Higgs Bundles

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Introduction

These are the slides from my lectures on Higgs bundles at the *International School on Geometry and Physics: moduli spaces in geometry, topology and physics* of the Spanish Semester on Moduli Spaces (January-June 2008).

I would like to thank the organizers of the school for the invitation to speak and for all their excellent work, and I would also like to thank the participants for creating a stimulating athmosphere.

These slides are provided "as is" and should be considered simple lecture notes. Nevertheless, I will be grateful if errors, omissions, missing references etc. are brought to my attention at `pbgothen@fc.up.pt`.

Harmonic maps

Classically, a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is *harmonic* if

$$\Delta(f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Important examples of harmonic maps are the real parts of holomorphic functions.
Let M be a smooth manifold and let $f\colon M \to \mathbb{R}$ be smooth. the differential of f is

$$df = \frac{\partial f}{\partial x^1}\, dx^1 + \cdots + \frac{\partial f}{\partial x^n}\, dx^n,$$

this has a natural extension to the *exterior differential* on p-forms $\omega = \sum f_{i_1\ldots i_p}\, dx^{i_1} \wedge \ldots \wedge dx^{i_p} \in A^p(M)$ given by

$$d\left(\sum f_{i_1\ldots i_p}\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}\right) = \sum df_{i_1\ldots i_p} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}.$$

de Rham cohomology

Since $d^2 = 0$, the exterior differential gives rise to the *de Rham complex*:

$$0 \to A^0(M) \xrightarrow{d} A^1(M) \xrightarrow{d} \ldots \xrightarrow{d} A^n(M) \to 0.$$

We have the de Rham cohomology groups:

$$H^P_{\mathrm{dR}}(M) = \ker(d\colon A^p \to A^{p+1}) / \operatorname{im}(d\colon A^{p-1} \to A^p).$$

When M is compact and riemannian, there is an L^2-inner product on $A^p(M)$ and d has a formal adjoint $d^*\colon A^p \to A^{p-1}$ defined by

$$\langle \omega, d\tau \rangle = \langle d^*\omega, \tau \rangle.$$

The *Laplace operator* is

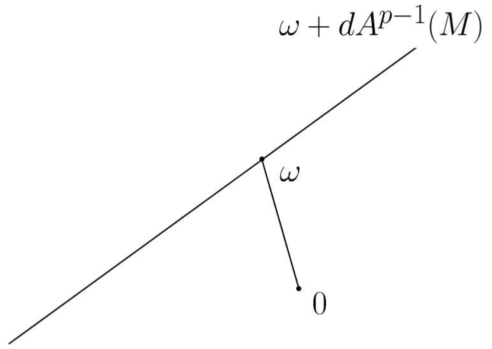
$$\Delta = d \circ d^* + d^* \circ d\colon A^p(M) \to A^p(M).$$

Harmonic representatives of cohomology classes

Assume that M is compact.

Question: Given $\omega \in A^p(M)$ with $d\omega = 0$, is there a "best" representative of the de Rham cohomology class $[\omega] \in H^p_{\mathrm{dR}}(M)$?

Answer: Minimize the L^2-norm $\|\omega\|^2 = \int_M |\omega|^2$.



Clearly, $\|\omega\|$ is minimal if and only if

$$\langle \omega, d\tau \rangle = 0 \ \forall \tau \iff d^*\omega = 0.$$

Harmonic forms

One easily sees that: $d\omega = 0$ and $d^*\omega = 0 \iff \Delta(\omega) = 0.$

Definition

$\omega \in A^p(M)$ is *harmonic* if $\Delta(\omega) = 0$.

On $\mathbb{R}^n$ with the standard euclidean metric,

$$\Delta = \frac{\partial^2}{\partial (x^1)^2} + \cdots + \frac{\partial^2}{\partial (x^n)^2}.$$

Complex manifolds

- X – complex manifold with coordinates $(z^1, \ldots, z^n)$.
- Holomorphic and antiholomorphic tangent and cotangent spaces:

$$T^{1,0}X = \mathbb{C}\langle \tfrac{\partial}{\partial z^1}, \ldots, \tfrac{\partial}{\partial z^n} \rangle, \qquad (T^{1,0})^*X = \mathbb{C}\langle\, dz^1, \ldots, \, dz^n \rangle,$$
$$T^{0,1}X = \mathbb{C}\langle \tfrac{\partial}{\partial \bar{z}^1}, \ldots, \tfrac{\partial}{\partial \bar{z}^n} \rangle, \qquad (T^{0,1})^*X = \mathbb{C}\langle\, d\bar{z}^1, \ldots, \, d\bar{z}^n \rangle.$$

- Complexified cotangent space:

$$T^*X_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{R}}(T^{1,0}X, \mathbb{C}) = (T^{1,0})^*X \oplus (T^{0,1})^*X$$

  ($\mathbb{C}$-linear and antilinear parts).
- $\mathbb{C}$-valued real differential forms decompose according to *type*:

$$(T^{p,q})^*X = \Lambda^p(T^{1,0}X)^* \wedge \Lambda^q(T^{0,1}X)^*,$$
$$A^{p,q}(X) = C^\infty(X, (T^{p,q})^*X),$$
$$\overset{\Psi}{\alpha} = \sum \alpha_I \, dz^{i_1} \wedge \ldots \wedge \, dz^{i_p} \wedge \, d\bar{z}^{j_1} \wedge \ldots \wedge \, d\bar{z}^{j_q}.$$

Exterior differential

The exterior differential $d\colon A^n(X) \to A^{n+1}(X)$ decomposes according to type $d = \partial + \bar{\partial}$, where

$$\partial\colon A^{p,q}(X) \to A^{p+1,q}(X) \quad \text{and} \quad \bar{\partial}\colon A^{p,q}(X) \to A^{p,q+1}(X).$$

Locally:

$$\bar{\partial}(\sum \alpha_I \, dz^{i_1} \wedge \ldots \wedge \, dz^{i_p} \wedge \, d\bar{z}^{j_1} \wedge \ldots \wedge \, d\bar{z}^{j_q})$$
$$= \sum \bar{\partial}(\alpha_I) \wedge \, dz^{i_1} \wedge \ldots \wedge \, dz^{i_p} \wedge \, d\bar{z}^{j_1} \wedge \ldots \wedge \, d\bar{z}^{j_q},$$

where

$$\bar{\partial}(\alpha_I) = \sum \frac{\partial \alpha_I}{\partial \bar{z}^i} \, d\bar{z}^i; \quad \frac{\partial}{\partial \bar{z}^i} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{1}{2}\frac{\partial}{\partial y}\right).$$

Harmonic theory

Analogously to the case of real manifolds, there is a harmonic theory for complex manifolds X endowed with a *hermitian metric*

$$h = g - 2i\omega,$$

where g is a riemannian metric and ω is a (non-degenerate) positive form of type $(1,1)$. (In fact, h can be recovered from ω.)

Since $\bar{\partial}^2 = 0$, the Dolbeault cohomology groups can be defined:

$$H^{p,q}(X) = \frac{\ker\left(\bar{\partial}\colon A^{p,q}(X) \to A^{p,q+1}(X)\right)}{\operatorname{im}\left(\bar{\partial}\colon A^{p,q-1}(X) \to A^{p,q}(X)\right)}.$$

When X is compact, there is a harmonic theory for the $\bar{\partial}$-*Laplacian*:

$$\Delta_{\bar{\partial}} = \bar{\partial}\circ\bar{\partial}^* + \bar{\partial}^*\circ\bar{\partial}\colon A^{p,q}(X) \to A^{p,q}(X).$$

Kähler manifolds

In general, the harmonic theories for Δ and $\Delta_{\bar{\partial}}$ are unrelated.

Definition

A hermitean metric on a complex manifold X is *Kähler* if its associated $(1,1)$-form is closed: $d\omega = 0$; in other words, (X,ω) is symplectic.

