

# Maximal $\mathrm{Sp}(4, \mathbb{R})$ representations and minimal surfaces

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UIUC

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Fixing  $J \in \mathcal{T}(S)$  breaks  $\operatorname{MCG}(S)$ -symmetry

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- $MCG(S) \curvearrowright \mathcal{X}^{\max}$  properly discontinuous

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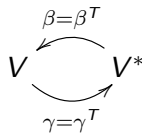
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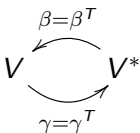
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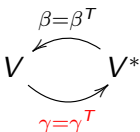
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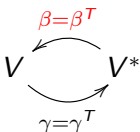
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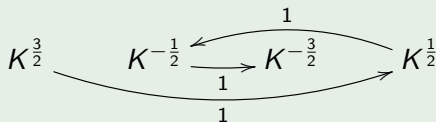
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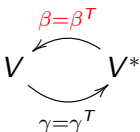
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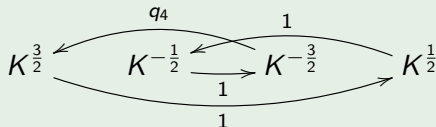
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If  $g - 1 < d$  and  $\lambda \in \mathbb{C}^*$  then  $(N, \mu, \nu, q_2) \cong (N, \lambda\mu, \lambda^{-1}\nu, q_2)$

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$$\begin{array}{ccccc} & & \xleftarrow{\nu} & & \xleftarrow{\mu} \\ N & \xleftarrow{N^{-1}K} & N^{-1} & \xrightarrow{N^{-1}} & NK^{-1} \\ & \xrightarrow{1} & & \xrightarrow{1} & \end{array}$$

If  $g-1 < d$  and  $\lambda \in \mathbb{C}^*$  then  $(N, \mu, \nu, q_2) \cong (N, \lambda\mu, \lambda^{-1}\nu, q_2)$

$$\text{Tr} \left( \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}^2 \right) = 4q_2.$$



$$\mathcal{F}_d^J = \{(N, \mu, \nu) | N \in \text{Pic}^d(S, J), 0 \neq \mu \in H^0(N^{-2}K^3), \nu \in H^0(N^2K)\} / \mathbb{C}^*$$

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If  $g - 1 < d \leq 3g - 3$  then there is a  $MCG(S)$ -invariant diffeomorphism

$$\Psi : \mathcal{F}_d \longrightarrow \mathcal{X}_d^{2g-2}$$

where  $\pi : \mathcal{F}_d \rightarrow \mathcal{T}(S)$  is the fiber bundle with  $\pi^{-1}(J) = \mathcal{F}_d^J$ .

# Harmonic maps and Energy

Fix  $\rho \in \mathcal{X}(\pi_1, G)$ , for each conformal structure  $\Sigma = (S, J)$ , let

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Shoen-Yau and Sacks-Uhlenbeck showed critical points of  $\mathcal{E}_\rho(J)$  are weakly conformal maps, or equivalently branched minimal immersions.

# Properness of Energy and Hitchin components

## Theorem (C.)

If  $\rho \in \mathcal{X}_d^{2g-2}(\pi_1, \mathrm{Sp}(4, \mathbb{R}))$  then there is a unique conformal structure  $J_\rho$  in which  $h_\rho : \widetilde{(S, J)} \rightarrow \mathrm{Sp}(4, \mathbb{R})/\mathrm{U}(2)$  is a minimal immersion.

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## Theorem (Labourie)

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## Corollary

If  $\rho \in \mathcal{X}_d^{2g-2}$  then there exists a conformal structure in which the harmonic metric  $h_\rho$  is a branched minimal immersion.

# Idea of proof (Fixed points)

Fix  $\rho \in \mathcal{X}_d^{2g-2}$  and let  $J$  be a conformal structure in which  $h_\rho$  is minimal.

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 & \swarrow & & \searrow & \\
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## Proposition

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$$\begin{array}{ccc} \mathrm{Sp}(4, \mathbb{C})/\mathbb{C}^* \times \mathbb{C}^* & \longleftarrow & \mathrm{Sp}(4, \mathbb{C})/U(1) \times U(1) \\ \uparrow & \nearrow f_\rho & \downarrow \\ \tilde{\Sigma} & \xrightarrow{h_\rho} & \mathrm{Sp}(4, \mathbb{C})/\mathrm{Sp}(4) \end{array}$$

*Obrigado!*