Balanced metrics on twisted Higgs bundles

Mario Garcia-Fernandez
Instituto de Ciencias Matemáticas (Madrid)

AMS-EMS-SPM International Meeting 2015
Special Session Higgs Bundles and Character Varieties
12 June 2015

Joint work with Julius Ross (University of Cambridge), arXiv:1401.7108

Twisted Higgs bundles

Let X be a compact complex manifold and L an ample line bundle over X.

Definition: A *twisted higgs bundle* over X is a pair (E, ϕ) consisting of a holomorphic vector bundle E over X and a holomorphic bundle morphism

$$\phi\colon M \otimes E \to E$$

for some holomorphic vector bundle M (the twist).

First considered by Hitchin when X is a curve and M is the tangent bundle of X, in his seminal paper 'The self-duality equations on a Riemann surface' (1986), and in this generality by Simpson in 'Higgs bundles and local systems' (1992).

Twisted Higgs bundles

Let X be a compact complex manifold and L an ample line bundle over X.

Definition: A *twisted higgs bundle* over X is a pair (E, ϕ) consisting of a holomorphic vector bundle E over X and a holomorphic bundle morphism

$$\phi \colon M \otimes E \to E$$

for some holomorphic vector bundle M (the twist).

First considered by Hitchin when X is a curve and M is the tangent bundle of X, in his seminal paper 'The self-duality equations on a Riemann surface' (1986), and in this generality by Simpson in 'Higgs bundles and local systems' (1992).

The Hitchin Equations

Let ω be a Kähler metric on X such that $[\omega] = c_1(L)$. For a choice of $0 < c \in \mathbb{R}$, there is a Hitchin-Kobayashi correspondence for twisted Higgs bundles, generalizing the Donaldson–Uhlenbeck–Yau Theorem.

- *Theorem* (Hitchin '87, Simpson '88, Garcia-Prada–Alvarez-Consul '03, Bradlow–GP–Mundet-Riera '03,): (E, ϕ) *is polystable if and only if* E *admits a hermitian metric* h *solving the Hitchin equations*

$$i\Lambda F_h + c[\phi, \phi^*] = \lambda \,\mathrm{Id}\,. \tag{1}$$

Remark: F_h denotes the curvature of h, $[\phi, \phi^*] = \phi\phi^* - \phi^*\phi$ with ϕ^* denoting the adjoint of ϕ taken fibrewise and $\lambda =$ is a topological constant.

The Hitchin–Kobayashi correspondence is a powerful tool to decide whether there exists a solution of (1), *but it provides little information as to the actual solution* h.

The Hitchin Equations

Let ω be a Kähler metric on X such that $[\omega] = c_1(L)$. For a choice of $0 < c \in \mathbb{R}$, there is a Hitchin-Kobayashi correspondence for twisted Higgs bundles, generalizing the Donaldson–Uhlenbeck–Yau Theorem.

- *Theorem* (Hitchin '87, Simpson '88, Garcia-Prada–Alvarez-Consul '03, Bradlow–GP–Mundet-Riera '03,): (E, ϕ) *is polystable if and only if* E *admits a hermitian metric* h *solving the Hitchin equations*

$$i\Lambda F_h + c[\phi, \phi^*] = \lambda \operatorname{Id}. \tag{1}$$

Remark: F_h denotes the curvature of h, $[\phi, \phi^*] = \phi\phi^* - \phi^*\phi$ with ϕ^* denoting the adjoint of ϕ taken fibrewise and $\lambda =$ is a topological constant.

The Hitchin–Kobayashi correspondence is a powerful tool to decide whether there exists a solution of (1)*, but it provides little information as to the actual solution* h*.*

The Hitchin Equations

Let ω be a Kähler metric on X such that $[\omega] = c_1(L)$. For a choice of $0 < c \in \mathbb{R}$, there is a Hitchin-Kobayashi correspondence for twisted Higgs bundles, generalizing the Donaldson–Uhlenbeck–Yau Theorem.

- *Theorem* (Hitchin '87, Simpson '88, Garcia-Prada–Alvarez-Consul '03, Bradlow–GP–Mundet-Riera '03,): (E, ϕ) *is polystable if and only if* E *admits a hermitian metric* h *solving the Hitchin equations*

$$i\Lambda F_h + c[\phi, \phi^*] = \lambda \operatorname{Id}. \tag{1}$$

Remark: F_h denotes the curvature of h, $[\phi, \phi^*] = \phi\phi^* - \phi^*\phi$ with ϕ^* denoting the adjoint of ϕ taken fibrewise and $\lambda =$ is a topological constant.

The Hitchin–Kobayashi correspondence is a powerful tool to decide whether there exists a solution of (1), but it provides little information as to the actual solution h.

The Hitchin Equations

Let ω be a Kähler metric on X such that $[\omega] = c_1(L)$. For a choice of $0 < c \in \mathbb{R}$, there is a Hitchin-Kobayashi correspondence for twisted Higgs bundles, generalizing the Donaldson–Uhlenbeck–Yau Theorem.

- *Theorem* (Hitchin '87, Simpson '88, Garcia-Prada–Alvarez-Consul '03, Bradlow–GP–Mundet-Riera '03,): (E, ϕ) *is polystable if and only if* E *admits a hermitian metric* h *solving the Hitchin equations*

$$i\Lambda F_h + c[\phi, \phi^*] = \lambda \,\mathrm{Id}\,. \tag{1}$$

Remark: F_h denotes the curvature of h, $[\phi, \phi^*] = \phi\phi^* - \phi^*\phi$ with ϕ^* denoting the adjoint of ϕ taken fibrewise and $\lambda =$ is a topological constant.

The Hitchin–Kobayashi correspondence is a powerful tool to decide whether there exists a solution of (1)*, but it provides little information as to the actual solution* h*.*

Approximate solutions via balanced metrics

In this lecture we study a 'quantization' of this problem that is expressed in terms of finite dimensional data and **balanced metrics**.

