Max Planck Institute for Mathematics
California Institute of Technology

A new TQFT from equivariant integration on the Hitchin moduli space

Sergei Gukov

based on joint work with Du Pei

arXiv: 1011.2218

arXiv: 1501.01310

Motivation

Phil. Trans. R. Soc. Lond. A **308**, 523–615 (1982) [523]
Printed in Great Britain

Complex

THE ~~YANG~~–MILLS EQUATIONS OVER RIEMANN SURFACES

BY M. F. ATIYAH, F.R.S.,† AND R. BOTT‡
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(*Received* 4 *March* 1982)

CONTENTS

2d Yang-Mills

- Mathematically well defined

THE YANG–MILLS EQUATIONS OVER RIEMANN SURFACES

BY M. F. ATIYAH, F.R.S.,† AND R. BOTT‡

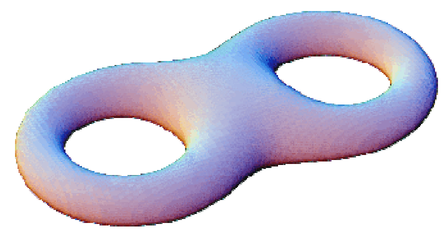
- Exactly solvable (solved by A.Migdal)
- Has many cousins and applications, ranging from Hurwitz theory to Chern-Simons theory

2d TQFT $Z \sim \mathrm{Vol}\,(\mathcal{M}_{\text{flat}})$ ← 2d Yang-Mills → Chern-Simons on $S^1 \times \Sigma$

2d Yang-Mills

G = (simple) compact Lie group e.g. **SU(2)**

A = connection on a **G**-bundle $E \to \Sigma$ over a genus-g Riemann surface Σ



$$S_{\mathrm{YM}}(A) = -\frac{1}{2e^2}\int_\Sigma d\mu\, \mathrm{Tr}\, F_A^2$$

Note,

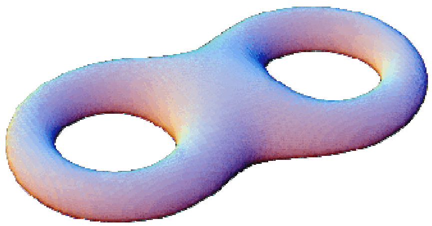
$$Z_{\mathrm{YM}}(\Sigma, e^2) = \int \mathcal{D}A\, e^{-S_{\mathrm{YM}}(A)}$$

is almost topological: depends only on $a = \int_\Sigma d\mu$

2d Yang-Mills

invariant under SDiff(Σ) ➡ "area-preserving quantum field theory" (cf. TQFT)

$$S_{\rm YM}(A) = -\frac{1}{2e^2}\int_\Sigma d\mu \, {\rm Tr}\, F_A^2$$



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$$Z_{\rm YM}(\Sigma, e^2) = \int \mathcal{D}A \, e^{-S_{\rm YM}(A)}$$

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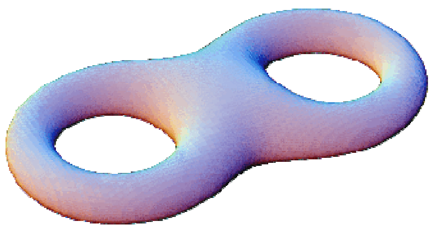
2d Yang-Mills

invariant under $\mathrm{SDiff}(\Sigma)$ "area-preserving quantum field theory" (cf. TQFT)



moreover, $S_{\mathrm{YM}}(A)$ is invariant under

$$e^2 \to u\, e^2, \qquad \mu \to \mu/u \quad \text{for} \quad u \in \mathbb{R}$$

$Z_{\mathrm{YM}}(\Sigma, e^2)$ depends only on $e^2 a$ and without loss of generality we can set $e^2 = 1$

2d Yang-Mills

Lie algebra valued scalar (0-form)

$$S(A,B) = -\frac{e^2}{2}\int_\Sigma d\mu\, \mathrm{Tr}\, B^2 - i\int_\Sigma \mathrm{Tr}\, B F_A$$

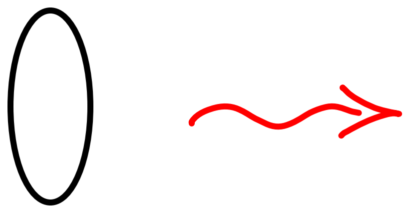
Integrating out B we get $S_{\mathrm{YM}}(A)$.

