Maximal $\mathrm{Sp}(4,\mathbb{R})$ representations and minimal surfaces

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UIUC

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Fixing $J \in \mathcal{T}(S)$ breaks $MCG(S)$-symmetry

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The spaces $\mathcal{M}^{2g-2}_{w_1,w_2}$ and $\mathcal{M}^{2g-2}_{d}$ are nonempty and connected for $d \neq 3g-3$. The space $\mathcal{M}^{2g-2}_{3g-3}$ consists of the 2^{2g} Hitchin components, giving $2(2^{2g}-1)+2g-2+2^{2g}$ connected components

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$$Tr\left(\begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}^2 \right) = 4q_2.$$

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If $g - 1 < d \leq 3g - 3$ then there is a $MCG(S)$-invariant diffeomorphism

$$\Psi : \mathcal{F}_d \longrightarrow \mathcal{X}_d^{2g-2}$$

where $\pi : \mathcal{F}_d \to \mathcal{T}(S)$ is the fiber bundle with $\pi^{-1}(J) = \mathcal{F}_d^J$.

Harmonic maps and Energy

Fix $\rho \in \mathcal{X}(\pi_1, \mathrm{G})$, for each conformal structure $\Sigma = (S, J)$, let

$$h_\rho : \widetilde{\Sigma} \longrightarrow (\mathrm{G}/\mathrm{H}, g)$$

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Shoen-Yau and Sacks-Uhlenbeck showed critical points of $\mathcal{E}_\rho(J)$ are weakely conformal maps, or equivalently branched minimal immersions.

Properness of Energy and Hitchin components

Theorem (C.)

If $\rho \in \mathcal{X}_d^{2g-2}(\pi_1, \mathsf{Sp}(4,\mathbb{R}))$ then there is a unique conformal structure J_ρ in which $h_\rho : \widetilde{(S,J)} \to \mathsf{Sp}(4,\mathbb{R})/\mathsf{U}(2)$ is a minimal immersion.

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Corollary

If $\rho \in \mathcal{X}_d^{2g-2}$ then there exists a conformal structure in which the harmonic metric h_ρ is a branched minimal immersion.

Idea of proof (Fixed points)

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$$\widetilde{\Sigma} \xrightarrow[h_\rho]{\hspace{4em}} \mathrm{Sp}(4, \mathbb{C})/\mathrm{Sp}(4)$$

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Obrigado!