On a Kähler manifold, the d- and $\bar{\partial}$-Laplacians are related:

$$\Delta = 2\Delta_{\bar{\partial}} = 2\Delta_{\partial}.$$

By looking at the harmonic representatives, this leads to the *Hodge decomposition*:

$$H^r(X) = \bigoplus_{p+q=r} H^{p,q}(X).$$

Philosophy

Recall de Rham's Theorem $H^r_{\text{sing}}(X, \mathbb{C}) \cong H^r_{dR}(X, \mathbb{C})$ which relates topology and geometry of real differential forms.
The representation of cohomology classes through harmonic differential forms on a compact Kähler manifold X, together with de Rham's Theorem, reveals an intimate interplay between

- Topology: $H^r(X, \mathbb{C})$ (singular cohomology).
- Geometry of differential forms: $H^r_{dR}(X, \mathbb{C})$ (de Rham cohomology).
- Holomorphic geometry: $H^{p,q}(X)$ (Dolbeault cohomology). *Recall:* $H^{p,q}(X) = H^q(X, \Omega^p)$, where Ω^p is the sheaf of holomorphic differential p-forms.

First cohomology

The first singular cohomology group is the abelianization of the fundamental group:

$$H^1(X, \mathbb{C}) \cong \text{Hom}(\pi_1(X), \mathbb{C}).$$

Thus Hodge theory provides an isomorphism

$$\text{Hom}(\pi_1(X), \mathbb{C}) \cong H^1(X, \mathcal{O}_X) \oplus H^0(X, \Omega^1).$$

"Integrate":
The *character variety* of the group $\pi_1(X)$ (or *Betti moduli space* of X) is

$$\mathcal{R}(\pi_1(X), \mathbb{C}^*) = \text{Hom}(\pi_1(X), \mathbb{C}^*) \cong H^1(X, \mathbb{C}^*).$$

The de Rham moduli space

Definition

The *de Rham moduli space* is the space of flat connections on the trivial complex line bundle on X:

$$\begin{aligned}\mathcal{M}_{dR} &:= \{\text{flat connections}\}/\{\text{gauge equivalence}\}\\ &= A^1(X,\mathbb{C})/\mathcal{G},\end{aligned}$$

where the gauge group of smooth gauge transformations is $\mathcal{G} = C^\infty(X, \mathbb{C}^*)$.

From this point of view, the analogue of de Rham's Theorem is

$$\begin{aligned}\mathcal{M}_{dR} &\xrightarrow{\cong} \mathcal{R}(\pi_1(X), \mathbb{C}^*)\\ B &\mapsto ([\gamma] \mapsto \text{holonomy of } B \text{ around } \gamma)\end{aligned}$$

Abelian Hodge Theory for H^1

Interpretation in Čech cohomology

Let $\check{H}^1(X, \mathbb{C}^*)$ be the first Čech cohomology group with coefficients in $\mathbb{C}^*$.

With respect to a cover $\mathcal{U} = \{U_\alpha\}$ of X, a Čech cohomology class is given by a cocycle $\{g_{\alpha\beta} \colon U_{\alpha\beta} \to \mathbb{C}^*\}$ of *locally constant* functions on the intersections $U_{\alpha\beta}$ satisfying the cocycle conditions:

$$\begin{cases} g_{\alpha\alpha} = 1 \\ g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma} \end{cases}$$

The $g_{\alpha\beta}$ can be interpreted as the transition functions defining a flat bundle with respect to trivializations over the U_α.

Thus $H^1(X, \mathbb{C}^*)$ can be identified with the space of isomorphism classes of flat complex line bundles on X.

The de Rham moduli space – 2

The short exact sequence of sheaves of locally constant functions

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to 0$$

gives an exact sequence in cohomology:

$$0 \to H^1(X,\mathbb{Z}) \to H^1(X,\mathbb{C}) \to H^1(X,\mathbb{C}^*) \xrightarrow{\delta} H^2(X,\mathbb{Z}).$$

The coboundary δ maps a flat bundle with trivial underlying topological bundle to zero. Hence, the de Rham moduli space is:

$$\begin{aligned} \mathcal{M}_{dR} &:= \{\text{flat connections}\}/\{\text{gauge equivalence}\} \\ &\cong H^1_{dR}(X,\mathbb{C})/H^1(X,\mathbb{Z}). \end{aligned}$$

Remark: This could of course also be seen directly via the action of $g \in \mathcal{G}$. This gives a canonical identification

$$T_B\mathcal{M}_{dR} = H^1_{dR}(X,\mathbb{C}). \tag{4.1}$$

Abelian Hodge Theory for H^1

Thus Hodge theory says that

$$T_B\mathcal{M}_{dR} \cong H^1(X,\mathcal{O}_X) \oplus H^0(X,\Omega^1)$$

at any flat connection B.

Question: What is the “integrated” version of this statement?

Clue: $H^1(X,\mathcal{O}_X)$ is the tangent space to the group of degree zero line bundles $\mathrm{Pic}^0(X) = H^1(X,\mathcal{O}_X^*)$ and *Serre duality* says that

$$H^0(X,\Omega^1) \cong H^1(X,\mathcal{O})^*.$$

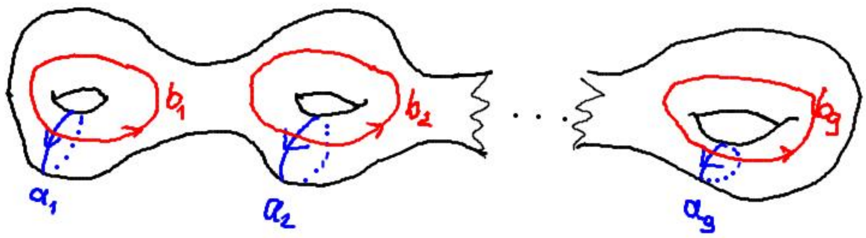
Answer: There is an isomorphism:

$$\mathcal{M}_{dR} \cong T^*\mathrm{Pic}^0(X). \tag{4.2}$$

Representations of $\pi_1(X)$ and flat connections

- Let X be a closed Riemann surface of genus $g \geqslant 2$.
- Let G be a connected reductive Lie group (real or complex).

The fundamental group of X is



$$\pi_1(X) = \langle a_1, b_1, \ldots, a_g, b_g \ : \ \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle$$

Basic object of interest: *Character variety* or *Betti moduli space*

$$\mathcal{R}(\pi_1(X), G) = \mathcal{M}_B(X, G) := \mathrm{Hom}^+(\pi_1 S, G)/G.$$

Flat connections

Smooth G-bundles on X are classified by a characteristic class

$$c(E) \in H^2(X, \pi_1(G)) \cong \pi_1(G).$$

Fix $d \in \pi_1(G)$ and let E be a fixed smooth G-bundle on X with $c(E) = d$.

Define the de Rham moduli space by
$\mathcal{M}_{dR}^d(X, G) := \{\text{reductive flat connections on } E\}/\{\text{gauge equivalence}\}.$

- A flat connection is *reductive* if its holonomy representation is reductive.
- A representation $\rho \colon \pi_1(X) \to G$ is *reductive* if the Zariski closure of its image is a reductive subgroup of G.

Holonomy

Let

$$\mathcal{R}_d(X,G) \subseteq \mathcal{R}(X,G)$$

be the subspace of representations such that that the corresponding flat bundle has characteristic class $d \in \pi_1(G)$.

The holonomy representation provides an identification

$$\mathcal{R}_d(\pi_1(X),G) \cong \mathcal{M}^d_{dR}(X,G).$$

Conversely, given $\rho\colon \pi_1(X) \to G$, the corresponding flat principal G-bundle E_ρ is given by

$$E_\rho = \tilde{X} \times_{\pi_1(X)} G,$$

where $\tilde{X} \to X$ is the universal cover and $\pi_1(X)$ acts on G via ρ.
Note: Everything we do can be generalized to the situation of connections with constant central curvature. These correspond to representations of a central extension of the fundamental group.