E.Witten

The limit $e^2 \to 0$ (equivalently $a \to 0$) gives a TQFT with the field equations $F_A = 0$

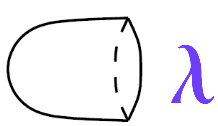
$$Z_{\mathrm{YM}}(e^2 a \to 0) = \mathrm{Vol}\left(\mathcal{M}_{\mathrm{flat}}(G,\Sigma)\right)$$

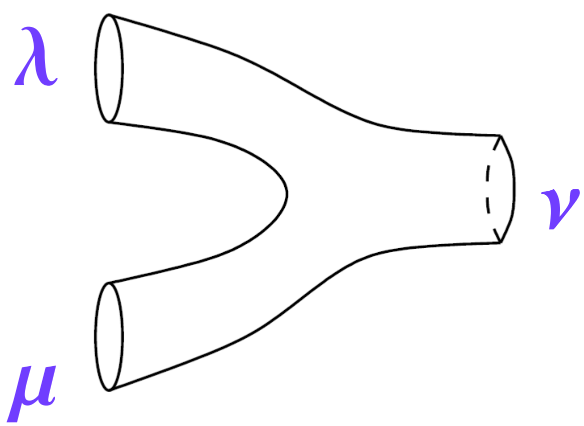
Atiyah-Segal axioms in 2d



$\mathcal{H}$ (Hilbert space)

States: $|\lambda\rangle \in \mathcal{H}$





$$= C_{\lambda\mu\nu}$$

Atiyah-Segal axioms in 2d

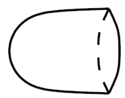
Associativity:

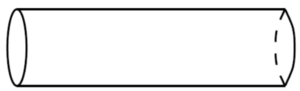
$$C_{\mu\nu\lambda}C^{\lambda}_{\rho\sigma} \; = \; C_{\mu\rho\lambda}C^{\lambda}_{\nu\sigma}$$

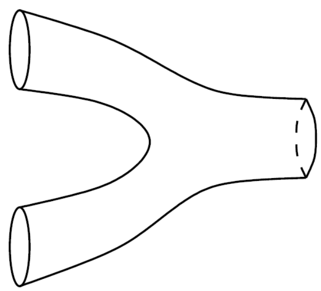
Topological invariance:

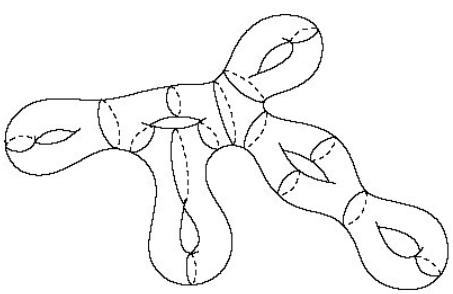
$$C_{\mu\nu\lambda} = \begin{cases} C_{\lambda\lambda\lambda}, & \text{if } \lambda = \mu = \nu \\ 0, & \text{otherwise} \end{cases}$$

Atiyah-Segal axioms in 2d

 $\lambda \;=\; \dfrac{1}{f(\lambda)}$

λ  $\mu \;=\; \delta_{\lambda,\mu}$

 $= \; C_{\lambda\mu\nu} \;=\; f(\lambda)\,\delta_{\lambda,\mu,\nu}$



$$\boxed{Z(\Sigma) \;=\; \sum_{\lambda} (C_{\lambda\lambda\lambda})^{2g-2+n}}$$

Atiyah-Segal axioms in 2d

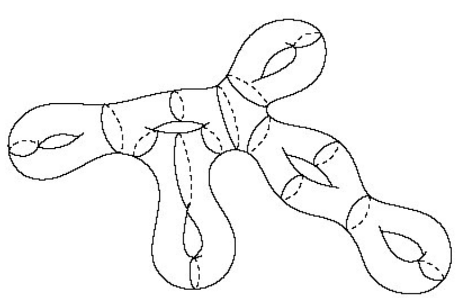
2d TQFT
$Z \sim \mathrm{Vol}\,(\mathcal{M}_{\mathrm{flat}})$

