such case we write

$$\int_A \varphi d\mu = \int_X \varphi \mathbf{1}_A d\mu.$$

We drop X when the integral is over the whole space.

In general, integration over continuous functions completely determines the Borel measure.

Theorem A.5. — *Let μ and ν be finite measures over the Borel sets of a compact metric space X. If $\int \varphi d\mu = \int \varphi d\nu$ for every continuous $\varphi : X \to \mathbb{R}$, then $\mu = \nu$.*

Let μ and ν be finite measures defined on a same σ-algebra $\mathcal{A}$. We say that ν is *absolutely continuous* with respect to μ, and write $\nu \ll \mu$, if $\nu(A) = 0$ whenever $\mu(A) = 0$. The measures μ and ν are said to be *equivalent* if both $\mu \ll \nu$ and $\nu \ll \mu$. We say that μ and ν are *singular measures*, and write $\mu \perp \nu$, if there exists $A \in \mathcal{A}$ such that $\mu(A) = 0 = \nu(X \setminus A)$.

Theorem A.6 **(Lebesgue decomposition)**. — *Let μ and ν be finite measures defined on a σ-algebra $\mathcal{A}$. There are finite measures μ_a and μ_s with $\mu_a \ll \nu$ and $\mu_s \perp \nu$ such that $\mu = \mu_a + \mu_s$.*

Let $(X, \mathcal{A}, \mu)$ be a measure space and φ be a nonnegative real function defined on X. The properties of the integral give that

$$\nu \colon \mathcal{A} \ni A \longmapsto \int_A \varphi d\mu$$

defines a measure on A. The next theorem shows that this is the only way of finding measures which are absolutely continuous with respect to a reference one. The function φ is called the *Radon-Nykodim derivative* of ν with respect to μ and is often denoted by $d\nu/d\mu$.

Theorem A.7 **(Radon-Nykodim)**. — *Let μ and ν be finite measures defined on a σ-algebra $\mathcal{A}$. Then ν is absolutely continuous with respect to μ if and only if there is $\varphi \colon X \to \mathbb{R}$ nonnegative and integrable with respect to μ such that*

$$\nu(A) = \int_A \varphi d\mu \quad \textit{for each } A \in \mathcal{A}.$$

Moreover, any two functions with this property coincide μ almost everywhere.

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