Harmonic theory in the non-abelian case

Harmonic maps

Let (M,g) and (N,h) be riemannian manifolds, with M compact.
A map $f\colon M \to N$ is *harmonic* if it is a critical point of the energy functional

$$E(f) = \int_M |df|^2 d\mathrm{vol}.$$

Note that df is a section of $f^*TN \to M$. Let ∇^h be the Levi–Civita connection on (N,h) and let $f^*\nabla^h$ be its pull-back to f^*TN. The Euler–Lagrange equations for $E(f)$ are

$$f^*\nabla^h(df) = 0.$$

When $\dim M = 2$, the equation only depends on the conformal class of the metric g on M. In particular, the notion of a harmonic map on a Riemann surface makes sense.
The fundamental work of Eells and Sampson proves the existence of harmonic maps under suitable conditions.

Harmonic metrics in flat bundles

Let $H \subseteq G$ be a maximal compact subgroup.

A *metric* on a G-bundle E is a section $\sigma \colon X \to E/H$ of the bundle $E/H \to X$.

Equivalently, a metric is a ρ-equivariant map

$$\sigma \colon \tilde{X} \to G/H,$$

Since G/H is riemannian and X is a Riemann surface, it makes sense to ask for σ to be a *twisted* harmonic map, i.e.,

$$\sigma^* \nabla(d\sigma) = 0, \tag{5.1}$$

where ∇ is the Levi–Civita connection on G/H.

Theorem (C. Corlette [4], S. Donaldson [6])

A flat bundle $E \to X$ admits a harmonic metric if and only if the flat connection is reductive. □

Lie groups and the isotropy representation

Let $H \subseteq G$ be a maximal compact subgroup. Take a *Cartan decomposition* of the Lie algebra of G:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}.$$

The restriction of the adjoint representation of G on $\mathfrak{g}$ to $H \subset G$ preserves the Cartan decomposition. In particular, we get the *isotropy representation*

$$\iota \colon H \to \mathrm{Aut}(\mathfrak{m}).$$

All this can be complexified to $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$.
There is a *Cartan involution* $\tau \colon \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ with $\tau_{|\mathfrak{h}^{\mathbb{C}}} = 1$ and $\tau_{|\mathfrak{m}^{\mathbb{C}}} = -1$.

Harmonic metrics – 1

A metric $\sigma\colon X \to E/H$ gives a reduction of structure group $i\colon E_H \hookrightarrow E$. Let B be a flat G-connection on E. Write

$$i^*B = A + \theta \in A^1(E, \mathfrak{g} \oplus \mathfrak{m}).$$

Then A is an H-connection on E_H and $\theta \in A^1(E_H, \mathfrak{m})$ is tensorial, i.e.,

$$\theta \in A^1(X, E_H(\mathfrak{m})),$$

where $E_H(\mathfrak{m}) = E_H \times_H \mathfrak{m}$ is the $\mathfrak{m}$-bundle associated to the isotropy representation.

Harmonic theory in the non-abelian case

Harmonic metrics – 2

Note that $\mathrm{Ad}(E) = E(\mathfrak{g}) = E_H(\mathfrak{g}) = E_H(\mathfrak{h}) \oplus E_H(\mathfrak{m})$. In terms of the data (A, θ), the harmonicity condition (5.1) on the metric σ is

$$d_A^*\theta = 0, \tag{5.2}$$

where $d_A\colon A^1(E_H(\mathfrak{m})) \to A^2(E_H(\mathfrak{m}))$ is the covariant derivative associated to the connection A and $d_A^*\colon A^1(E_H(\mathfrak{m})) \to A^0(E_H(\mathfrak{m}))$ is its adjoint.

Interpretation in holomorphic terms
Write $d_A = \bar{\partial}_A + \partial_A$ and $\theta = \phi + \tau(\phi)$, where $\tau\colon A^1(E_H(\mathfrak{m}^{\mathbb{C}})) \to A^1(E_H(\mathfrak{m}^{\mathbb{C}}))$ denotes the combination of complex conjugation on the form component with the Cartan involution.
Then (5.2) and the flatness condition $F(B) = 0$ become *Hitchin's equations*:

$$\begin{aligned} F(A) - [\phi, \tau(\phi)] &= 0 \\ \bar{\partial}_A\phi &= 0. \end{aligned} \tag{5.3}$$

Higgs bundles

Note that $\bar{\partial}_A \phi = 0$ means that ϕ is a holomorphic one-form with values in $E_H(\mathfrak{m}^{\mathbb{C}})$, endowed with the holomorphic structure defined by $\bar{\partial}_A$.

Definition

A *G-Higgs bundle* is a pair (E, ϕ), where E is a holomorphic principal $H^{\mathbb{C}}$-bundle and $\phi \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K)$.

Remark

- Consider the case of complex G with maximal compact $H \subseteq G$ (so $G = H^{\mathbb{C}}$). Then a G-Higgs bundle is (E, ϕ) where $\phi \in H^0(X, E(\mathfrak{g}) \otimes K) = H^0(X, \mathrm{Ad}(E) \otimes K)$.

Higgs bundles – 2

Whenever G is a (real reductive) subgroup of $\mathrm{SL}(n, \mathbb{C})$, there is a Higgs vector bundle associated to the data of the principal Higgs bundle.

Example

1. Let $G = \mathrm{SL}(n, \mathbb{C})$. Then a G-Higgs bundle is a pair (E, Φ), where $E \to X$ is a holomorphic rank n vector bundle with $\det(E) = \mathcal{O}$ and $\Phi \in H^0(X, \mathrm{End}_0(E) \otimes K)$ (traceless endomorphisms).
2. Let $G = \mathrm{SL}(2, \mathbb{R})$. Then a G-Higgs bundle is a pair (L, ϕ) with L a holomorphic line bundle and $\phi \in H^0((L^2 \oplus L^{-2}) \otimes K)$. The associated Higgs vector bundle is

$$\left(L \oplus L^{-1}, \phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}\right)$$

with $\beta \in H^0(X, L^2 K)$ and $\gamma \in H^0(X, L^{-2} K)$.

Higgs bundles – 3

Remark

- One can consider Higgs bundles (E, Φ), where E is a holomorphic bundle (of any determinant) and $\Phi \in H^0(X, \mathrm{End}(E) \otimes K)$. Instead of flat connections, one must then consider connections with constant central curvature and introduce a corresponding term in the first of Hitchin's equations (5.3).

Stability

Basic question: When does a G-Higgs bundle (E, ϕ) come from a flat G-bundle E_G?
In other words, when can we find a metric $\sigma' \colon X \to E/H$ such that (A, ϕ) satisfy Hitchin's equations (5.3)? (With $A = A^{0,1} + A^{1,0}$ and $A^{1,0}$ defined via the metric.)
Answer: "stability"

Recall the *degree* of a holomorphic vector bundle $V \to X$:
$\deg(V) = \deg(\det(V)) \in \mathbb{Z}$.

Alternatively, $\deg(V)$ can be defined via Chern–Weil theory as
$\deg(V) = \frac{i}{2\pi} \int_X \operatorname{tr} F(A)$ for any unitary connection A on V.

The *slope* of a vector bundle V is, by definition, $\mu(V) = \frac{\deg(V)}{\mathrm{rk}(V)}$.

Stability – 2

Let (V, Φ) be a Higgs bundle. Think of $\phi \in H^0(X, \mathrm{End}(V) \otimes K)$ as a K-twisted endomorphism $\phi \colon V \to V \otimes K$.