2d Yang-Mills

Chern-Simons
on $S^1 \times \Sigma$

$$C_{\lambda\lambda\lambda} = \frac{1}{\dim \lambda}$$

λ = unitary irreducible representation of G



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Atiyah-Segal axioms in 2d

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2d Yang-Mills

Chern-Simons on $S^1 \times \Sigma$

$$C_{\lambda\lambda\lambda} = \frac{1}{\dim \lambda} \quad \rightsquigarrow \quad \frac{1}{\dim_q \lambda}$$

λ = unitary irreducible representation of G

$$\lambda \in \Lambda_{G,k} = \left(\frac{\Lambda_{wt}}{W \times (k+h)\Lambda_{rt}} \right)'$$

λ = highest weight integrable representation of the loop group $\mathcal{L}G$ at level k

Atiyah-Segal axioms in 2d

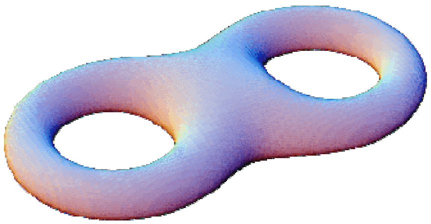
2d TQFT
$Z \sim \mathrm{Vol}\,(\mathcal{M}_{\text{flat}})$

$\longleftarrow$ 2d Yang-Mills $\longrightarrow$

Chern-Simons
on $S^1 \times \Sigma$

$$C_{\lambda\lambda\lambda} = \frac{1}{\dim \lambda} \quad \rightsquigarrow \quad \frac{1}{\dim_q \lambda} \quad \rightsquigarrow \quad \frac{1}{\dim_{q,t} \lambda}$$

Atiyah-Segal axioms in 3d



$$\rightsquigarrow \quad \mathcal{H}^{\mathrm{CS}}(G, \Sigma, k)$$

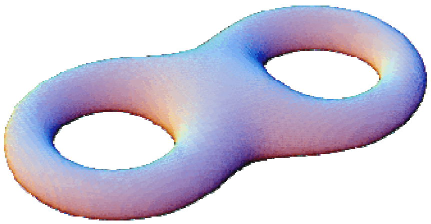
$$\dim \mathcal{H}^{\mathrm{CS}}(G, \Sigma, k) \;=\; Z_{\mathrm{CS}}(S^1 \times \Sigma) \;=\; \sum_{\lambda} \frac{1}{(S_{\emptyset\lambda})^{2g-2}}$$

<u>Example:</u> **G=SU(2)**

$$S_{\lambda\mu} \;=\; \sqrt{\frac{2}{k+2}} \sin \frac{\pi\lambda\mu}{k+2}$$

E.Verlinde

Atiyah-Segal axioms in 3d



$$\rightsquigarrow \quad \mathcal{H}^{\mathrm{CS}}(G, \Sigma, k)$$

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Example: **G=SU(2)** g=2

$$\dim \mathcal{H} = \frac{1}{6}k^3 + k^2 + \frac{11}{6}k + 1$$

E.Verlinde

Atiyah-Segal axioms in 3d

$$\begin{tikzcd} \begin{array}{c}\text{2d TQFT}\\ Z \sim \mathrm{Vol}\,(\mathcal{M}_{\text{flat}})\end{array} & \text{2d Yang-Mills} \arrow[l, "a=0"'] \arrow[r] & \begin{array}{c}\text{Chern-Simons}\\ \text{on } \mathbf{S}^1 \times \Sigma\end{array} \arrow[ll, bend left, "k\to\infty"] \end{tikzcd}$$

$$\dim \mathcal{H}^{\mathrm{CS}}(G,\Sigma,k) \;=\; Z_{\mathrm{CS}}(S^1 \times \Sigma) \;=\; \sum_{\lambda} \frac{1}{(S_{\emptyset\lambda})^{2g-2}}$$

