In a mechanical device, like a robot arm, the forward kinematic map relates the configuration space of the joints to the working space of the device. A typical kinematic map is smooth but with certain singularities, and it admits only partial inverses. We are going to describe a general setting for the study of the motion planning problem in this context, and discuss the complexity of some simple joint configurations.
TOPOLOGICAL COMPLEXITY

Motion planning in configuration spaces:

\[ \mathcal{C} - \text{configuration space} \] of a mechanical device

Find robust rules that product paths connecting any two given configurations in \( \mathcal{C} \).

\[ (c_{\text{init}}, c_{\text{fin}}) \in \mathcal{C} \times \mathcal{C} \quad \text{query} \]

\[ \rho : \mathcal{C} \times \mathcal{C} \to PC, \quad \rho(c_{\text{init}}, c_{\text{fin}})(0) = c_{\text{init}}, \quad \rho(c_{\text{init}}, c_{\text{fin}})(1) = c_{\text{fin}} \quad \text{roadmap} \]

(Farber) Global roadmap exists \( \iff \mathcal{C} \) is contractible.

Consider partial roadmaps, defined on subsets of \( \mathcal{C} \times \mathcal{C} \).

\[ TC(\mathcal{C}) : = \text{minimal number of partial roadmaps, needed to satisfy all queries.} \]
Partial roadmaps are partial sections of $\pi$. $\pi$ is a fibration $\Rightarrow$ $TC(\mathcal{C})$ is the Schwarz genus of $\pi$.

In particular, $TC(\mathcal{C})$ is a homotopy invariant of $\mathcal{C}$.

Extensive computations of $TC(\mathcal{C})$ based on explicit roadmaps (upper bounds) and cohomology obstructions (lower bounds).
Actual motion planning is more complex:

- **configuration space** \( C \) (position of joints)
- **working space** \( W \) (spatial position of the device)

**robot device**

**planar joints**

\[
C = T^2 = \{(e^{i\varphi}, e^{i\psi}) \in S^1 \times S^1\}
\]

\[
W = \{(x, y) \in \mathbb{R}^2 | (R - r)^2 \leq x^2 + y^2 \leq (R + r)^2\}
\]

\[
f(e^{i\varphi}, e^{i\psi}) = (R \cos \varphi + r \cos(\varphi + \psi),
R \sin \varphi + r \sin(\varphi + \psi))
\]

**universal joint**

\[
C = T^2
\]

\[
W = S^2
\]

\[
x = r \cos \psi \cos \varphi
\]

\[
y = r \cos \psi \sin \varphi
\]

\[
z = r \sin \psi
\]
TOPOLOGICAL COMPLEXITY OF KINEMATIC MAPS

INTRODUCTION

\[ C = T^3 \quad \mathcal{W} = SO(3) \]

Forward kinematic map \( f : T^3 \rightarrow SO(3) \) is given in terms of Euler angles as

\[
f(e^{i\phi}, e^{i\psi}, e^{i\theta}) = \begin{bmatrix}
cos \phi \cos \psi \cos \theta - \sin \phi \sin \theta & -\cos \theta \sin \phi - \cos \phi \cos \psi \sin \theta & \cos \phi \sin \psi \\
cos \phi \sin \theta + \cos \psi \cos \theta \sin \phi & \cos \phi \cos \theta - \cos \psi \sin \phi \sin \theta & \sin \phi \sin \psi \\
-\cos \psi \sin \phi & \sin \psi \sin \theta & \cos \psi
\end{bmatrix}
\]
**MOTION PLANNING**

\[ f : \mathcal{C} \to \mathcal{W} \]  
forward kinematic map from a configuration space \( \mathcal{C} \) to a working space \( \mathcal{W} \)

Find robust rules to move the device from a given joint configuration to a desired position in the working space.

\[(c_{init}, w_{fin}) \in \mathcal{C} \times \mathcal{W} \quad \text{query}\]

\[\rho : \mathcal{C} \times \mathcal{W} \to PC, \quad \rho(c_{init}, w_{fin})(0) = c_{init}, \quad f(\rho(c_{init}, w_{fin})(1)) = w_{fin} \quad \text{roadmap}\]

(Dranishnikov) Global roadmap exists ⇒ \( f \) admits a section.

Proof: given \( \rho : \mathcal{C} \times \mathcal{W} \to PC, \quad w \mapsto \rho(c_0, w)(1) \) is a section of \( f \).

If \( f \) does not admit a section consider partial roadmaps, defined on subsets of \( \mathcal{C} \times \mathcal{W} \).

\[ TC(f) : = \text{minimal number of partial roadmaps, needed to satisfy all queries.} \]
Assume $C, W, Q$ are ENRs:

If $f$ is a fibration, then $TC(f)$ is the Schwarz genus of $(1 \times f)\pi$.

However, forward kinematic maps that arise in practice:

1. do not admit inverse kinematic maps (i.e. sections), and
2. are smooth but always have singular points (where Jacobian does not have maximal rank). Therefore are not fibrations. (D. Gottlieb, 1986, 1988)

In general $TC(f) \geq$ Schwarz genus of $(1 \times f)\pi$ (and the difference can be arbitrarily big).

Problem: compute $TC(f)$ for basic robot arm configurations.
Observe: $TC(1_X) = TC(X)$.