Balanced metric (Luo '98, Donaldson '01): Hermitian metric on a finite dimensional vector space, which gives an approximate solution of suitable geometric PDE (moment map interpretation), as for example Hitchin equations.

- Arise in infinite sequences.
- Intimately related with:
  - geometric quantization,
  - Gieseker stability ('02 X. Wang).
- Amenable to numerical methods (Donaldson '09, Lukic–Keller '15).

Previous work: L. Wang '97 (vortices), Keller '07 (untwisted quiver sheaves), GF–Ross '13.

Approximate solutions via balanced metrics

In this lecture we study a 'quantization' of this problem that is expressed in terms of finite dimensional data and **balanced metrics**.

Balanced metric (Luo '98, Donaldson '01): Hermitian metric on a finite dimensional vector space, which gives an approximate solution of suitable geometric PDE (moment map interpretation), as for example Hitchin equations.

- Arise in infinite sequences.
- Intimately related with:
  - geometric quantization,
  - Gieseker stability ('02 X. Wang).
- Amenable to numerical methods (Donaldson '09, Lukic–Keller '15).

Previous work: L. Wang '97 (vortices), Keller '07 (untwisted quiver sheaves), GF–Ross '13.

Approximate solutions via balanced metrics

In this lecture we study a 'quantization' of this problem that is expressed in terms of finite dimensional data and **balanced metrics**.

Balanced metric (Luo '98, Donaldson '01): Hermitian metric on a finite dimensional vector space, which gives an approximate solution of suitable geometric PDE (moment map interpretation), as for example Hitchin equations.

- Arise in infinite sequences.
- Intimately related with:
  - geometric quantization,
  - Gieseker stability ('02 X. Wang).
- Amenable to numerical methods (Donaldson '09, Lukic–Keller '15).

Previous work: L. Wang '97 (vortices), Keller '07 (untwisted quiver sheaves), GF–Ross '13.

Approximate solutions via balanced metrics

In this lecture we study a 'quantization' of this problem that is expressed in terms of finite dimensional data and **balanced metrics**.

Balanced metric (Luo '98, Donaldson '01): Hermitian metric on a finite dimensional vector space, which gives an approximate solution of suitable geometric PDE (moment map interpretation), as for example Hitchin equations.

- Arise in infinite sequences.
- Intimately related with:
  - geometric quantization,
  - Gieseker stability ('02 X. Wang).
- Amenable to numerical methods (Donaldson '09, Lukic–Keller '15).

Previous work: L. Wang '97 (vortices), Keller '07 (untwisted quiver sheaves), GF–Ross '13.

Approximate solutions via balanced metrics

In this lecture we study a 'quantization' of this problem that is expressed in terms of finite dimensional data and **balanced metrics***.*

Balanced metric (Luo '98, Donaldson '01): Hermitian metric on a finite dimensional vector space, which gives an approximate solution of suitable geometric PDE (moment map interpretation), as for example Hitchin equations.

- Arise in infinite sequences.
- Intimately related with:
  - geometric quantization,
  - Gieseker stability ('02 X. Wang).
- Amenable to numerical methods (Donaldson '09, Lukic–Keller '15).

Previous work: L. Wang '97 (vortices), Keller '07 (untwisted quiver sheaves), GF–Ross '13.

Balanced metrics on twisted Higgs bundles

In this lecture we study **balanced metrics** on twisted Higgs bundles (E, ϕ)

$$\phi \colon M \otimes E \to E$$

to give approximate solutions of Hitchin equations.

Motivation:

- Donagi–Wijnholt (JHEP '13) propose to study balanced metrics for twisted Higgs bundles on surfaces with $M = K_X^{-1}$, motivated by physical quantities which depend on detailed knowledge of the solution (Vafa–Witten equations).
- Numerical approximation of hyperKähler metric on Hitchin's moduli space.

Anticlimax assumption: by now, need globally generated M.

Applies to: Vafa-Witten equations, co-Higgs bundles (Rayan), but also to vortices, holomorphic triples, quiver sheaves, ...

Balanced metrics on twisted Higgs bundles

In this lecture we study **balanced metrics** on twisted Higgs bundles (E, ϕ)

$$\phi \colon M \otimes E \to E$$

to give approximate solutions of Hitchin equations.

Motivation:

- Donagi–Wijnholt (JHEP '13) propose to study balanced metrics for twisted Higgs bundles on surfaces with $M = K_X^{-1}$, motivated by physical quantities which depend on detailed knowledge of the solution (Vafa–Witten equations).
- Numerical approximation of hyperKähler metric on Hitchin's moduli space.

Anticlimax assumption: *by now, need globally generated M.*

Applies to: Vafa-Witten equations, co-Higgs bundles (Rayan), but also to vortices, holomorphic triples, quiver sheaves, ...

Balanced metrics on twisted Higgs bundles

In this lecture we study **balanced metrics** on twisted Higgs bundles (E, ϕ)

$$\phi \colon M \otimes E \to E$$

to give approximate solutions of Hitchin equations.

Motivation:

- Donagi–Wijnholt (JHEP '13) propose to study balanced metrics for twisted Higgs bundles on surfaces with $M = K_X^{-1}$, motivated by physical quantities which depend on detailed knowledge of the solution (Vafa–Witten equations).
- Numerical approximation of hyperKähler metric on Hitchin's moduli space.

Anticlimax assumption: *by now, need globally generated M.*

Applies to: Vafa-Witten equations, co-Higgs bundles (Rayan), but also to vortices, holomorphic triples, quiver sheaves, ...

Balanced metrics on twisted Higgs bundles

In this lecture we study **balanced metrics** on twisted Higgs bundles (E, ϕ)

$$\phi\colon M \otimes E \to E$$

to give approximate solutions of Hitchin equations.

Motivation:

- Donagi–Wijnholt (JHEP '13) propose to study balanced metrics for twisted Higgs bundles on surfaces with $M = K_X^{-1}$, motivated by physical quantities which depend on detailed knowledge of the solution (Vafa–Witten equations).
- Numerical approximation of hyperKähler metric on Hitchin's moduli space.