Note: $S_{\emptyset\lambda} \;\sim\; \dim_q \lambda \;\xrightarrow{k\to\infty}\; \dim \lambda$

$$\boxed{Z_{\mathrm{YM}}(\Sigma; a) \;=\; \sum_{\lambda} (\dim \lambda)^{2-2g} e^{-\frac{a}{2} C_2(\lambda)}}$$

Hirzebruch-Riemann-Roch

2d TQFT
$Z \sim \mathrm{Vol}\,(\mathcal{M}_{\text{flat}})$

2d Yang-Mills

Chern-Simons
on $S^1 \times \Sigma$

$$\begin{aligned}
\dim \mathcal{H}^{\mathrm{CS}}(G,\Sigma,k) &= Z_{\mathrm{CS}}(S^1\times\Sigma) \\
&= \dim H^0(\mathcal{M},\mathcal{L}) \\
&= \int_{\mathcal{M}} e^{c_1(\mathcal{L})}\wedge \mathrm{Td}(\mathcal{M}) \\
&= Z_{G/G}(\Sigma) \\
&= Z_{\text{A-model}}(\mathrm{Gr}(N,k)) \\
&= \dim \mathrm{Hom}(\mathcal{B}',\mathcal{B}_{cc})
\end{aligned}$$

$\mathcal{L}$ = line bundle
over $\mathcal{M} = \mathcal{M}_{\text{flat}}(G,\Sigma)$
with $c_1(\mathcal{L}) = k\omega$

$$\omega = \frac{1}{4\pi^2}\int_\Sigma \mathrm{Tr}\,\delta A\wedge\delta A$$

Complexification

$$A \rightsquigarrow \mathcal{A} = A + i\phi$$

$\mathbf{G}_{\mathbb{C}}$-valued connection

$$\mathcal{M}_{\mathrm{flat}}(G,\Sigma) \rightsquigarrow \mathcal{M}_{\mathrm{flat}}(G_{\mathbb{C}},\Sigma) \cong \mathcal{M}_H(G,\Sigma)$$

hyper-Kahler moduli space of Higgs bundles

$\mathbf{I}$ $\mathbf{J}$ $\mathbf{K}$

ω_I ω_J ω_K

$$\left\{ (A,\Phi) \,\middle|\, \begin{aligned} F_A + [\Phi,\Phi^*] &= 0 \\ \overline{D}\Phi &= 0 \end{aligned} \right\} \Big/ \text{conj.}$$

Complexification

$$A \rightsquigarrow \mathcal{A} = A + i\phi$$

$\mathbf{G}_{\mathbb{C}}$-valued connection

$$\mathcal{M}_{\text{flat}}(G, \Sigma) \rightsquigarrow \mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma) \cong \mathcal{M}_H(G, \Sigma)$$

circle action $\circlearrowright$

$$(A, \Phi) \mapsto (A, e^{i\theta}\Phi)$$

$$\text{Vol}(\mathcal{M}) = \int_{\mathcal{M}} e^{\omega} \rightsquigarrow \text{Vol}_{\beta}(\mathcal{M}_H) = \int_{\mathcal{M}_H} e^{\omega_I(\beta)}$$

Equivariant Volume

$$H = \int_\Sigma \mathrm{Tr}\, |\Phi|^2 d^2 z$$

$$\mathrm{Vol}_\beta(\mathcal{M}_H) := \int_{\mathcal{M}_H} e^{\omega_I(\beta)} = \int_{\mathcal{M}_H} e^{\omega_I - \beta H}$$

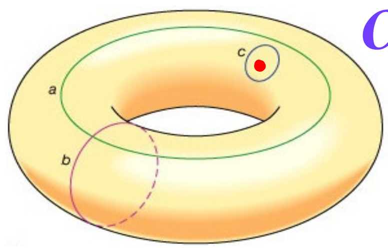
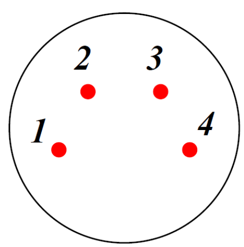
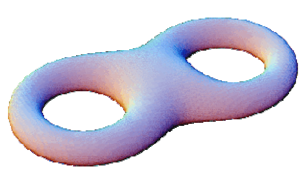
Atiyah-Bott localization formula:

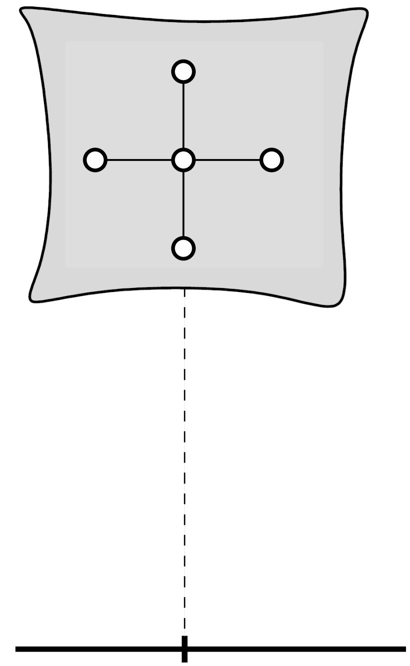
$$\boxed{\mathrm{Vol}_\beta(\mathcal{M}_H) = \sum_s e^{-\beta H(F_s)} \int_{F_s} \frac{e^{\omega_I}}{\mathrm{Eu}_\beta(F_s)}}$$

equivariant Euler class of the normal bundle of the critical manifolds of H

$$\prod_{i=1}^{\mathrm{codim}_{\mathbb{C}} F_s} (x_i + \beta n_i)$$

Simple Examples

Σ	$\mathrm{Vol}_\beta(\mathcal{M}_H)$
 α	$\frac{1-2\alpha}{\beta} + \frac{2-2e^{-\alpha\beta}}{\beta^2}$
	$\frac{\mathrm{Vol}(\mathcal{M})}{\beta} + \frac{4-\sum_i e^{-H_i\beta}}{2\beta^2}$
	$\frac{1}{\beta^3}\left[\frac{1}{6}+\frac{1}{\beta}-\frac{2}{\beta^2}+\frac{2-2e^{-\beta}}{\beta^3}\right]$



Quantization

$$A \rightsquigarrow \mathcal{A} = A + i\phi$$

$G_{\mathbb{C}}$-valued connection

$$\mathcal{M}_{\text{flat}}(G,\Sigma) \rightsquigarrow \mathcal{M}_{\text{flat}}(G_{\mathbb{C}},\Sigma) \cong \mathcal{M}_H(G,\Sigma)$$

circle action $\circlearrowright$

$$(A,\Phi) \mapsto (A, e^{i\theta}\Phi)$$

$$\boxed{\mathcal{H}(\Sigma; G_{\mathbb{C}}, k) \; = \; \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n}$$

Quantization

$$\mathcal{M} = \mathcal{M}_{\text{flat}}(G, \Sigma) \quad \rightsquigarrow \quad \mathcal{M}_H(G, \Sigma)$$

"Prequantum" line bundle and the Hilbert space:

$$c_1(\mathcal{L}) = k\omega \quad \rightsquigarrow \quad c_1(\mathcal{L}) = k\omega_I$$

$$\mathcal{H}^{\text{CS}}(G, \Sigma, k) \quad \rightsquigarrow \quad \mathcal{H}^{\text{CS}}(G_{\mathbb{C}}, \Sigma, k)$$

Index theorem:

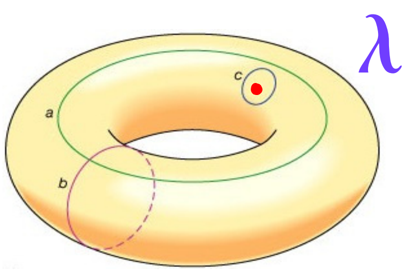
$$\int_{\mathcal{M}} e^{c_1(\mathcal{L})} \wedge \text{Td}(\mathcal{M}) \rightsquigarrow \int_{\mathcal{M}_H} \text{ch}(\mathcal{L}, \beta) \wedge \text{Td}(\mathcal{M}_H, \beta)$$