Definition

A subbundle $F \subseteq V$ is *ϕ-invariant* if $\phi(F) \subseteq F \otimes K$.

- A Higgs bundle (V, ϕ) is *stable* if $\mu(F) < \mu(V)$ for any proper ϕ-invariant subbundle $F \subseteq V$.
- A Higgs bundle (V, ϕ) is *semistable* if $\mu(F) \leqslant \mu(V)$ for any ϕ-invariant subbundle $F \subseteq V$.
- A Higgs bundle (V, ϕ) is *polystable* if $(V, \phi) = (F_1 \oplus \cdots \oplus F_r, \phi_1 \oplus \cdots \oplus \phi_r)$, where each (F_j, ϕ_j) is a stable Higgs bundle of slope $\mu(F_i) = \mu(V)$.

Remark

The correct definition of stability in the principal bundle setting is subtle (and will be treated in the course by Mundet i Riera). Here we shall stick to the simpler vector bundle case.

The Hitchin–Kobayashi correspondence

Theorem (Hitchin [10], Simpson [16], Bradlow–García-Prada–Mundet [3])

A G-Higgs bundle (E, ϕ) admits a solution to Hitchin's equations (5.3) if and only if it is polystable. $\square$

Moduli spaces

Fix the topological invariant $d \in \pi_1(G)$. The following moduli spaces intervene in the theory:

- $\mathcal{R}_d(\pi_1(X), G)$ – the character variety.
- $\mathcal{M}^d_{dR}(X, G)$ – the moduli space of flat G-connections on X, or *de Rham moduli space*.
- $\mathcal{M}^d_{\text{gauge}}(X, G)$ – the gauge theory moduli space of solutions to Hitchin's equations:

$$\mathcal{M}^d_{\text{gauge}}(X, G) = \left\{ (A, \phi) \ : \ \begin{array}{r} F(A) - [\phi, \tau(\phi)] = 0 \\ \bar{\partial}_A \phi = 0. \end{array} \right\} / \mathcal{G}_H,$$

  where $\mathcal{G}_H = A^0(E_H \times_{\text{Ad}} H)$ is the gauge group of H-gauge transformations.
- $\mathcal{M}^d_{\text{Dol}}(X, G)$ – the moduli space of polystable (or better, semistable) G-Higgs bundles constructed via GIT:

$$\mathcal{M}^d_{\text{Dol}} = \{(E, \phi) \ : \ \text{polystable } G\text{-Higgs bundles}\} / \cong .$$

Identifications between the moduli spaces

With more care, the above correspondences give identifications

$$\mathcal{M}_B \cong \mathcal{M}_{dR} \cong \mathcal{M}_{\text{gauge}} \cong \mathcal{M}_{\text{Dol}}.$$

When G is complex, the moduli spaces $\mathcal{M}_B$ and $\mathcal{M}_{dR}$ are naturally complex varieties. Let J be the complex structure on $\mathcal{M}_{dR}$.

The moduli space $\mathcal{M}_{\text{Dol}}$ is also a complex variety, since X is an algebraic curve; let I be its complex structure.

Fact: The complex structures I and J are inequivalent. One way to see this: $(\mathcal{M}_{dR}, J)$ is affine, while $(\mathcal{M}_{\text{Dol}}, I)$ contains the projective moduli space of principal G-bundles. This gives rise to the hyper-Kähler structure on the moduli space.

Deformation theory of flat bundles

Let E be a smooth G-bundle. The deformation theory of flat connections on E goes as follows:
Linearize the flatness condition $F(B) = 0$:

$$\frac{d}{dt}F(B + bt)_{|t=0} = d_B(b)$$

for $b \in A^1(X, \mathrm{Ad}(E))$.
Linearize the action of the gauge group $B \mapsto g \cdot B = gBg^-1 + dg\, g^{-1}$.
For $g(t) = \exp(\psi t)$ with $\psi \in A^0(X, \mathrm{Ad}(E))$,

$$\frac{d}{dt}(g(t) \cdot B)_{t=0} = d_B(\psi).$$

Thus the infinitesimal deformation space is H^1 of the complex

$$0 \to A^0(X, \mathrm{Ad}(E)) \xrightarrow{d_B} A^1(X, \mathrm{Ad}(E)) \xrightarrow{d_B} A^2(X, \mathrm{Ad}(E)) \to 0.$$

Note that $F(B) = d_B \circ d_B = 0$ means that this is in fact a complex.

Deformation theory of Hitchin's equations

In a similar way, one calculates the deformation theory of Hitchin's equations for a pair (A, ϕ):

$$\begin{aligned} F(A) + [\phi, \tau(\phi)] &= 0, \\ \bar{\partial}_A \phi &= 0, \end{aligned}$$

where A is a unitary connection on a fixed smooth principal H-bundle $E_H \to X$ and $\phi \in A^{1,0}(X, E_H(\mathfrak{m}^{\mathbb{C}}))$.
The linearized equations are:

$$\begin{aligned} d_A(\dot{A}) - [\dot{\phi}, \tau(\phi)] - [\phi, \tau(\dot{\phi})] &= 0, \\ \bar{\partial}_A \dot{\phi} + [\dot{A}^{0,1}, \phi] &= 0, \end{aligned}$$

for $\dot{A} \in A^1(X, E_h(\mathfrak{h}))$ and $\dot{\phi} \in A^{1,0}(X, E_H(\mathfrak{m}^{\mathbb{C}}))$.

Deformation theory of Hitchin's equations – 2

The infinitesimal action of $\psi \in A^0(X, E_H(\mathfrak{h})) = \mathrm{Lie}(\mathcal{G}_H)$ is

$$(A, \phi) \mapsto (d_A\psi, [\phi, \psi]).$$

Thus the deformation theory of Hitchin's equations is governed by the (elliptic) complex

$$C^\bullet_{\mathrm{g}}(A, \phi)\colon A^0(X, E_H(\mathfrak{h})) \xrightarrow{d_0} A^1(X, E_H(\mathfrak{h})) \oplus A^{1,0}(X, E_H(\mathfrak{m}^{\mathbb{C}}))$$
$$\xrightarrow{d_1} A^2(X, E_H(\mathfrak{h})) \oplus A^{1,1}(X, E_H(\mathfrak{m}^{\mathbb{C}})),$$

where the maps are

$$d_0(\psi) = (d_A\psi, [\phi, \psi])$$

and

$$d_1(\psi) = (d_A(\dot{A}) - [\dot{\phi}, \tau(\phi)] - [\phi, \tau(\dot{\phi})], \bar{\partial}_A\dot{\phi} + [\dot{A}^{0,1}, \phi]).$$

The fact that (A, ϕ) is a solution to the equations guarantees that $d_1 \circ d_0 = 0$.

Deformation theory of Hitchin's equations – 3

Denote by $H^i(C^\bullet_{\mathrm{g}}(A, \phi))$ the cohomology groups of the gauge theory deformation complex $C^\bullet_{\mathrm{g}}(A, \phi)$.

Theorem

Assume that $H^0(C^\bullet_g(A, \phi)) = H^2(C^\bullet_g(A, \phi)) = 0$ and that (A, ϕ) has no non-trivial automorphisms. Then $\mathcal{M}_{gauge}$ is smooth at $[A, \phi]$ and the tangent space is

$$T_{[A,\phi]}\mathcal{M}_{gauge} \cong H^1(C^\bullet_g(A, \phi)).$$

$\square$

Metric on $\mathcal{M}_{\text{gauge}}$

The fact that the structure group of E_H is the maximal compact $H \subseteq G$ means that the vector bundles $E_H(\mathfrak{h}^{\mathbb{C}})$ and $E_H(\mathfrak{m}^{\mathbb{C}})$ have induced hermitean metrics. Thus a hermitean metric can be defined on $\mathcal{M}_{\text{gauge}}$ by

$$\langle(\dot{A}_1, \dot{\phi}_1), (\dot{A}_2, \dot{\phi}_2)\rangle = i \int_X (\langle \dot{A}_1^{0,1}, \dot{A}_2^{0,1}\rangle + \langle \dot{\phi}_1, \dot{\phi}_2\rangle),$$

where we are combining with conjugation on the form component in the second factors. This turns out to be Kähler with respect to I.