Anticlimax assumption: *by now, need globally generated M.*

Applies to: Vafa-Witten equations, co-Higgs bundles (Rayan), but also to vortices, holomorphic triples, quiver sheaves, ...

Balanced metrics on twisted Higgs bundles

In this lecture we study **balanced metrics** on twisted Higgs bundles (E, ϕ)

$$\phi \colon M \otimes E \to E$$

to give approximate solutions of Hitchin equations.

Motivation:

- Donagi–Wijnholt (JHEP '13) propose to study balanced metrics for twisted Higgs bundles on surfaces with $M = K_X^{-1}$, motivated by physical quantities which depend on detailed knowledge of the solution (Vafa–Witten equations).
- Numerical approximation of hyperKähler metric on Hitchin's moduli space.

Anticliimax assumption: *by now, need globally generated M.*

Applies to: Vafa-Witten equations, co-Higgs bundles (Rayan), but also to vortices, holomorphic triples, quiver sheaves, ...

Definition of balanced metrics

Let X a projective manifold, with ample line L and fixed hermitian metric h_L with Kähler form $\omega = iF_{h_L}$. Let (E, ϕ) be a twisted Higgs bundle with globally generated M, and fixed hermitian metric h_M.

- For $\mathbb{Z} \ni k > 0$, consider

$$\phi_* = \phi_{*,k} \colon H^0(M) \otimes H^0(E \otimes L^k) \to H^0(M \otimes E \otimes L^k) \xrightarrow{\phi} H^0(E \otimes L^k)$$

  where the first map is the natural multiplication.
- Define $\chi(k) = \frac{h^0(E \otimes L^k)}{r_E[\omega]^n}$.
- Choose sequence of positive rationals $\delta = \delta(k) = O(k^{n-1})$.

Since $H^0(M)$ is hermitian (L^2-metric), given a hermitian metric $\langle \cdot, \cdot \rangle_k$ on $H^0(E \otimes L^k)$, we can define an endomorphism of $H^0(E \otimes L^k)$ by

$$P := \chi^{-1} \left(\mathrm{Id} + \frac{\delta[\phi_*, (\phi_*)^*]}{1 + \|\phi_*\|^2} \right),$$

where $\|\phi_*\|^2 := \mathrm{tr}\left((\phi_*)^* \phi_*\right)$ (Frobenius norm).

Definition of balanced metrics

Let X a projective manifold, with ample line L and fixed hermitian metric h_L with Kähler form $\omega = iF_{h_L}$. Let (E, ϕ) be a twisted Higgs bundle with globally generated M, and fixed hermitian metric h_M.

- For $\mathbb{Z} \ni k > 0$, consider

$$\phi_* = \phi_{*,k} \colon H^0(M) \otimes H^0(E \otimes L^k) \to H^0(M \otimes E \otimes L^k) \xrightarrow{\phi} H^0(E \otimes L^k)$$

where the first map is the natural multiplication.

- Define $\chi(k) = \frac{h^0(E \otimes L^k)}{r_E [\omega]^n}$.
- Choose sequence of positive rationals $\delta = \delta(k) = O(k^{n-1})$.

Since $H^0(M)$ is hermitian (L^2-metric), given a hermitian metric $\langle \cdot, \cdot \rangle_k$ on $H^0(E \otimes L^k)$, we can define an endomorphism of $H^0(E \otimes L^k)$ by

$$P := \chi^{-1} \left(\mathrm{Id} + \frac{\delta [\phi_*, (\phi_*)^*]}{1 + \|\phi_*\|^2} \right),$$

where $\|\phi_*\|^2 := \operatorname{tr}((\phi_*)^* \phi_*)$ (Frobenius norm).

Definition of balanced metrics

Let X a projective manifold, with ample line L and fixed hermitian metric h_L with Kähler form $\omega = iF_{h_L}$. Let (E, ϕ) be a twisted Higgs bundle with globally generated M, and fixed hermitian metric h_M.

- For $\mathbb{Z} \ni k > 0$, consider

  $$\phi_* = \phi_{*,k} \colon H^0(M) \otimes H^0(E \otimes L^k) \to H^0(M \otimes E \otimes L^k) \xrightarrow{\phi} H^0(E \otimes L^k)$$

  where the first map is the natural multiplication.
- Define $\chi(k) = \frac{h^0(E \otimes L^k)}{r_E[\omega]^n}$.
- Choose sequence of positive rationals $\delta = \delta(k) = O(k^{n-1})$.

Since $H^0(M)$ is hermitian (L^2-metric), given a hermitian metric $\langle \cdot, \cdot \rangle_k$ on $H^0(E \otimes L^k)$, we can define an endomorphism of $H^0(E \otimes L^k)$ by

$$P := \chi^{-1} \left(\mathrm{Id} + \frac{\delta[\phi_*, (\phi_*)^*]}{1 + |||\phi_*|||^2} \right),$$

where $|||\phi_*|||^2 := \mathrm{tr}\left((\phi_*)^* \phi_*\right)$ (Frobenius norm).

Definition of balanced metrics

Let X a projective manifold, with ample line L and fixed hermitian metric h_L with Kähler form $\omega = iF_{h_L}$. Let (E, ϕ) be a twisted Higgs bundle with globally generated M, and fixed hermitian metric h_M.

- For $\mathbb{Z} \ni k > 0$, consider

  $$\phi_* = \phi_{*,k} \colon H^0(M) \otimes H^0(E \otimes L^k) \to H^0(M \otimes E \otimes L^k) \xrightarrow{\phi} H^0(E \otimes L^k)$$

  where the first map is the natural multiplication.
- Define $\chi(k) = \frac{h^0(E \otimes L^k)}{r_E [\omega]^n}$.
- Choose sequence of positive rationals $\delta = \delta(k) = O(k^{n-1})$.