Atiyah-Segal-Singer index for complex Chern-Simons

$$\dim_\beta \mathcal{H}^{\mathrm{CS}}(G_{\mathbb{C}}, \Sigma, k) := \mathrm{index}_{S^1}(\not{\partial}_{\mathcal{L}})$$

$$= \sum_s e^{-\beta k H(F_s)} \int_{F_s} \frac{\mathrm{Td}(F_s) \wedge e^{k\omega_I}}{\prod (1 - e^{-x_i - \beta n_i})}$$

<u>Example:</u>

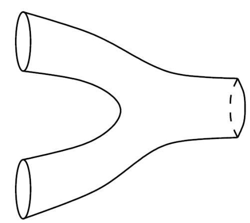


$$\frac{k - \lambda + 1}{1 - e^{-\beta}} + \frac{2e^{-\beta}}{(1 - e^{-\beta})^2} + \frac{4e^{-\beta\lambda/2}}{(1 - e^{\beta})(1 - e^{-2\beta})}$$

New 2d TQFT

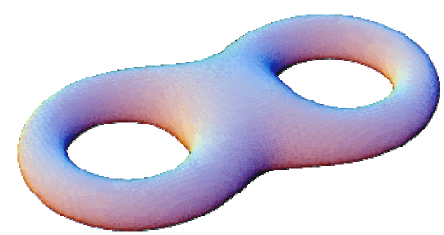
$$\dim_\beta \mathcal{H}(\Sigma; G_\mathbb{C}, k) \; := \; \sum_n t^n \dim \mathcal{H}_n \; = \sum_\lambda (C_{\lambda\lambda\lambda})^{2g-2}$$

<u>Example:</u> **G=SU(2)**



$$= \frac{1}{\dim_{q,\beta} \lambda}$$

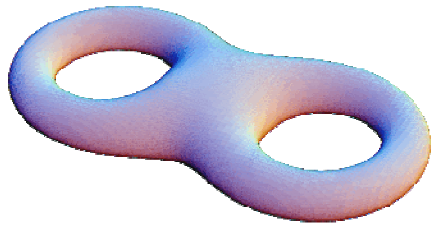
$$\begin{aligned}\dim_\beta \mathcal{H}(\Sigma; G_\mathbb{C}, k) &= \frac{1}{6}k^3 + k^2 + \frac{11}{6}k + 1 \\ &+ \left(\frac{1}{2}k^3 + 3k^2 - \frac{1}{2}k - 3\right) t \\ &+ \left(k^3 + 8k^2 - 3k + 6\right) t^2 \\ &+ \left(\frac{5}{3}k^3 + 16k^2 - \frac{71}{3}k + 6\right) t^3 \\ &+ \left(\frac{5}{2}k^3 + 29k^2 - \frac{109}{2}k + 63\right) t^4 \\ &+ \dots\end{aligned}$$



$$t = e^{-\beta}$$

New TQFTs

$$C_{\lambda\lambda\lambda} = \frac{1}{\dim \lambda}$$



$$\frac{1}{\dim_q \lambda}$$

2d TQFT

$Z \sim \mathrm{Vol}\,(\mathcal{M}_{\text{flat}})$

$Z \sim \mathrm{Vol}_\beta\,(\mathcal{M}_{\mathbf{H}})$

$\beta \to \infty$

2d Yang-Mills

2d Yang-Mills

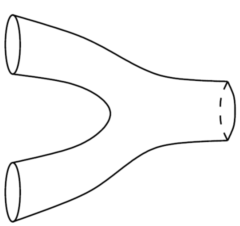
$$Z = \sum_\lambda (C_{\lambda\lambda\lambda})^{g-2}$$