When G is a complex group there is another complex structure on $H^1(C^\bullet_{\mathfrak{g}}(A,\phi))$ coming from the complex structure on G:

$$J \colon H^1(C^\bullet_{\mathfrak{g}}(A,\phi)) \to H^1(C^\bullet_{\mathfrak{g}}(A,\phi)), \quad J^2 = -1.$$

Note that $IJ = -JI$. Then $K = IJ$ is a complex structure and I, J and K satisfy the identities of the quaternions.

The moduli space as a hyper-Kähler quotient – 1

For simplicity, consider the case of Higgs vector bundles (V, ϕ).

Thus we are considering solutions (A, ϕ) to Hitchin's equations, where A is a unitary connection on V and $\phi \in A^{1,0}(X, \mathrm{End}_0(V))$.

Using the correspondence between unitary connections and $\bar{\partial}$-operators, the space $\mathcal{C}$ of pairs (A, ϕ) is an affine space modeled on the tangent space at (A, ϕ):

$$T_{(A,\phi)}\mathcal{C} \cong A^{0,1}(X, \mathrm{End}_0(V)) \oplus A^{1,0}(X, \mathrm{End}_0(V)).$$

There is a Kähler metric on $\mathcal{C}$ (as above), given by:

$$\langle(\alpha_1, \dot{\phi}_1), (\alpha_2, \dot{\phi}_2)\rangle = i \int_X \mathrm{tr}(\alpha_1^*\alpha_2 + \dot{\phi}_1\dot{\phi}_2^*).$$

There is also a complex symplectic form on $\mathcal{C}$, defined by:

$$\Omega(\alpha_1, \dot{\phi}_1), (\alpha_2, \dot{\phi}_2) = \int_X \mathrm{tr}(\dot{\phi}_1\alpha_2 - \dot{\phi}_2\alpha_1).$$

The moduli space as a hyper-Kähler quotient – 2

Let ω_1 be the Kähler form of the hermitean metric on $\mathcal{C}$ just defined and write $\Omega = \omega_2 + i\omega_3$.

One can then show:

- ω_1, ω_2 and ω_3 are the Kähler forms of a *hyper-Kähler metric* on $\mathcal{C}$, with respect to complex structures I, J and K respectively.

The action of the gauge group $\mathcal{G}_H$ is hamiltonian for all three Kähler forms and the corresponding moment maps are:

$$\begin{aligned} \mu_1(A,\phi) &= F(A) + [\phi, \phi^*] \\ (\mu_2 + i\mu_3)(A,\phi) &= \bar{\partial}_A \phi. \end{aligned}$$

Thus Hitchin's equations are equivalent to the simultaneous vanishing of the three moment maps.

Theorem (Hitchin [10])

The almost complex structures I, J and K are integrable and form a hyper-Kähler structure on $\mathcal{M}_{gauge}$. □

Deformation theory of G-Higgs bundles

Next we consider the deformation theory of a G-Higgs bundle (E,ϕ) as an algebraic (or holomorphic) object.

Consider the complex of sheaves (we identify a bundle with its sheaf of holomorphic sections):

$$\begin{aligned} C^\bullet(E,\phi)\colon E(\mathfrak{h}^{\mathbb{C}}) &\to E(\mathfrak{m}^{\mathbb{C}}) \otimes K \\ \psi &\mapsto [\phi,\psi] = \mathrm{ad}(\phi)(\psi). \end{aligned}$$

Hypercohomology of a complex of sheaves $\mathcal{F}^\bullet\colon \cdots \to \mathcal{F}^i \to \mathcal{F}^{i+1} \to \dots$ is calculated as follows:

1. Create a double complex by taking vertically over each $\mathcal{F}^i$ your favourite resolution for calculating sheaf cohomology.
2. The ith hypercohomology group $\mathbb{H}^i(\mathcal{F}^\bullet)$ is the ith cohomology group of the resulting total complex.

Deformation theory of G-Higgs bundles – 2

Theorem

The infinitesimal deformation space of (E, ϕ) is canonically isomorphic to the first hypercohomology group $\mathbb{H}^1(C^\bullet(E, \phi))$.

This can be proved in several ways:

- Use Dolbeault resolution and differential geometry;
- Use Čech cohomology to represent an infinitesimal deformation of (E, ϕ) as an object over $\mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2))$.

Hypercohomology enjoys nice properties, such as a long exact sequence associated to a short exact sequence of complexes. This gives a long exact sequence:

$$\begin{aligned} 0 &\to \mathbb{H}^0(C^\bullet(E,\phi)) \to H^0(E(\mathfrak{h}^{\mathbb{C}})) \xrightarrow{\mathrm{ad}(\phi)} H^0(E(\mathfrak{m}^{\mathbb{C}}) \otimes K) \\ &\to \mathbb{H}^1(C^\bullet(E,\phi)) \to H^1(E(\mathfrak{h}^{\mathbb{C}})) \xrightarrow{\mathrm{ad}(\phi)} H^1(E(\mathfrak{m}^{\mathbb{C}}) \otimes K) \\ &\to \mathbb{H}^2(C^\bullet(E,\phi)) \to 0. \end{aligned} \tag{9.1}$$

Comparison of deformation theories

For a proper understanding of many aspects (hyper-Kähler structure, Morse theory) of the geometry of the moduli space of Higgs bundles, one needs to consider the moduli space as the gauge theory moduli space $\mathcal{M}_{\mathrm{gauge}}$. On the other hand, the formulation of the deformation theory in terms of hypercohomology is very convenient. Fortunately:

Proposition

At a smooth point of the moduli space, there is a natural isomorphism of infinitesimal deformation spaces

$$H^1(C^\bullet_g(A, \phi)) \cong \mathbb{H}^1(C^\bullet(E, \phi)),$$

where the holomorphic structure on the Higgs bundle (E, ϕ) is given by $\bar{\partial}_A$.

Fix the notation $\mathcal{M}(X, G)$ when we want to blur the distinction between the Dolbeault and the gauge theory moduli spaces.

Serre duality

This is an important tool for the study of the deformation theory of Higgs bundles.

Observation: There is a non-degenerate quadratic form on $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$, invariant under the adjoint action of $H^{\mathbb{C}}$ and such that the direct sum decomposition is orthogonal. It follows that

$$E(\mathfrak{h}^{\mathbb{C}}) \cong E(\mathfrak{h}^{\mathbb{C}})^*, \quad E(\mathfrak{m}^{\mathbb{C}}) \cong E(\mathfrak{m}^{\mathbb{C}})^*.$$

The *dual complex* of $C^\bullet(E,\phi)\colon E(\mathfrak{h}^{\mathbb{C}}) \to E(\mathfrak{m}^{\mathbb{C}}) \otimes K$ is therefore

$$C^\bullet(E,\phi)^*\colon E(\mathfrak{m}^{\mathbb{C}}) \otimes K^{-1} \to E(\mathfrak{h}^{\mathbb{C}}).$$

Serre duality for hypercohomology says that

$$\mathbb{H}^i(C^\bullet(E,\phi)) \cong \mathbb{H}^{2-i}(C^\bullet(E,\phi)^* \otimes K)^*.$$

The complex symplectic form

Consider the case of G complex. Then $\mathfrak{m}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}}$ and Serre duality tells us that

$$\mathbb{H}^1(C^\bullet(E,\phi)) \cong \mathbb{H}^1(C^\bullet(E,\phi))^*.$$

Proposition

This duality defines a complex symplectic form Ω on the moduli space of G-Higgs bundles for complex G.