Since $H^0(M)$ is hermitian (L^2-metric), given a hermitian metric $\langle \cdot, \cdot \rangle_k$ on $H^0(E \otimes L^k)$, we can define an endomorphism of $H^0(E \otimes L^k)$ by

$$P := \chi^{-1} \left(\mathrm{Id} + \frac{\delta[\phi_*, (\phi_*)^*]}{1 + |||\phi_*|||^2} \right),$$

where $|||\phi_*|||^2 := \mathrm{tr}\left((\phi_*)^* \phi_*\right)$ (Frobenius norm).

Definition of balanced metrics

Let X a projective manifold, with ample line L and fixed hermitian metric h_L with Kähler form $\omega = iF_{h_L}$. Let (E, ϕ) be a twisted Higgs bundle with globally generated M, and fixed hermitian metric h_M.

- For $\mathbb{Z} \ni k > 0$, consider

$$\phi_* = \phi_{*,k} \colon H^0(M) \otimes H^0(E \otimes L^k) \to H^0(M \otimes E \otimes L^k) \stackrel{\phi}{\to} H^0(E \otimes L^k)$$

  where the first map is the natural multiplication.
- Define $\chi(k) = \frac{h^0(E \otimes L^k)}{r_E [\omega]^n}$.
- Choose sequence of positive rationals $\delta = \delta(k) = O(k^{n-1})$.

Since $H^0(M)$ is hermitian (L^2-metric), given a hermitian metric $\langle \cdot, \cdot \rangle_k$ on $H^0(E \otimes L^k)$, we can define an endomorphism of $H^0(E \otimes L^k)$ by

$$P := \chi^{-1} \left(\mathrm{Id} + \frac{\delta [\phi_*, (\phi_*)^*]}{1 + \lVert\!|\phi_*|\!\rVert^2} \right),$$

where $\lVert\!|\phi_*|\!\rVert^2 := \operatorname{tr}((\phi_*)^* \phi_*)$ (Frobenius norm).

Definition of balanced metrics

Given a hermitian metric on $H^0(E \otimes L^k)$, we define

$$P := \chi^{-1}\left(\mathrm{Id} + \frac{\delta[\phi_*, (\phi_*)^*]}{1 + \|\phi_*\|^2}\right),$$

where $\|\phi_*\|^2 := \operatorname{tr}((\phi_*)^* \phi_*)$ (Frobenius norm).

Kodaira embbeding: global sections of $E \otimes L^k$ give an embedding

$$\iota\colon X \hookrightarrow \mathbb{G}(H^0(E \otimes L^k), r_E)$$

- *Definition 1: A hermitian metric on $H^0(E \otimes L^k)$ is balanced if there exists an orthonormal basis $\underline{s} = (s_j)$ such that*

$$\int_X (s_l, s_j)_{\iota^* h_{FS}} \omega^n = P_{jl}$$

- *Definition 2: A hermitian metric h on E is balanced if there exists a balanced metric on $H^0(E \otimes L^k)$ such that $h \otimes h_L^k = \iota^* h_{FS}$.*

Definition of balanced metrics

Given a hermitian metric on $H^0(E \otimes L^k)$, we define

$$P := \chi^{-1}\left(\mathrm{Id} + \frac{\delta[\phi_*, (\phi_*)^*]}{1 + \|\|\phi_*\|\|^2}\right),$$

where $\|\|\phi_*\|\|^2 := \mathrm{tr}\left((\phi_*)^*\phi_*\right)$ (Frobenius norm).

Kodaira embbeding: global sections of $E \otimes L^k$ give an embedding

$$\iota\colon X \hookrightarrow \mathbb{G}(H^0(E \otimes L^k), r_E)$$

- *Definition 1*: *A hermitian metric on $H^0(E \otimes L^k)$ is balanced if there exists an orthonormal basis $\underline{s} = (s_j)$ such that*

$$\int_X (s_l, s_j)_{\iota^* h_{FS}} \omega^n = P_{jl}$$

- *Definition 2*: *A hermitian metric h on E is balanced if there exists a balanced metric on $H^0(E \otimes L^k)$ such that $h \otimes h_L^k = \iota^* h_{FS}$.*

Definition of balanced metrics

Given a hermitian metric on $H^0(E \otimes L^k)$, we define

$$P := \chi^{-1}\left(\mathrm{Id} + \frac{\delta[\phi_*, (\phi_*)^*]}{1 + \|\!|\phi_*|\!\|^2}\right),$$

where $\|\!|\phi_*|\!\|^2 := \mathrm{tr}\left((\phi_*)^*\phi_*\right)$ (Frobenius norm).

Kodaira embbeding: global sections of $E \otimes L^k$ give an embedding

$$\iota \colon X \hookrightarrow \mathbb{G}(H^0(E \otimes L^k), r_E)$$

- *Definition 1*: *A hermitian metric on $H^0(E \otimes L^k)$ is balanced if there exists an orthonormal basis $\underline{s} = (s_j)$ such that*

$$\int_X (s_l, s_j)_{\iota^* h_{FS}} \omega^n = P_{jl}$$

- *Definition 2*: *A hermitian metric h on E is balanced if there exists a balanced metric on $H^0(E \otimes L^k)$ such that $h \otimes h_L^k = \iota^* h_{FS}$.*

Main results

Assume that M is globally generated.

Theorem (G-F, Ross '14)

A twisted Higgs bundle (E, ϕ) is Gieseker-polystable if and only if for all k sufficiently large it carries a balanced metric at level k.

- *Definition (Schmitt '04)*: (E, ϕ) *Gieseker stable if for any proper subsheaf $F \subset E$ with $\phi(F \otimes M) \subset F$, have $\chi(F \otimes L^k) < \chi(E \otimes L^k)$.*

Remark: stability used for moduli constuction, when $n = \dim_{\mathbb{C}} X > 1$.

