Homotopy groups of character varieties

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Outline

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   - Ex: $\mathrm{SL}(n, \mathbb{C})$ or $\mathrm{SL}(n, \mathbb{R})$ or $\mathrm{SU}(n)$ or $\mathrm{SO}(n)$

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Let $R_\Gamma(G) = \mathrm{Hom}(\Gamma, G)$, and denote $\overbrace{G \times G \times \cdots \times G}^{r}$ by G^r.

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- Since G is admits a faithful linear representation $G \hookrightarrow \mathrm{GL}(V)$, $\mathrm{Hom}(\Gamma, G) \subset \mathrm{Hom}(\Gamma, \mathrm{GL}(V))$ is subspace of traditional representations.

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Examples: Punctured Torus T

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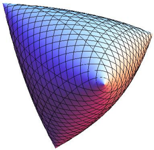

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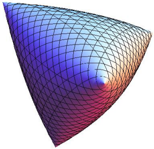
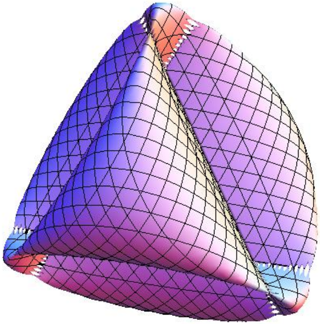


Figure : Solid closed 3-ball.

3. Thus, $\mathfrak{X}_{\mathbb{Z}^2}(\mathrm{SU}(2)) \cong \partial\mathfrak{X}_{\pi_1(T)}(\mathrm{SU}(2)) \cong S^2$.

4. $\mathfrak{X}_{\mathbb{Z}^3}(\mathrm{SU}(2))$ is a 3 dimensional orbifold with 8 singularities; each locally $\mathcal{C}_{\mathbb{R}}(\mathbb{R}P^2)$



If you Google any of the following key words,

you will find that the study of character varieties *at least* touches their corresponding theories:

flat G-bundles, G-Higgs bundles, holomorphic vector bundles, (G, X)-structures, Mirror symmetry, String vacua, Yang-Mills connections, knot invariants, Geometric Langlands, Quantization, Spin Networks, A-polynomial, hyperbolic manifolds

They are even getting famous now.

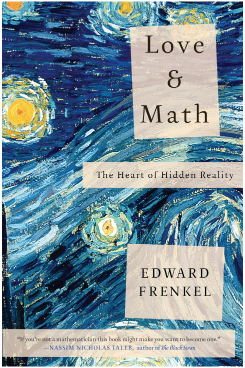


Figure : Character varieties are featured in the recent love story about the Geometric Langlands Program.

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Let G be a real reductive algebraic group, K a maximal compact subgroup, and Γ a (finitely generated) free group or Abelian group. Then $\mathfrak{X}_\Gamma(G)$ strongly deformation retracts onto $\mathfrak{X}_\Gamma(K)$. In particular, they have the same homotopy type.

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- Although particular counter-examples were known much earlier (via Atiyah-Bott 1983 and Hitchin 1987), Biswas-Florentino showed using Higgs bundle theory (2011) that for a closed surface of genus $g \geq 2$ the moduli spaces $\mathfrak{X}_\Gamma(G)$ and $\mathfrak{X}_\Gamma(K)$ are *never* homotopic.

Poincaré Polynomials

The Poincaré polynomial of $\mathfrak{X}_{\mathbb{Z}^{\star r}}(\mathrm{SU}(2))$ was calculated by T. Baird, using methods of equivariant cohomology. His result is that $P_t(\mathfrak{X}_{\mathbb{Z}^{\star r}}(\mathrm{SU}(2))) = 1 + t - \frac{t(1+t^3)^r}{1-t^4} + \frac{t^3}{2}\left(\frac{(1+t)^r}{1-t^2} - \frac{(1-t)^r}{1+t^2}\right).$

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- $P_t(\mathfrak{X}_{\mathbb{Z}^{\star r}}(\mathsf{U}(2,2))) = P_t(\mathfrak{X}_{\mathbb{Z}^{\star r}}(\mathsf{U}(2)))^2$
- $P_t(\mathfrak{X}_{\mathbb{Z}^{\star r}}(\mathsf{O}(3,\mathbb{C}))) = P_t(\mathfrak{X}_{\mathbb{Z}^{\star r}}(\mathsf{O}(3))) = 2^r P_t(\mathfrak{X}_{\mathbb{Z}^{\star r}}(\mathsf{SU}(2)))$

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2. $\pi_0(\mathfrak{X}_\Gamma(G)) = 0$ if the surface is open.
3. Assume Γ is Abelian and G is semisimple. Then, $\pi_0(\mathfrak{X}_\Gamma(G))) = 0$ iff Γ does not have torsion, and one of the following is true: (a) $r := \mathrm{Rank}(\Gamma) = 1$, (b) $r = 2$ and G is simply connected, or (c) $r > 2$ and G is a product of simply connected groups of type A_n or C_n.

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Theorem (Biswas, L-, Ramras, 2014)

Let G be either a connected reductive $\mathbb{C}$-group, or a connected compact Lie group, and let Γ be one the following:

1. *a free group,*
2. *a free Abelian group, or*
3. *the fundamental group of a closed orientable surface.*

Then $\pi_1(\mathfrak{X}^0_\Gamma(G)) = \pi_1(G/[G,G])^r$, where $r = \mathrm{Rank}\,(\Gamma/[\Gamma,\Gamma])$.

Theorem (Florentino, L-, Ramras, 2014)

Let G_n be any of $\mathrm{GL}(n,\mathbb{C}), \mathrm{SL}(n,\mathbb{C}), \mathrm{SU}(n)$ *or* $\mathrm{U}(n)$*. Then*

$$\pi_2(\mathfrak{X}_r(G_n)) = 0.$$

Theorem (Florentino, L-, Ramras, 2014)

Let G_n be any of $\mathrm{GL}(n,\mathbb{C}), \mathrm{SL}(n,\mathbb{C}), \mathrm{SU}(n)$ *or* $\mathrm{U}(n)$*. Then*

$$\pi_2(\mathfrak{X}_r(G_n)) = 0.$$

Theorem (Florentino, L-, Ramras, 2014)

Assume $(r-1)(n-1) \geq 2$ *and* $1 < k < 2(r-1)(n-1)-1$*. Then*

$$\pi_k(\mathfrak{X}_r(G_n)^{irr}) = \begin{cases} \mathbb{Z}/n\mathbb{Z}, & \text{if } k = 2 \\ \mathbb{Z}^r, & \text{if } k \text{ is odd and } k < 2n \\ \mathbb{Z}, & \text{if } k \text{ is even and } 2 < k < 2n \\ (\mathbb{Z}/n!\mathbb{Z})^r \oplus \mathbb{Z}, & \text{if } k = 2n \end{cases}$$

Moreover, $\pi_k(\mathfrak{X}_r(G_n)^{irr})$ *is finite for* $k > 2n$*.*

Thank you!

- References are at

  `http://arxiv.org/a/lawton_s_1`.

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