Introduction to the geometry of moduli spaces of Higgs bundles

Jochen Heinloth
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$\mathcal{M}_{Betti} := \{(A_i,B_i)\in \mathrm{GL}_n^{2g} | \prod_{i=1}^{g}[A_i,B_i]=1\}/\text{conjugation}$

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Some variants

- Can look at bundles with level structures $\rightsquigarrow$ Repr of $\pi_1(C - \text{pts})$ with prescribed monodromy at punctures.
- G non-split, e.g. for $k = \mathbb{R}$.
- G/C family of groups over C.

Questions (Hitchin)

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- How are the results for different groups related?

First results:

- $n = 2$ Hitchin, $n = 3$ Gothen: Computed e.g. $H^*(\mathcal{M}_{Dol})$.

Plan:

1. $\mathcal{M}_{Betti}$ - **Method used by Hausel–Rodriguez-Villegas**
2. $\mathcal{M}_{Dol}$ - Two geometric methods
3. P=W – A conjecture relating the extra structure on H^*'s

Point-counting on $\mathcal{M}_{Betti}$ Part I

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Weil conjectures allow to deduce $H^*(X(\mathbb{C}))$ from counting $X(\mathbb{F}_q)$ if X is smooth projective.

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Warning: This does not apply to $\mathcal{M}_{Betti}$!

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Frobenius knew (G finite group, $F\colon G^k \to G$):

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- Alternative geometric argument for $\mathrm{GL}_2, \mathrm{SL}_2$ (Logares-Muñoz-Newstead,Muñoz-Martínez)
- Bad news: There are cancellations, i.e. this does not allow to deduce the Poincaré polynomial as computed by Hitchin.

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Example (Hausel-Rodriguez-Villegas)

The rational function:

$$\begin{aligned} H(q,t) = {} & \frac{(q^2t^3+1)^{2g}}{(1-q^2t^2)(1-q^2t^4)} + \frac{(qt^2)^{2g-2}(1+q^2t)^{2g}}{(1-q^2)(1-(qt)^2))} - \\ & -\frac{1}{2}(qt^2)^{2g-2}\left(\frac{(1-qt)^{2g}}{(1-q)(1-qt^2)} + \frac{(1+qt)^{2g}}{(1+q)(1+qt^2)}\right) \end{aligned}$$

specializes to $\mathcal{M}_{Betti}$ for $t = -1$ and to $E_{\not{c}}(\mathcal{M}_{Dol})$ for $q = 1$.

HRV conjecture a similar formula for all n. — Still open.

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Exponents of $q = xy \leftrightarrow$ weight filtration on $H^*(\mathcal{M}_{Betti})$.

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2. $\mathcal{M}_{Dol}$ **- Two geometric methods** (a brief sketch)
3. P=W – A conjecture relating the extra structure on H^*'s

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1. Proper, flat

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Consequence:

$$H^*(\text{Higgs}^{sst}) = H^*(h^{-1}(0))$$

Thus:

E-Polynomial & point counting do determine H^* for $\mathcal{M}_{Dol}$!

Use coprimality and GL_n / assume smoothness

$\mathcal{M}_{DoI}$ – Method 1

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$$X = (X - Z) \coprod Z$$

$\rightsquigarrow$ compute in $\widetilde{K_0(\mathrm{Var})}$.
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2. Use that only obviously unstable points are unstable for *large* parameters.
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This produces an algorithm.

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This produces a formula, similar – but even more complicated than the one conjectured by H–RV.
In the first article the problem is first reformulated in terms of isomorphism classes of indecomposable bundles.

Plan:

1. $\mathcal{M}_{Betti}$ - Method used by Hausel–Rodriguez-Villegas
2. $\mathcal{M}_{Dol}$ - Two geometric methods
3. **P=W – A conjecture relating the extra structure on H^*'s**

More structure obtained from Hitchin's fibration

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P_a perverse sheaves (i.e. complexes determined by a local system on some locally closed subset a of $\mathcal{A}$).

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Example: The local system $H^i(J_a) = \wedge^i H^1(J_a)$ occurs.

$$H^*(\text{Higgs}^{\text{sst}}, \mathbb{C}) = \bigoplus \oplus (P_a[-i])_0$$

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A consequence of $P = W$-conjecture:

$a = 0$ does not occur / Intersection form vanishes.
This can be checked for all n.