Theorem (G-F, Ross '14)

Suppose h_k is a sequence of hermitian metrics on E such that 1) converges (in C^∞) to h as $k \to \infty$, 2) h_k is balanced at level k and 3) the sequence of corresponding balanced metrics on $H^0(E(k))$ is "weakly geometric". Then h is (up to conformal change) a solution of Hitchin equations.

Remark: sequences of L^2-metrics for convergent sequences of metrics on E are weakly geometric (case $\phi = 0$).

Main results

Assume that M is globally generated.

Theorem (G-F, Ross '14)

A twisted Higgs bundle (E, ϕ) is Gieseker-polystable if and only if for all k sufficiently large it carries a balanced metric at level k.

- *Definition (Schmitt '04)*: (E, ϕ) *Gieseker stable if for any proper subsheaf $F \subset E$ with $\phi(F \otimes M) \subset F$, have $\chi(F \otimes L^k) < \chi(E \otimes L^k)$.*

Remark: stability used for moduli constuction, when $n = \dim_{\mathbb{C}} X > 1$.

Theorem (G-F, Ross '14)

Suppose h_k is a sequence of hermitian metrics on E such that 1) converges (in C^∞) to h as $k \to \infty$, 2) h_k is balanced at level k and 3) the sequence of corresponding balanced metrics on $H^0(E(k))$ is "weakly geometric". Then h is (up to conformal change) a solution of Hitchin equations.

Remark: sequences of L^2-metrics for convergent sequences of metrics on E are weakly geometric (case $\phi = 0$).

Main results

Assume that M is globally generated.

Theorem (G-F, Ross '14)

A twisted Higgs bundle (E, ϕ) is Gieseker-polystable if and only if for all k sufficiently large it carries a balanced metric at level k.

- *Definition (Schmitt '04)*: (E, ϕ) *Gieseker stable if for any proper subsheaf $F \subset E$ with $\phi(F \otimes M) \subset F$, have $\chi(F \otimes L^k) < \chi(E \otimes L^k)$.*

Remark: stability used for moduli constution, when $n = \dim_{\mathbb{C}} X > 1$.

Theorem (G-F, Ross '14)

Suppose h_k is a sequence of hermitian metrics on E such that 1) converges (in C^∞) to h as $k \to \infty$, 2) h_k is balanced at level k and 3) the sequence of corresponding balanced metrics on $H^0(E(k))$ is "weakly geometric". Then h is (up to conformal change) a solution of Hitchin equations.

Remark: sequences of L^2-metrics for convergent sequences of metrics on E are weakly geometric (case $\phi = 0$).

Proof of Theorem 1

Step 1: Let $N_k = h^0(E \otimes L^k)$ and consider the parameter space

$$Maps(X, \mathbb{G}(\mathbb{C}^{N_k}, r_E)) \times \overline{Z}_k$$

where $\overline{Z}_k = \mathbb{P}(Z_k \oplus \mathbb{C})$ and

$$Z_k = Hom(H^0(M) \otimes \mathbb{C}^{N_k}, \mathbb{C}^{N_k}).$$

This has a natural Kähler structure (smooth locus), preserved by the $U(N_k)$-action.

Proposition (GF-Ross '14): *The $U(N_k)$-action is Hamiltonian, with equivariant moment map μ_k. A balanced metric at level k corresponds to a zero of μ_k.*

Step 2: For a choice of isomorphism, $H^0(E \otimes L^k) \cong \mathbb{C}^{N_k}$ consider the Gieseker-type point

$$([T_E], \phi_*) \in \mathbb{P}(Hom(\Lambda^{r_E}\mathbb{C}^{N_k}, H^0(\det(E \otimes L^k)))) \times \overline{Z}_k,$$

where $T_E \colon \Lambda^{r_E}\mathbb{C}^{N_k} \to H^0(\det(E \otimes L^k))$ is induced by multiplication map.

Proof of Theorem 1

Step 1: Let $N_k = h^0(E \otimes L^k)$ and consider the parameter space

$$Maps(X, \mathbb{G}(\mathbb{C}^{N_k}, r_E)) \times \overline{Z}_k$$

where $\overline{Z}_k = \mathbb{P}(Z_k \oplus \mathbb{C})$ and

$$Z_k = Hom(H^0(M) \otimes \mathbb{C}^{N_k}, \mathbb{C}^{N_k}).$$

This has a natural Kähler structure (smooth locus), preserved by the $U(N_k)$-action.

Proposition (GF-Ross '14): *The $U(N_k)$-action is Hamiltonian, with equivariant moment map μ_k. A balanced metric at level k corresponds to a zero of μ_k.*

Step 2: For a choice of isomorphism, $H^0(E \otimes L^k) \cong \mathbb{C}^{N_k}$ consider the Gieseker-type point

$$([T_E], \phi_*) \in \mathbb{P}(Hom(\Lambda^{r_E}\mathbb{C}^{N_k}, H^0(\det(E \otimes L^k)))) \times \overline{Z}_k,$$

where $T_E \colon \Lambda^{r_E}\mathbb{C}^{N_k} \to H^0(\det(E \otimes L^k))$ is induced by multiplication map.

Proof of Theorem 1

Step 1: Let $N_k = h^0(E \otimes L^k)$ and consider the parameter space

$$Maps(X, \mathbb{G}(\mathbb{C}^{N_k}, r_E)) \times \overline{Z}_k$$

where $\overline{Z}_k = \mathbb{P}(Z_k \oplus \mathbb{C})$ and

$$Z_k = Hom(H^0(M) \otimes \mathbb{C}^{N_k}, \mathbb{C}^{N_k}).$$

This has a natural Kähler structure (smooth locus), preserved by the $U(N_k)$-action.

Proposition (GF-Ross '14): *The $U(N_k)$-action is Hamiltonian, with equivariant moment map μ_k. A balanced metric at level k corresponds to a zero of μ_k.*

Step 2: For a choice of isomorphism, $H^0(E \otimes L^k) \cong \mathbb{C}^{N_k}$ consider the Gieseker-type point

$$([T_E], \phi_*) \in \mathbb{P}(Hom(\Lambda^{r_E}\mathbb{C}^{N_k}, H^0(\det(E \otimes L^k)))) \times \overline{Z}_k,$$

where $T_E \colon \Lambda^{r_E}\mathbb{C}^{N_k} \to H^0(\det(E \otimes L^k))$ is induced by multiplication map.