Chern-Simons

on $\mathbf{S^1} \times \Sigma$

on $\mathbf{S^1} \times \Sigma$

$\beta \to \infty$

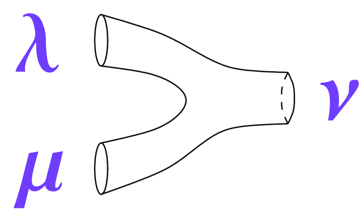


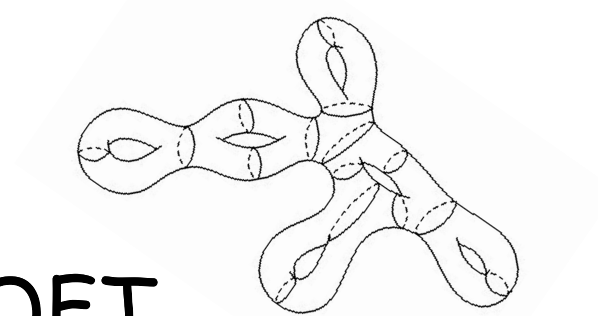
$$= \frac{1}{\dim_\beta \lambda}$$

$$\frac{1}{\dim_{q,\beta} \lambda}$$

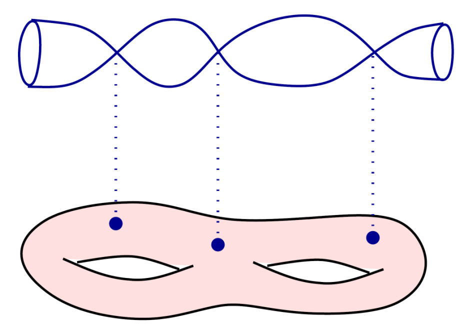
Conclusions

- Two new TQFTs in Atiyah-Segal fashion:
  - Equivariant volume of the Hitchin space
  - Atiyah-Segal-Singer equivariant index





- One new "area-preserving" QFT
- Localization for complex Chern-Simons theory:
  - one extra parameter, β
  - Seifert 3-manifolds



D-branes in Hitchin moduli space

$$\mathcal{H}^{\mathrm{CS}}(G_{\mathbb{C}}, \Sigma, k) = \mathrm{Ext}^*_{\mathcal{M}_H}(\mathcal{B}_{cc}, \mathcal{B}')$$

$$\int_{\mathcal{M}_H} \mathrm{ch}(\mathcal{L}) \wedge \mathrm{Td}(\mathcal{M}_H) = \chi(\mathcal{B}_{cc}, \mathcal{B}')$$

objects in the derived
category of coherent sheaves,
(B,A,A) branes

Mirror Symmetry

integrable system:

$$\begin{tikzcd} \mathcal{M}_H(G,\Sigma) \arrow[dr, "\pi"'] & & \mathcal{M}_H({}^L G,\Sigma) \arrow[dl, "\widetilde{\pi}"] \\ & B & \end{tikzcd}$$

[N.Hitchin]

[T.Hausel, M.Thaddeus]

U(N)	**U(N)**
SO(2N)	**SO(2N)**
SO(2N+1)	**Sp(2N)**
E6	**E6/Z_3**
E8	**E8**

Robert Langlands

Mirror Symmetry

$$\begin{array}{ccc} \text{(B,A,A) branes} & \longleftrightarrow & \text{(B,B,B) branes} \\ \mathcal{M}_H(G,\Sigma) & & \mathcal{M}_H({}^LG,\Sigma) \\ \pi \searrow & & \swarrow \widetilde{\pi} \\ & \boldsymbol{B} & \end{array}$$

$$\mathcal{H}^{\mathrm{CS}}(G_{\mathbb{C}},\Sigma,k) = \mathrm{Ext}^*_{\mathcal{M}_H({}^LG,\Sigma)}(\widetilde{\mathcal{B}}_{cc},\widetilde{\mathcal{B}}')$$

$$\int_{\mathcal{M}_H(G,\Sigma)} \mathrm{ch}(\mathcal{L}) \wedge \mathrm{Td}(\mathcal{M}_H(G,\Sigma)) = \chi(\widetilde{\mathcal{B}}_{cc},\widetilde{\mathcal{B}}')$$