Remark

The complex symplectic form Ω is of course the same as the one previously defined from the gauge theory point of view.

The moduli space of stable G-bundles

Assume that G is complex.

Let $\mathbf{M}(X; G)$ denote the moduli space of semistable principal G-bundles.

Let E be a principal holomorphic G-bundle. The infinitesimal deformation space of E is

$$H^1(X, E(\mathfrak{g}^{\mathbb{C}}))$$

Since $E(\mathfrak{g}^{\mathbb{C}}) \cong E(\mathfrak{g}^{\mathbb{C}})^*$, Serre duality says that

$$H^1(X, E(\mathfrak{g}^{\mathbb{C}}))^* = H^0(X, E(\mathfrak{g}^{\mathbb{C}}) \otimes K).$$

In other words, if E is stable as a G-bundle then a G-Higgs bundle (E, ϕ) represents a cotangent vector to the moduli space $\mathbf{M}(X, G)$. It follows that there is an inclusion

$$T^*\mathbf{M}(X, G) \hookrightarrow \mathcal{M}_{\mathrm{Dol}}(X, G).$$

Proposition

The complex symplectic form Ω on $\mathcal{M}_{Dol}$ restricts to the standard complex symplectic form on the cotangent bundle $T^*\mathbf{M}(X, G)$. $\square$

Invariant polynomials and the Hitchin map

Let $p_1, \ldots, p_k$ be a basis of the invariant polynomials on $\mathfrak{g}$; write $d_i = \deg(p_i)$. The *Hitchin map* on $\mathcal{M}(X, G)$ is defined by:

$$H \colon \mathcal{M}(X, G) \to \bigoplus_{i=1}^{k} H^0(X, K^{d_i}),$$
$$(E, \phi) \mapsto \big(p_i(\phi)\big)_{i=1}^{k}.$$

Example

If $G = \mathrm{SL}(2, \mathbb{C})$, a G-Higgs bundle is (V, ϕ) with $\mathrm{rk}(V) = 2$, $\det(V) = \mathcal{O}$ and $\phi \in H^0(X, \mathrm{End}_0(V))$. The Hitchin map is simply:

$$H(V, \phi) = \det(\phi) \in H^0(X, K^2).$$

Important observation:

$$\dim\Big(\bigoplus_{i=1}^{k} H^0(X, K^{d_i})\Big) = \sum (2d_i - 1)(g - 1) = (g - 1)\dim G.$$

The Hitchin map for $\mathrm{SL}(2,\mathbb{R})$

Recall: an $\mathrm{SL}(2,\mathbb{R})$-Higgs bundle is given by (L,β,γ), with $\beta \in H^0(X, L^2K)$ and $\gamma \in H^0(X, L^{-2}K)$.

The Hitchin map is

$$H(L,\beta,\gamma) = \beta\gamma \in H^0(X, K^2).$$

Consider the case of $\deg(L) = g-1$, then $\gamma = 1$, $L^2 = K$ and $H(L,\beta,1) = \beta$. Thus, fixing the square root L of K,

$$\begin{aligned} H^0(X,K^2) &\to \mathcal{M}(X, \mathrm{SL}(2,\mathbb{R})), \\ \beta &\mapsto (L,\beta,1) \end{aligned}$$

gives a section of H. This identifies

$$\mathcal{M}_{g-1,L}(X, \mathrm{SL}(2,\mathbb{R})) \cong H^0(X,K^2)$$

and shows that Teichmüller space $\mathcal{M}_{g-1,L}(X, \mathrm{SL}(2,\mathbb{R}))$ is homeomorphic to a euclidean space of dimension $6g-6$.

The Hitchin system

Assume that G is complex. Since $\dim \mathcal{M}(X,G) = (2g-2)\dim G$, the Hitchin space $\mathcal{H} = \bigoplus_{i=1}^{k} H^0(X, K^{d_i})$ has dimension $n := \dim \mathcal{H} = \frac{1}{2}\dim \mathcal{M}(X,G)$.

Proposition

The n functions defined by the Hitchin map Poisson commute.

This can be proved by considering $\mathcal{M}(X,G)$ as an (infinite dimensional) symplectic quotient.

Finally, it can be shown that the generic fibre of H is an abelian variety of dimension $\frac{1}{2}\dim \mathcal{M}(X,G)$, on which the Hamiltonian vector fields of the n Poisson commuting functions are linear. In other words:

The Hitchin map $H\colon \mathcal{M}(X,G) \to \mathcal{H}$
is an algebraically completely integrable system.

As an illustration, we shall do the case $G = \mathrm{SL}(2,\mathbb{C})$ in a bit more detail.

$\mathrm{SL}(2,\mathbb{C})$-Higgs bundles and spectral curves

(Following Beauville–Narasimhan–Ramanan [1])

Consider the Hitchin map

$$\begin{aligned} H\colon \mathcal{M}(X,\mathrm{SL}(2,\mathbb{C}) &\to H^0(X,K^2),\\ (V,\phi) &\mapsto \det(\phi). \end{aligned}$$

Idea: Think of the characteristic polynomial $\chi_\phi(y) = \det(\phi) + y^2$ as a section on the total space of $\pi\colon K \to X$ and express (V,ϕ) in terms of abelian data on the *spectral curve:* $\{\chi(\phi) = 0\}$.

- Let $S = \mathcal{P}(\mathcal{O}\oplus K) \xrightarrow{\pi} X$ be the fibrewise compactification of K.
- Let $\mathcal{O}(1) \to S$ be the hyperplane bundle along the fibres.
- Let $x, y \in H^0(S,\mathcal{O}(1))$ be the sections given by projecting on K and $\mathcal{O}$, respectively; i.e., $[x : y]$ are homogeneous coordinates on the fibres of $S \to X$.

The spectral curve – 2

The characteristic polynomial of ϕ can now be viewed as the section

$$\chi(\phi) = \pi^*\det(\phi)\cdot y^2 + x^2 \in H^0(S,\pi^*K^2\otimes\mathcal{O}(2)).$$

The *spectral curve* $X_{\chi(\phi)} \subseteq S$ is defined as the zero locus of $\chi(\phi)$.

When $X_{\chi(\phi)}$ is integral, we have a ramified double cover:

$$\pi\colon X_{\chi(\phi)} \to X,$$

with ramification divisor $D = \mathrm{div}(\det(\phi))$. If D has no multiple points, $X_{\chi(\phi)}$ is smooth.

Note: The restriction of y to $X_{\chi(\phi)}$ is nowhere vanishing. Thus $\mathcal{O}(1)_{|X_{\chi(\phi)}}$ is trivial and we can view $x \in H^0(X_{\chi(\phi)},\pi^*K_X)$; the two values of x are the square roots of $\det(\phi)$.

The spectral curve – 3

Assume $X_{\chi(\phi)}$ is smooth. There is a line bundle $M \to X_{\chi(\phi)}$ such that

$$V = \pi_* M \quad \text{and} \quad \phi = \pi_* x,$$

where we interpret $x \in H^0(X, \mathrm{End}(M) \otimes \pi^* K_X)$. One way to define M is note that $M(-D)$ is the kernel

$$0 \to M(-D) \to \pi^* V \xrightarrow{\pi^*\phi - x} \pi^*(V \otimes K_X).$$

Let $\sigma \colon X_{\chi(\phi)} \to X_{\chi(\phi)}$ be the involution interchanging the sheets of the double cover. Then

$$M \otimes \sigma^* M \cong \pi^* \det(V) = \mathcal{O}$$

because $\det(V) = \mathcal{O}$.