Proof of Theorem 1

Step 1: Let $N_k = h^0(E \otimes L^k)$ and consider the parameter space

$$Maps(X, \mathbb{G}(\mathbb{C}^{N_k}, r_E)) \times \overline{Z}_k$$

where $\overline{Z}_k = \mathbb{P}(Z_k \oplus \mathbb{C})$ and

$$Z_k = Hom(H^0(M) \otimes \mathbb{C}^{N_k}, \mathbb{C}^{N_k}).$$

This has a natural Kähler structure (smooth locus), preserved by the $U(N_k)$-action.

Proposition (GF-Ross '14): *The $U(N_k)$-action is Hamiltonian, with equivariant moment map μ_k. A balanced metric at level k corresponds to a zero of μ_k.*

Step 2: For a choice of isomorphism, $H^0(E \otimes L^k) \cong \mathbb{C}^{N_k}$ consider the Gieseker-type point

$$([T_E], \phi_*) \in \mathbb{P}(Hom(\Lambda^{r_E}\mathbb{C}^{N_k}, H^0(\det(E \otimes L^k)))) \times \overline{Z}_k,$$

where $T_E \colon \Lambda^{r_E}\mathbb{C}^{N_k} \to H^0(\det(E \otimes L^k))$ is induced by multiplication map.

Proof of Theorem 1

Step 2: For a choice of isomorphism, $H^0(E \otimes L^k) \cong \mathbb{C}^{N_k}$ consider the Gieseker-type point

$$([T_E], \phi_*) \in \mathbb{P}(Hom(\Lambda^{r_E}\mathbb{C}^{N_k}, H^0(\det(E \otimes L^k)))) \times \overline{Z}_k,$$

where $T_E \colon \Lambda^{r_E}\mathbb{C}^{N_k} \to H^0(\det(E \otimes L^k))$ is induced by multiplication map.

Theorem (G-F, Ross '14)

There is an $\epsilon_0 \geq 0$ such that for all rational function $\epsilon = \epsilon(k)$ with $\epsilon \geq \epsilon_0 k^{-1}$ and for all k sufficiently large, the following holds: (E, ϕ) is Gieseker-polystable if and only if the Gieseker point $([T_E], \phi_*)$ is GIT polystable with respect to $\mathcal{O}(1) \boxtimes \mathcal{O}_{\overline{Z}_k}(\epsilon)$.

Remark: we shall apply the previous Theorem with $\epsilon = \delta(k)/\chi(k)$.

Proof of Theorem 1

Step 2: For a choice of isomorphism, $H^0(E \otimes L^k) \cong \mathbb{C}^{N_k}$ consider the Gieseker-type point

$$([T_E], \phi_*) \in \mathbb{P}(Hom(\Lambda^{r_E}\mathbb{C}^{N_k}, H^0(\det(E \otimes L^k)))) \times \overline{Z}_k,$$

where $T_E \colon \Lambda^{r_E}\mathbb{C}^{N_k} \to H^0(\det(E \otimes L^k))$ is induced by multiplication map.

Theorem (G-F, Ross '14)

There is an $\epsilon_0 \geq 0$ such that for all rational function $\epsilon = \epsilon(k)$ with $\epsilon \geq \epsilon_0 k^{-1}$ and for all k sufficiently large, the following holds: (E, ϕ) is Gieseker-polystable if and only if the Gieseker point $([T_E], \phi_*)$ is GIT polystable with respect to $\mathcal{O}(1) \boxtimes \mathcal{O}_{\overline{Z}_k}(\epsilon)$.

Remark: we shall apply the previous Theorem with $\epsilon = \delta(k)/\chi(k)$.

Proof of Theorem 1

Step 3: Note that the Gieseker space is naturally a Kähler manifold, with Hamiltonian $U(N_k)$-action. Then, given a basis $\overline{s}$ of $H^0(E \otimes L^k)$, compare the *integral of the moment map* μ_k for

$$(\iota_{\overline{s}}, \phi_*) \in Maps(X, \mathbb{G}(\mathbb{C}^{N_k}, r_E)) \times \overline{Z}_k$$

with the integral of the moment map for the Gieseker point

$$([T_E], \phi_*) \in \mathbb{P}(Hom(\Lambda^{r_E}\mathbb{C}^{N_k}, H^0(\det(E \otimes L^k)))) \times \overline{Z}_k,$$

They match! (for Phong-Sturm choice of metric on $H^0(\det(E \otimes L^k))$).

- The proof follows from GIT Theorem 3 combined with the Kempf-Ness Theorem. ■

Remark: for Higgs bundles over a curve with genus > 1, the parameter space $Z_k = Hom(H^0(M) \otimes \mathbb{C}^{N^k}, \mathbb{C}^{N_k})$ is replaced by the total space of a bundle W over a Grassmannian. Positivity of the Kähler metric requires Nakano semi-negativity of W (hard to prove).

Proof of Theorem 1

Step 3: Note that the Gieseker space is naturally a Kähler manifold, with Hamiltonian $U(N_k)$-action. Then, given a basis $\overline{s}$ of $H^0(E \otimes L^k)$, compare the *integral of the moment map* μ_k for

$$(\iota_{\overline{s}}, \phi_*) \in Maps(X, \mathbb{G}(\mathbb{C}^{N_k}, r_E)) \times \overline{Z}_k$$

with the integral of the moment map for the Gieseker point

$$([T_E], \phi_*) \in \mathbb{P}(Hom(\Lambda^{r_E}\mathbb{C}^{N_k}, H^0(\det(E \otimes L^k)))) \times \overline{Z}_k,$$

They match! (for Phong-Sturm choice of metric on $H^0(\det(E \otimes L^k))$).