Turning on a

cf. Generalized Volume Conjecture

2d TQFT
$Z \sim \mathrm{Vol}_\beta(\mathcal{M}_{\mathbf{H}})$

$\xleftarrow{a\,=\,0}$

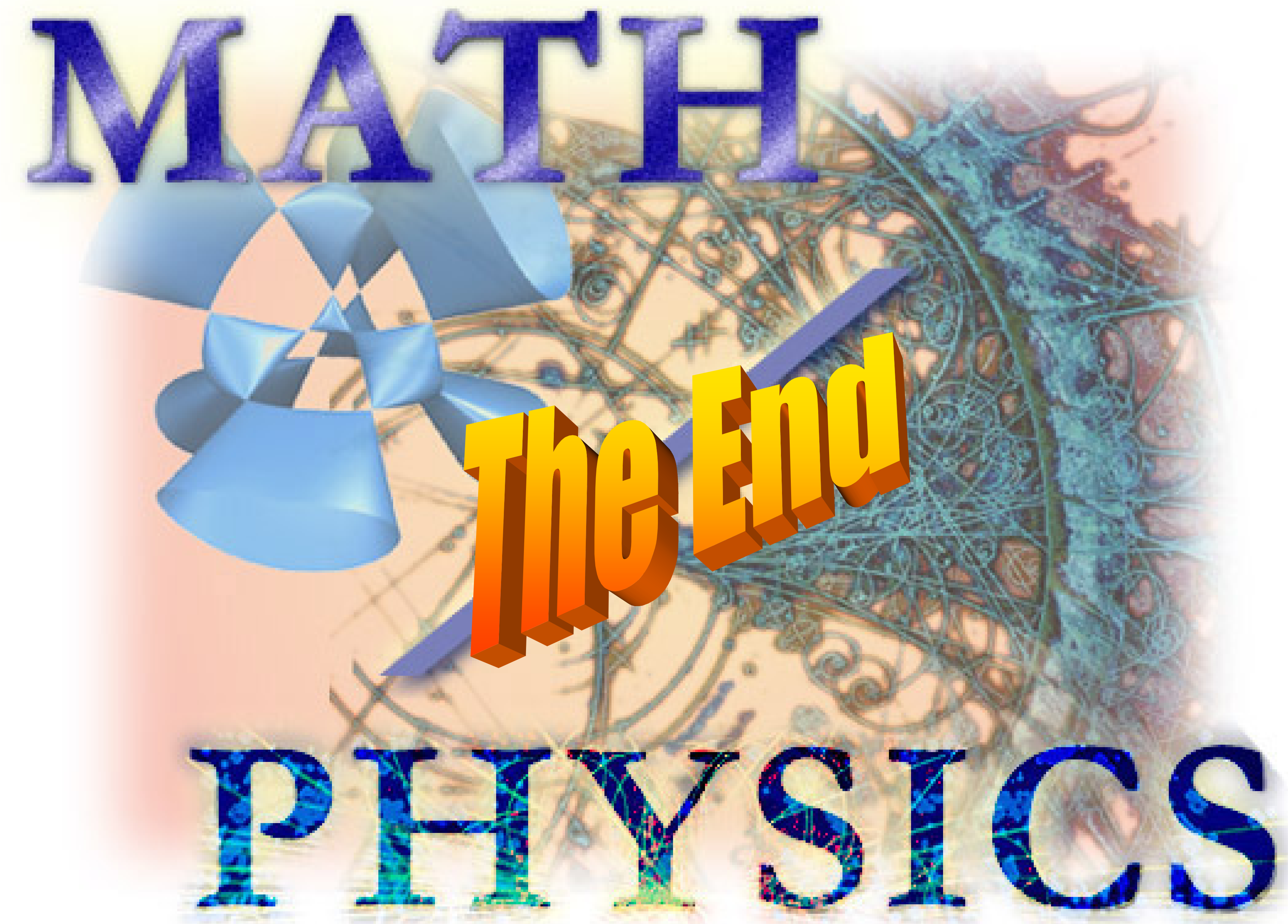
Complex
2d Yang-Mills

$\longrightarrow$

Complex
Chern-Simons
on $\mathbf{S}^1 \times \Sigma$

as a limit of complex Chern-Simons on a degree-d circle bundle over Σ (without singular fibers):

$$\boxed{q \to 1 \ , \qquad q^d = a}$$


MATH
The End
PHYSICS