Integrable systems

The Prym variety

The *Prym variety* of a double cover $X_s \to X$ is

$$\{L \ : \ L \otimes \sigma^* L \cong \mathcal{O}\} \subseteq \mathrm{Jac}(X).$$

Since $M \otimes \sigma^* M \cong \mathcal{O}$, it follows that M is in the Prym of the spectral cover.
Conversely, given $s \in H^0(X, K^2)$, one can define the spectral curve $X_s \to X$ as above and, for M with $M \otimes \sigma^* M \cong \mathcal{O}$,

$$(V, \phi) := \pi_*(M, x)$$

defines an $\mathrm{SL}(2, \mathbb{C})$-Higgs bundle, which turns out to be stable,

Algebraically completely integrable systems

Thus we have

Theorem

The fibre of the Hitchin map $\mathcal{M}(X, \mathrm{SL}(2,\mathbb{C}) \to H^0(X, K^2)$ over an s with simple zeros is isomorphic to the Prym variety of the spectral curve $X_s \to X$. □

It follows that the hamiltonian vector fields associated to the Poisson commuting coordinate functions of H are linear.

Theorem

The Hitchin map $H\colon \mathcal{M}(X, \mathrm{SL}(2,\mathbb{C})) \to H^0(X, K^2)$ is an algebraically completely integrable system. □

Generalizations:

- Replace $\mathrm{SL}(2,\mathbb{C})$ with any semisimple group.
- Replace K_X with $K_X(D)$ – this makes $\mathcal{M}$ into a Poisson manifold.

For (much) more on integrable systems see, e.g., Hitchin [11], Bottacin [2], Markman [15], Donagi–Markman [5].

The circle action on the moduli space

An essential aspect of the moduli space of Higgs bundles is that there is an action of $\mathbb{C}^*$:

$$\begin{aligned} \mathbb{C}^* \times \mathcal{M}_{\mathrm{Dol}}(X, G) &\to \mathcal{M}_{\mathrm{Dol}}(X, G), \\ (\lambda, (E, \phi)) &\mapsto (E, \lambda\phi). \end{aligned}$$

From the gauge theory point of view, to preserve solutions to Hitchin's equations, one restricts to the compact $S^1 \subseteq \mathbb{C}^*$.

Proposition

The S^1-action $(A, \phi) \mapsto (A, e^{i\theta}\phi)$ on $\mathcal{M}_{gauge}(X, G)$ is hamiltonian with respect to the Kähler form ω_1 associated to the complex structure I.

Geometry of hamiltonian circle actions

Let (M, ω) be a Kähler manifold with a hamiltonian circle action. A *moment map* for the action is $\tilde{f}: M \to \mathbb{R}$ such that

$$\nabla(\tilde{f}) = I \cdot Z,$$

where $Z \in \mathcal{X}(M)$ is the vector field generating the circle action and I is the complex structure on M.

Theorem (Frankel [7])

Let $\tilde{f}: M \to \mathbb{R}$ be a proper moment map for a Hamiltonian circle action on a Kähler manifold M. Then $\tilde{f}$ is a perfect Bott–Morse function. □

A *Bott–Morse function* is an $\tilde{f}$ whose critical points form submanifolds $N_i \subseteq M$ such that the Hessian of $\tilde{f}$ defines a non-degenerate quadratic form on the normal bundle to each N_i in M.

The *index* λ_i of N_i is the the dimension of the negative weightspace of the Hessian.

Bott–Morse theory

A Bott–Morse function $\tilde{f}$ is *perfect* when the Betti numbers of M are given by

$$P_t(M) := \sum t^j \dim(H^j(M)) = \sum_i t^{\lambda_i} P_t(N_i).$$

Proposition

(1) The critical points of $\tilde{f}$ are the fixed points of the circle action.

(2) The eigenvalue l subspace for the Hessian of $\tilde{f}$ is the same as the weight $-l$ subspace for the infinitesimal circle action on the tangent space. Thus the Morse index of $\tilde{f}$ at a critical point equals the dimension of the positive weight space of the circle action on the tangent space.

Bott–Morse theory on the moduli space of Higgs bundles

For simplicity, we consider the case of Higgs vector bundles (V, ϕ) (Recall: $(\mathrm{rk}(V), \deg(V)) = 1$ implies that $\mathcal{M}$ is smooth.)

Proposition

A Higgs bundle (V, ϕ) is fixed under the circle action if and only if it is a complex variation of Hodge structure, i.e., $V = V_1 \oplus \cdots \oplus V_r$, with $\phi_i = \phi_{|V_i} \colon V_i \to V_{i+1} \otimes K$.

This is proved by letting the V_i be the eigenbundles of the isomorphism $(V, \phi) \xrightarrow{\cong} (V, \lambda\phi)$ for λ which is not a root of unity.

Remark

1. Let (V, ϕ) be a complex variation of Hodge structure. Then ϕ is nilpotent, so $H(V, \phi) = 0$, where $H \colon \mathcal{M}(X, G) \to \mathcal{H}$ is the Hitchin map. It follows that the fixed locus of the circle action is contained in the *nilpotent cone* $H^{-1}(0)$.
2. The fixed loci of S^1 and $\mathbb{C}^*$ coincide on $\mathcal{M}_{\mathrm{Dol}}$.

Weights of the circle action

Recall the deformation complex

$$C^\bullet(V, \phi) \colon \; \mathrm{End}(V) \xrightarrow{\mathrm{ad}(\phi)} \mathrm{End}(V) \otimes K.$$

The decomposition $V = \bigoplus V_i$ induces a decomposition $C^\bullet(V, \phi) = \bigoplus_l C^\bullet_l(V, \phi)$, where, letting $\mathrm{End}(V)_l = \bigoplus_{l=j-i} \mathrm{Hom}(V_i, V_j)$,

$$C^\bullet_l(V, \phi) \colon \; \mathrm{End}(V)_l \xrightarrow{\mathrm{ad}(\phi)} \mathrm{End}(V)_{l+1} \otimes K.$$

This gives a decomposition $\mathbb{H}^1(C^\bullet(V, \phi)) \cong \bigoplus_l \mathbb{H}^1(C^\bullet_l(V, \phi))$ of the infinitesimal deformation space.

Proposition

The subspace $\mathbb{H}^1(C^\bullet_l(V, \phi))$ is the weight $-l$ subspace of the infinitesimal S^1-action on $\mathcal{M}$.

The Morse function

The moment map for the S^1-action on $\mathcal{M} = \mathcal{M}_{\text{gauge}}$ is given by

$$\tilde{f}[(A,\phi)] = -\tfrac{1}{2}\|\phi\|^2 = -i\int_X \operatorname{tr}(\phi\phi^*).$$

We find it more natural to work with the positive function

$$f([A,\phi]) := \tfrac{1}{2}\|\phi\|^2.$$

Keeping track of the signs we have the following.

Proposition

The eigenvalue l subspace of the Hessian of f at a complex variation of Hodge structure (V,ϕ) is

$$T_{(V,\phi)}\mathcal{M}_l \cong \mathbb{H}^1(C^\bullet(V,\phi)_{-l}).$$

□

Morse indices – 1

The Riemann–Roch Theorem allows to calculate the Euler characteristic

$$\chi(C^\bullet(V,\phi)_l) = \sum(-1)^i \dim \mathbb{H}^i(C^\bullet(V,\phi)_l).$$

Recall the long exact sequence hypercohomology sequence (9.1):

$$\begin{aligned} 0 &\to \mathbb{H}^0(C^\bullet(V,\phi)) \to H^0(\operatorname{End}(V)) \xrightarrow{\operatorname{ad}(\phi)} H^0(\operatorname{End}(V)\otimes K) \\ &\to \mathbb{H}^1(C^\bullet(V,\phi)) \to H^1(\operatorname{End}(V)) \xrightarrow{\operatorname{ad}(\phi)} H^1(\operatorname{End}(V))\otimes K) \\ &\to \mathbb{H}^2(C^\bullet(V,\phi)) \to 0. \end{aligned}$$

From this we immediately get that $\operatorname{End}(V,\phi) = \mathbb{H}^0(C^\bullet(V,\phi))$.