- The proof follows from GIT Theorem 3 combined with the Kempf-Ness Theorem. ■

Remark: for Higgs bundles over a curve with genus > 1, the parameter space $Z_k = Hom(H^0(M) \otimes \mathbb{C}^{N^k}, \mathbb{C}^{N_k})$ is replaced by the total space of a bundle W over a Grassmannian. Positivity of the Kähler metric requires Nakano semi-negativity of W (hard to prove).

Proof of Theorem 1

Step 3: Note that the Gieseker space is naturally a Kähler manifold, with Hamiltonian $U(N_k)$-action. Then, given a basis $\overline{s}$ of $H^0(E \otimes L^k)$, compare the *integral of the moment map* μ_k for

$$(\iota_{\overline{s}}, \phi_*) \in Maps(X, \mathbb{G}(\mathbb{C}^{N_k}, r_E)) \times \overline{Z}_k$$

with the integral of the moment map for the Gieseker point

$$([T_E], \phi_*) \in \mathbb{P}(Hom(\Lambda^{r_E}\mathbb{C}^{N_k}, H^0(\det(E \otimes L^k)))) \times \overline{Z}_k,$$

They match! (for Phong-Sturm choice of metric on $H^0(\det(E \otimes L^k))$).

- The proof follows from GIT Theorem 3 combined with the Kempf-Ness Theorem. ∎

Remark: for Higgs bundles over a curve with genus > 1, the parameter space $Z_k = Hom(H^0(M) \otimes \mathbb{C}^{N^k}, \mathbb{C}^{N_k})$ is replaced by the total space of a bundle W over a Grassmannian. Positivity of the Kähler metric requires Nakano semi-negativity of W (hard to prove).

Proof of Theorem 1

Step 3: Note that the Gieseker space is naturally a Kähler manifold, with Hamiltonian $U(N_k)$-action. Then, given a basis $\overline{s}$ of $H^0(E \otimes L^k)$, compare the *integral of the moment map* μ_k for

$$(\iota_{\overline{s}}, \phi_*) \in Maps(X, \mathbb{G}(\mathbb{C}^{N_k}, r_E)) \times \overline{Z}_k$$

with the integral of the moment map for the Gieseker point

$$([T_E], \phi_*) \in \mathbb{P}(Hom(\Lambda^{r_E}\mathbb{C}^{N_k}, H^0(\det(E \otimes L^k)))) \times \overline{Z}_k,$$

They match! (for Phong-Sturm choice of metric on $H^0(\det(E \otimes L^k))$).

- The proof follows from GIT Theorem 3 combined with the Kempf-Ness Theorem. ■

Remark: for Higgs bundles over a curve with genus > 1, the parameter space $Z_k = Hom(H^0(M) \otimes \mathbb{C}^{N^k}, \mathbb{C}^{N_k})$ is replaced by the total space of a bundle W over a Grassmannian. Positivity of the Kähler metric requires Nakano semi-negativity of W (hard to prove).

Proof of Theorem 1

Step 3: Note that the Gieseker space is naturally a Kähler manifold, with Hamiltonian $U(N_k)$-action. Then, given a basis $\overline{s}$ of $H^0(E \otimes L^k)$, compare the *integral of the moment map* μ_k for

$$(\iota_{\overline{s}}, \phi_*) \in Maps(X, \mathbb{G}(\mathbb{C}^{N_k}, r_E)) \times \overline{Z}_k$$

with the integral of the moment map for the Gieseker point

$$([T_E], \phi_*) \in \mathbb{P}(Hom(\Lambda^{r_E}\mathbb{C}^{N_k}, H^0(\det(E \otimes L^k)))) \times \overline{Z}_k,$$

They match! (for Phong-Sturm choice of metric on $H^0(\det(E \otimes L^k))$).

- The proof follows from GIT Theorem 3 combined with the Kempf-Ness Theorem. ■

Remark: for Higgs bundles over a curve with genus > 1, the parameter space $Z_k = Hom(H^0(M) \otimes \mathbb{C}^{N^k}, \mathbb{C}^{N_k})$ is replaced by the total space of a bundle W over a Grassmannian. Positivity of the Kähler metric requires Nakano semi-negativity of W (hard to prove).

Proof of Theorem 1

Step 3: Note that the Gieseker space is naturally a Kähler manifold, with Hamiltonian $U(N_k)$-action. Then, given a basis $\overline{s}$ of $H^0(E \otimes L^k)$, compare the *integral of the moment map* μ_k for

$$(\iota_{\overline{s}}, \phi_*) \in Maps(X, \mathbb{G}(\mathbb{C}^{N_k}, r_E)) \times \overline{Z}_k$$

with the integral of the moment map for the Gieseker point

$$([T_E], \phi_*) \in \mathbb{P}(Hom(\Lambda^{r_E}\mathbb{C}^{N_k}, H^0(\det(E \otimes L^k)))) \times \overline{Z}_k,$$

They match! (for Phong-Sturm choice of metric on $H^0(\det(E \otimes L^k))$).

- The proof follows from GIT Theorem 3 combined with the Kempf-Ness Theorem. ■

Remark: for Higgs bundles over a curve with genus > 1, the parameter space $Z_k = Hom(H^0(M) \otimes \mathbb{C}^{N^k}, \mathbb{C}^{N_k})$ is replaced by the total space of a bundle W over a Grassmannian. Positivity of the Kähler metric requires Nakano semi-negativity of W (hard to prove).

