# Schottky Bundles

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# Contents

# Notation

- $X$ compact Riemann surface with genus $g$.
- $\pi_1 := \pi_1(X) = \left\langle \alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g : \prod_{i=1}^{g} [\alpha_i, \beta_i] = 1 \right\rangle$.
- $\widetilde{X}$ a universal cover of the compact Riemann surface $X$.
- $G$ connected reductive complex algebraic group.
- $Z$ center of $G$.
- $K$ a maximal compact subgroup of $G$.
- $E$ a (holomorphic) principal $G$-bundle over $X$.
- $E_\rho = \widetilde{X} \times_\rho G$ where $\rho : \pi_1(X) \to G$ is a representation.

# Principal $G$-bundles

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  Example: When $K = U(n)$ and $G = GL(n, \mathbb{C})$, this is the Narasimhan-Seshadri theorem (1964) for vector bundles of degree 0.
- $\mathcal{M}_G^{ss,\delta}$ is an irreducible projective variety with dimension $(g-1)\dim G + \dim Z$.

# Schottky representations

Let $X$ be a compact Riemann surface with genus $g$.

$$\boxed{X = \Omega / \Sigma}$$

where $\Sigma \subset PSL(2, \mathbb{C})$ is a Schottky group and $\Omega \subset \mathbb{CP}^1$ is its region of discontinuity in the Riemann sphere.

**Definition (Schottky representation)**

Let $G$ denote a complex connected reductive algebraic group, $Z$ be its center and let $\rho : \pi_1(X) \to G$ be a representation.

If $\rho(\alpha_i) \in Z$, $\rho$ is said to be a **Schottky representation**.

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- $\mathbb{S} \cong \mathbb{S}^\circ \times (Z_f)^g$ where $\mathbb{S}^\circ \cong \mathrm{Hom}\left(F_g, G \times Z^\circ\right) /\!\!/ G$ and $Z_f = Z / Z^\circ$

# Schottky $G$-bundles

## Definition (Schottky $G$-bundle)

- A holomorphic principal $G$-bundle $E$ over $X$ is called a Schottky $G$-bundle if it is isomorphic to $E_\rho$ where $\rho : \pi_1 \to G$ is a Schottky representation, that is, $\rho(\alpha_i) \in Z$ for all $i = 1, \cdots, g$.
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Example: Schottky vector bundles $V_\rho$ are strict $GL(n, \mathbb{C})$ Schottky bundles.

# Topological type

- $\mathcal{M}_G^{ss} = \bigsqcup_{\delta \in \pi_1(G)} \mathcal{M}_G^{ss,\delta}$
- $\rho : \pi_1(X) \to G$, $\rho\left(\prod_{i=1}^{g} [\alpha_i, \beta_i]\right) = \prod_{i=1}^{g} [a_i, b_i] = e$
- the characteristic class $\nu(E) = \nu \in H^2\left(X, \underline{\pi_1(G)}\right) \cong \pi_1(G)$ is defined as **topological type** of $E_\rho$ and coincides with the element $\delta = \prod_{i=1}^{g} \left[\tilde{a}_i, \tilde{b}_i\right]$.

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 & & \pi_1(X) \arrow[ur, "\rho"] & &
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# Topological type of a Schottky $G$-bundle

**Theorem**

*Every Schottky $G$-bundle over $X$ has trivial topological type.*

**Corollary**

*The moduli space of semistable Schottky $G$-bundles over a compact Riemann surfaces with $g \geq 2$ is contained inside the connected component of the trivial $G$-bundle $\mathcal{M}_G^{ss,0}$ in $\mathcal{M}_G^{ss}$.*

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$$\begin{array}{rccc} \mathbb{E}: & \mathrm{Hom}\left(\pi_1(X), G\right) /\!\!/ G & \to & \mathcal{M}_G \\ & [\rho] & \mapsto & [E_\rho] \end{array} \longrightarrow \mathbb{W} := \mathbb{E}|\mathbb{S}$$

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where $\mathbb{S}^\sharp := \mathbb{W}^{-1}\left(\mathcal{M}_G^{ss}\right)$.

Example: in the case of $\mathbb{C}^*$-bundles, this map can be written as $\mathbb{S} \to Jac(X)$. This is non-injective since $\mathbb{S} = \mathrm{Hom}(F_g, \mathbb{C}^* \times \mathbb{C}^*)$.

Remark: in general, this map is non-injective.

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# Analytic equivalence

$\rho, \sigma \in \mathrm{Hom}(\pi_1, G)$ are **analytically equivalent** representations if there is a map $\psi : \widetilde{X} \to G$ such that

$$\psi(\gamma z)\,\sigma(\gamma) = \rho(\gamma)\,\psi(z) \text{ for all } \gamma \in \Gamma, z \in \Omega.$$

**Lemma**

1. $\rho$ and $\sigma$ are analytically equivalent;
2. $E_\rho \cong E_\sigma$;
3. there is $\omega \in H^0(X, \mathrm{Ad}(E_\rho) \otimes \Omega^1_X)$ such that

$$\sigma(\gamma) = h_\omega(\tilde{x})^{-1}\rho(\gamma)h_\omega(\tilde{x}\cdot\gamma)$$

where $h_\omega : \widetilde{X} \to G$ is the unique solution of the differential equation $h^{-1}dh = \omega$ satisfying $h(\tilde{x}_0) = I$.