Vanishing of hypercohomology

Analogously to the case of vector bundles:

$$(V,\phi) \text{ stable} \implies \mathbb{H}^0(C^\bullet(V,\phi)) = \mathbb{C},$$

i.e., "stable implies simple". Note also:
$\mathbb{H}^0(C^\bullet(V,\phi)_0) = \mathbb{H}^0(C^\bullet(V,\phi)) = \mathbb{C}$
Now, as noted before, $C^\bullet(V,\phi)^* \cong C^\bullet(V,\phi) \otimes K^{-1}$. Thus, Serre duality of complexes implies that

$$\mathbb{H}^2(X, C^\bullet(V,\phi)) \cong \mathbb{H}^0(X, C^\bullet(V,\phi))^* = \mathbb{C}.$$

Applying duality to the complexes $C^\bullet(V,\phi)_l$, we see

$$\begin{aligned} &C^\bullet(V,\phi)_l^* = C^\bullet(V,\phi)_{-l-1} \otimes K^{-1} \\ \implies &\mathbb{H}^i(C^\bullet(V,\phi)_l) \cong \mathbb{H}^{2-i}(C^\bullet(V,\phi)_{-l-1})^* \\ \implies &T_{(V,\phi)}\mathcal{M}_l \cong (T_{(V,\phi)}\mathcal{M}_{1-l})^*. \end{aligned}$$

Morse indices – 2

With all this at our disposal, we obtain:

Proposition

The Morse index at a fixed point $(V,\phi) = \bigoplus (V_i,\phi_i)$ *is*

$$\tfrac{1}{2}\lambda = -\sum_{l>0} \chi(C^\bullet(V,\phi)_l),$$

which can be calculated explicitly in terms of the ranks and degrees of the V_i *using Riemann–Roch.* □

Remark

This essentially works for any group G (real or complex). Care must be taken with stability and vanishing of hypercohomology.

Morse indices – 3

Important observations:

- The dimension of the critical submanifold through $\bigoplus(V_i, \phi_i)$ is $\dim \mathbb{H}^1(C^\bullet(V,\phi)_0)$.
- (For complex G): By duality,

$$\mathbb{H}^1(\bigoplus_{l\geqslant 0} C^\bullet(V,\phi)_l) \cong \mathbb{H}^1(\bigoplus_{l<0} C^\bullet(V,\phi)_l)^*.$$

  In particular, the Morse index of the critical submanifold $\mathcal{N} \subseteq \mathcal{M}$ is

$$\lambda = \dim \mathcal{M} - 2\dim \mathcal{N}.$$

Remark

The main difficulty in determining the Betti numbers of $\mathcal{M}$ lies in determining the Betti numbers of the moduli spaces of complex variation of Hodge structure (the critical submanifolds). This has only been carried out for rank 2 and 3. On the other hand, using number theoretic methods, the *mixed Hodge polynomial* of $\mathcal{M}$ has been determined by Hausel and Rodriguez-Villegas [9].

The downwards Morse flow

Restrict to the case of Higgs vector bundles (V,ϕ) with $\det(V)$ fixed and $(\deg(V), \mathrm{rk}(V)) = 1$.

Let $\{\mathcal{N}_\lambda\}_{\lambda\in A}$ denote the critical submanifolds $\mathcal{N}_\lambda \subseteq \mathcal{M}$ of the Morse function. In particular, $\mathcal{N}_0 \subseteq \mathcal{M}$ denotes the moduli space of stable bundles.

- The downwards Morse flow D_λ of $\mathcal{N}_\lambda$ is the set of points which flow to $\mathcal{N}_\lambda$ at time $-\infty$ under the gradient flow of the Morse function.
- The *downwards Morse flow* is $\bigcup D_\lambda$

Our calculation of Morse indices shows that

$$\dim(\bigcup D_\lambda) = \frac{1}{2}\dim \mathcal{M}.$$

Laumon's Theorem, following Hausel

We shall relate this to the $\mathbb{C}^*$-action $(t, x) \mapsto t \cdot x$ for $t \in \mathbb{C}^*$ and $x \in \mathcal{M}$.
Fact: $D_\lambda = \{x \in \mathcal{M} \ : \ \lim_{t\to\infty} t \cdot x \in \mathcal{N}_\lambda\}$
From this it follows that:

Proposition (Hausel [8])

The downwards Morse flow coincides with the nilpotent cone $H^{-1}(0)$. □

Since the nilpotent cone is coisotropic, it follows from our calculation of the dimension that

Theorem (Laumon [14])

The nilpotent cone is Lagrangian in $\mathcal{M}$. □

Further reading

- Some basic references on the foundations of the theory are: Corlette [4], Donaldson [6], Hitchin [10, 11, 12], Simpson[16, 17, 18, 19].
- For an example of the recent interest of Higgs bundles in physics see, e.g., Kapustin–Witten [13].

[1] A. Beauville, M. S. Narasimhan, and S. Ramanan, *Spectral curves and the generalised theta divisor*, J. Reine Angew. Math. **398** (1989), 169–179.

[2] F. Bottacin, *Symplectic geometry on moduli spaces of stable pairs*, Ann. Sci. Ec. Norm. Super., IV. Ser. **28** (1995), 391–433.

[3] S. B. Bradlow, O. García-Prada, and I. Mundet i Riera, *Relative Hitchin-Kobayashi correspondences for principal pairs*, Q. J. Math. **54** (2003), 171–208.

[4] K. Corlette, *Flat G-bundles with canonical metrics*, J. Differential Geom. **28** (1988), 361–382.

[5] R. Donagi and E. Markman, *Spectral covers, algebraically completely integrable, Hamiltonian systems, and moduli of bundles*, Integrable systems and quantum groups (Montecatini Terme, 1993), Lecture Notes in Math., vol. 1620, Springer, Berlin, 1996, pp. 1–119.

[6] S. K. Donaldson, *Twisted harmonic maps and the self-duality equations*, Proc. London Math. Soc. (3) **55** (1987), 127–131.

References

[7] T. Frankel, *Fixed points and torsion on Kähler manifolds*, Ann. of Math. (2) **70** (1959), 1–8.

[8] T. Hausel, *Compactification of moduli of Higgs bundles*, J. Reine Angew. Math. **503** (1998), 169–192.

[9] T. Hausel and F. Rodriguez-Villegas, *Mixed hodge polynomials of character varieties*, 2006, `arXiv.org:math/0612668`.

[10] N. J. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. (3) **55** (1987), 59–126.

[11] ———, *Stable bundles and integrable systems*, Duke Math. J. **54** (1987), 91–114.

[12] ———, *Lie groups and Teichmüller space*, Topology **31** (1992), 449–473.

[13] A. Kapustin and E. Witten, *Electric-magnetic duality and the geometric Langlands program*, Commun. Number Theory Phys. **1** (2007), 1–236.

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[14] G. Laumon, *Un analogue global du cone nilpotent*, Duke Math. J. **57** (1988), 647–671.

[15] E. Markman, *Spectral curves and integrable systems*, Compositio Math. **93** (1994), 255–290.

[16] C. T. Simpson, *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization*, J. Amer. Math. Soc. **1** (1988), 867–918.

[17] ______, *Higgs bundles and local systems*, Inst. Hautes Études Sci. Publ. Math. **75** (1992), 5–95.

[18] ______, *Moduli of representations of the fundamental group of a smooth projective variety I*, Publ. Math., Inst. Hautes Étud. Sci. **79** (1994), 47–129.

[19] ______, *Moduli of representations of the fundamental group of a smooth projective variety II*, Publ. Math., Inst. Hautes Étud. Sci. **80** (1995), 5–79.