Sketch of Theorem 2

Step 1: The balanced condition interacts with the Kähler geometry of X, as it implies

$$\sum_j (Ps_j')(\cdot, s_j')_{H_k} = \mathrm{Id} \in C^\infty(End\ E \otimes L^k) \tag{2}$$

for $H_k = \iota^* h_{FS}$, $\underline{s}$ orthonormal with respect to the balanced metric on $H^0(E \otimes L^k)$, and $\underline{s}'$ an L^2-orthonormal basis for H_k and

$$P := \chi^{-1}\left(\mathrm{Id} + \frac{\delta[\phi_*, (\phi_*)^*]}{1 + |||\phi_*|||^2}\right),$$

Step 2: Using the weakly geometric hypothesis we proof an asymptotic expansion for P, with error measured in the L^2-norm with respect to H_k

$$\chi P = Id + k^{-1}\left(\frac{\delta k[\phi_*, (\phi_*)^{*_{L^2}}]}{1 + |||\phi_*|||^2}\right) + \sum_{j=2}^{N} k^{-j} A_j + O(k^{-N-1})$$

Sketch of Theorem 2

Step 1: The balanced condition interacts with the Kähler geometry of X, as it implies

$$\sum_j (Ps'_j)(\cdot, s'_j)_{H_k} = \mathrm{Id} \in C^\infty(End\ E \otimes L^k) \tag{2}$$

for $H_k = \iota^* h_{FS}$, $\underline{s}$ orthonormal with respect to the balanced metric on $H^0(E \otimes L^k)$, and $\underline{s}'$ an L^2-orthonormal basis for H_k and

$$P := \chi^{-1} \left(\mathrm{Id} + \frac{\delta[\phi_*, (\phi_*)^*]}{1 + |||\phi_*|||^2} \right),$$

Step 2: Using the weakly geometric hypothesis we proof an asymptotic expansion for P, with error measured in the L^2-norm with respect to H_k

$$\chi P = Id + k^{-1} \left(\frac{\delta k[\phi_*, (\phi_*)^{*_{L^2}}]}{1 + |||\phi_*|||^2} \right) + \sum_{j=2}^{N} k^{-j} A_j + O(k^{-N-1})$$

Sketch of Theorem 2

Step 2: Using the weakly geometric hypothesis we proof an asymptotic expansion for P, with error measured in the L^2-norm with respect to H_k

$$\chi P = Id + k^{-1}\left(\frac{\delta k[\phi_*, (\phi_*)^{*_{L^2}}]}{1 + \|\|\phi_*\|\|^2}\right) + \sum_{j=2}^{N} k^{-j} A_j + O(k^{-N-1})$$

Weakly geometric: *A sequence of metrics* $\langle \cdot, \cdot \rangle_k$ *on* $H^0(E \otimes L^k)$ *is weakly geometric if there exists* $c' > 0$ *such that*

- $c' k^n \leq \|\|\phi_*\|\|_k^2$ (Frobenius norm),
- $\|\phi_*\|_k \leq c'$ (operator norm).

Step 3: Using the asymptic expansion (4) and the *Hormander Estimate*, we obtain an asymptotic expansion in L^2 norm (with respect to H_k)

$$B_k + ck^{n-1}[\phi, \phi^{*_k}] = \chi Id + O(k^{n-2}),$$

where $B_k = \sum_j s_j'(\cdot, s_j')_{H_k}$ is the Bergman Kernel of H_k (diagonal of the integration kernel for orthogonal projection $C^\infty(E \otimes L^k) \to H^0(E \otimes L^k)$).

Sketch of Theorem 2

Step 2: Using the weakly geometric hypothesis we proof an asymptotic expansion for P, with error measured in the L^2-norm with respect to H_k

$$\chi P = Id + k^{-1}\left(\frac{\delta k[\phi_*, (\phi_*)^{*_{L^2}}]}{1 + |||\phi_*|||^2}\right) + \sum_{j=2}^{N} k^{-j} A_j + O(k^{-N-1})$$

Weakly geometric: *A sequence of metrics $\langle\cdot,\cdot\rangle_k$ on $H^0(E \otimes L^k)$ is weakly geometric if there exists $c' > 0$ such that*

- $c' k^n \leq |||\phi_*|||_k^2$ (Frobenius norm),
- $\|\phi_*\|_k \leq c'$ (operator norm).

Step 3: Using the asymptic expansion (4) and the *Hormander Estimate*, we obtain an asymptotic expansion in L^2 norm (with respect to H_k)

$$B_k + ck^{n-1}[\phi, \phi^{*_k}] = \chi Id + O(k^{n-2}),$$

where $B_k = \sum_j s_j'(\cdot, s_j')_{H_k}$ is the Bergman Kernel of H_k (diagonal of the integration kernel for orthogonal projection $C^\infty(E \otimes L^k) \to H^0(E \otimes L^k)$).

Proof of Theorem 2

Step 3: Using the asymptic expansion (4) and the *Hormander Estimate*, we obtain an asymptotic expansion in L^2 norm (with respect to H_k)

$$B_k + ck^{n-1}[\phi, \phi^{*_k}] = \chi \, \mathrm{Id} + O(k^{n-2}),$$

Step 4: Use the asymptotic expansion of the Bergman Kernel

Theorem (Fefferman '74, Yau '86, Tian '90, Catlin '97, Zelditch '98, Ma–Marinescu '07)

The Bergman Kernel has C^∞ asymptotic expansion over the diagonal, which is uniform under variations of the metric on E

$$B_k = k^n \, \mathrm{Id} + k^{n-1}(i\Lambda_\omega F + S_\omega/2 \, \mathrm{Id}) + O(k^{n-2})$$

Perspectives:

- Higgs bundles over curves of genus > 2 (need to prove Nakano semi-negativity of bundles),
- Any solution of Hitchin equations is approximated by a weakly geometric sequence of balanced metrics,

Perspectives:

- Higgs bundles over curves of genus > 2 (need to prove Nakano semi-negativity of bundles),
- Any solution of Hitchin equations is approximated by a weakly geometric sequence of balanced metrics,

Perspectives:

- Higgs bundles over curves of genus > 2 (need to prove Nakano semi-negativity of bundles),
- Any solution of Hitchin equations is approximated by a weakly geometric sequence of balanced metrics,

What is the actual shape of this?

Figure: Real points of moduli of parabollic Higgs on $T^2\backslash\{p\}$

GRACIAS!