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# Local submersion of Schottky moduli map

**Theorem**

*The Schottky moduli map is a local submersion at unitary and good Schottky representations.*

**Local derivative**

$$\mathbb{W} : \mathbb{S}^\sharp \to \mathcal{M}_G^{ss,0} \quad \longrightarrow \quad d_{[\rho]}\mathbb{W} : T_{[\rho]}\mathbb{S}^\sharp \to T_{[E_\rho]}\mathcal{M}_G^{ss,0}$$

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# Tangent Spaces

## Local Derivative

$$d_{[\rho]}\mathbb{E} : T_{[\rho]}\mathbb{G}^\sharp \to T_{[E_\rho]}\mathcal{M}_G^{ss} \quad \longrightarrow \quad d_{[\rho]}\mathbb{W} : T_{[\rho]}\mathbb{S}^\sharp \to T_{[E_\rho]}\mathcal{M}_G^{ss,0}$$

- if $\rho : \pi_1(X) \to G$ is a good representation then
$$T_{[\rho]}\mathbb{G} \cong H^1\left(\pi_1\left(X\right), \mathfrak{g}_{\mathrm{Ad}_\rho}\right).$$
- if $\rho \in \mathcal{S}$ is a good representation then
$$T_{[\rho]}\mathbb{S} \cong H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) \oplus \mathfrak{z}^g$$
- $\dim \mathbb{S} = (g-1)\dim G + (g+1)\dim Z$

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# Local Derivative

According to the analytic equivalence between representations and the corresponding induced bundles:

$$
\begin{tikzcd}
 & & T_{[\rho]}\mathbb{G} \arrow[r, "d_{[\rho]}\mathbb{E}"] & T_{[E_\rho]}\mathcal{M}_G^{ss} \\
H^0\left(X,\, \mathrm{Ad}E_\rho \otimes \Omega_X^1\right) \arrow[r, "P_{\mathrm{Ad}_\rho}"] & H^1\left(\pi_1,\, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) \arrow[ur, "\cong"] & &
\end{tikzcd}
$$

$$\omega \longmapsto [P_{\mathrm{Ad}\rho}(\omega)]$$

$$H^0\left(X,\, \mathrm{Ad}E_\rho \otimes \Omega_X^1\right) \xrightarrow{Q_\rho} \mathbb{G}$$

$$\omega \longmapsto \left[h_\omega(\tilde{x})^{-1}\rho(\gamma)h_\omega\right]$$

# Main theorem

If $\rho$ is a unitary and good representation, we obtain

$$\ker \mathrm{d}\left(\mathbb{W}_\rho\right) \cong \mathrm{Im}\left(\mathrm{H}^1\left(\mathrm{F}_\mathrm{g}, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) \oplus \mathfrak{z}^\mathrm{g}\right) \bigcap \mathrm{Im} d\left(Q_\rho\right)_0$$

**Theorem**

*Let $\rho$ be a good and unitary Schottky representation. Then, the derivative of the Schottky map $d\left(\mathbb{W}\right)_\rho : T_{[\rho]}\mathbb{S} \to \mathcal{M}_G^{ss}$ has maximal rank. In particular, the Schottky map $\mathbb{W} : \mathbb{S}^\sharp \to \mathcal{M}_G^{ss}$ is a local submersion and* $\dim\left(\mathbb{W}^{-1}\left([E_\rho]\right)\right) = g \dim Z$.

Special Case: if $G$ is semisimple, or if we consider the strict case, the derivative is a local isomorphism.

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Special Case: if $G$ is semisimple, or if we consider the strict case, the derivative is a local isomorphism.

# Elliptic curve

**Theorem**

*Let $X$ be an elliptic curve and let $G$ be a connected reductive algebraic group. Then $E$ is a flat principal $G$-bundle over $X$ if and only if $E$ is Schottky.*

- If $G$ is reductive then
$$\mathbb{W} : \mathbb{S} \to \mathcal{M}^{flat}$$
is (globally) surjective.
- If $G$ is semisimple,
$$\mathbb{W} : \mathbb{S} \to \mathcal{M}^{ss,0}$$
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## Future work

By Florentino-Mourão-Nunes work: $G = SL(n,\mathbb{C})$, $X$ elliptic curve

$$\begin{array}{rcl} \mathcal{M}_G^{ss} & \xrightarrow{\cong} & \mathbb{P}^{n-1} = X/S_n \\ E \cong_S \ L_1 \oplus \cdots \oplus L_n & \mapsto & p_1 + \cdots + p_n \end{array}$$

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