Two special cases:

(1) Assume $f: C \to W$ admits section $s: W \to C$:

Then, to move from $c \in C$ to $w \in W$ it is sufficient to move from $c$ to $s(w)$ in $C$, and to move from $w$ to $w'$ in $W$ it is sufficient to move from $s(w)$ to $w'$.

$f$ sectioned $\Rightarrow TC(W) \leq TC(f) \leq TC(C)$

(2) Assume $f: C \to W$ fibration:

Then, to move from $c \in C$ to $w \in W$ it is sufficient to find a path in $W$ from $f(c)$ to $w$, and lift it to $C$ starting at $c$.

$f$ fibration $\Rightarrow TC(f) \leq TC(W)$
In general, avoid singularities and split into subsets that admit local sections.

**RELATIVE COMPLEXITY**

\[ f: \mathcal{C} \rightarrow \mathcal{W}, \ Q \subseteq \mathcal{C} \times \mathcal{W} \]

**\( TC(f|Q) \)** = minimal number of partial roadmaps, needed to satisfy all queries in \( Q \).

Properties:

1. \( TC(f|\Gamma(f)) = 1 \)
2. \( Q \subseteq Q' \Rightarrow TC(f|Q) \leq TC(f|Q') \)
3. \( TC(f|Q \cup Q') \leq TC(f|Q) + TC(f|Q') \)
4. If \( Q' \) can be horizontally deformed into \( Q \), then \( TC(f|Q) \geq TC(f|Q') \)

In particular, if \( Q' \) can be horizontally deformed to a subset \( Q \), then \( TC(f|Q) = TC(f|Q') \).
TOPOLOGICAL COMPLEXITY OF KINEMATIC MAPS

\[ f: \mathcal{C} \rightarrow \mathcal{W} \text{ smooth} \]

\[ \mathcal{W}^r := \text{regular values of } f, \quad \mathcal{C}^r := f^{-1}(\mathcal{W}^r) \]

\[
\begin{align*}
TC(f|\mathcal{C}^r \times \mathcal{W}^r) & \leq TC(f|\mathcal{W}^r) \\
TC(f|\mathcal{C} \times \mathcal{W}^r) & \leq \text{cat}(\mathcal{C} \times \mathcal{W}^r)
\end{align*}
\]

RELATIVE COMPLEXITY

\[ f: \mathcal{C} \rightarrow \mathcal{W} \text{ admits a partial section } s \text{ over } \mathcal{W}', \quad \mathcal{C}': = f^{-1}(\mathcal{W}') \]

\[
\begin{align*}
TC(\mathcal{W}|\mathcal{W} \times \mathcal{W}') & \leq TC(f|\mathcal{C} \times \mathcal{W}') \leq TC(\mathcal{C}|\mathcal{C} \times s(\mathcal{W}'))
\end{align*}
\]

Moreover,

if \( \mathcal{C}' \) can be deformed to \( s(\mathcal{W}') \), then 
\[ TC(f|\mathcal{C} \times \mathcal{W}') = TC(\mathcal{C}|\mathcal{C} \times s(\mathcal{W}')) \]

and

if \( \mathcal{W}' \subseteq \mathcal{W}^r \), then 
\[ TC(f|\mathcal{C}' \times \mathcal{W}') = TC(\mathcal{W}') \]

If \( \mathcal{C} \) is a topological group, then 
\[ TC(\mathcal{C}|\mathcal{C} \times s(\mathcal{W}')) = TC(\mathcal{C}) = \text{cat}(\mathcal{C}). \]
Kinematic map is sectioned and is regular except for $\psi = 0, \pi$.

$$\Rightarrow \quad 2 = TC(S^1 \times I) \leq TC(f) \leq TC(T^2) = 3$$

Furthermore, for $C' = S^1 \times 0$ we have

$$TC(f) \geq TC(f|T^2 \times f(C')) = TC(T^2|T^2 \times C') = 3.$$ 

Therefore $TC(f) = 3$.

Kinematic map is regular except for $\psi = \pm \frac{\pi}{2}$, and is sectioned over $S^2 - \{\text{poles}\}$.

Previous results yield $3 \leq TC(f) \leq 5$, ...

... which may be improved to $4 \leq TC(f) \leq 5$, by a careful analysis of extensions of roadmaps around singular points.
Singular values of $f$ are rotations whose Z-axis coincides with the Z-axis of the reference frame. These are matrices of the form
\[
\begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
corresponding to points on a circle in $SO(3)$.

Singular points are of the form $(e^{i\varphi}, 1, e^{i\theta})$ and $(e^{i\varphi}, -1, e^{i\theta})$, corresponding to points in a disjoint union of two two-dimensional tori in $T^3$.

$f$ admits a section on the complement of singular points, and the previous estimates yield $4 \leq TC(f) \leq 6$. 

robot arm wrist
Further results

We implicitly used cohomological lower bounds for the topological complexity of various configuration and working spaces.

One can also consider the diagram for partial liftings

\[
\begin{array}{ccc}
Q & \longrightarrow & C \times \mathcal{W} \\
\rho \downarrow & & \downarrow (1 \times f)\pi \\
P C & \longrightarrow & \mathcal{Q} \times \mathcal{C}
\end{array}
\]

and use the standard approach to derive the estimate

\[ TC(f) \geq \text{nil}(\ker(1, f)^*: H^*(C \times \mathcal{W}) \to H^*(C)) \]

(Also follows from the fact that \( TC(f) \) is bigger than the Schwarz genus of \((1 \times f)\pi\).)
THANKS!