Asymptotics of certain families of Higgs bundles

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(joint with Brian Collier, UIUC)

AMS-EMS-SPM Porto Meeting– Higgs Bundles and Character Varieties

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Composing with the the unique irreducible representation $PSL(2, \mathbb{R}) \to PSL(n, \mathbb{R})$ we obtain a distinguished component of $Rep_{PSL(n,\mathbb{R})}$, called *Hitchin component*.

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where K is the canonical line bundle over Σ.

The proof of this theorem uses Higgs bundle techniques.The Higgs bundle parametrization of $Hit_n(S)$ is as follows:

$$\text{Bundle: } E = K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus \cdots \oplus K^{-\frac{n-3}{2}} \oplus K^{-\frac{n-1}{2}}$$

$$\text{Higgs field: } \quad \phi = \begin{pmatrix} 0 & \frac{n-1}{2}q_2 & & \cdots & q_{n-1} & q_n \\ 1 & 0 & \frac{n-3}{2}q_2 & \cdots & q_{n-2} & q_{n-1} \\ & \ddots & & \ddots & & \\ & & & & \frac{n-3}{2}q_2 & \\ & & & 1 & 0 & \frac{n-1}{2}q_2 \\ & & & & 1 & 0 \end{pmatrix}$$

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Let (E, ϕ) be a stable $SL(n, \mathbb{C})$-Higgs bundle, then there exists a unique metric h on E, solving the Hitchin equation

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Conversely, if (A_h, ϕ) solves Hitchin equation, then the Higgs bundle (E, ϕ) is polystable.

If (A_h, ϕ) solves Hitchin equation, then $A_h + \phi + \phi^{*_h}$ is a **flat** $SL(n, \mathbb{C})$-connection.

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• To answer the above questions, the first step is to solve Hitchin equation asymptotically. This is in general impossible.

• However, we manage to understand the two cases:
(1) $t(0, \cdots, q_n)$
(2) $t(0, \cdots, q_{n-1}, 0)$.

Why these two cases?

Theorem (Baraglia n, Collier $n-1$)

For $(0, \cdots, q_n)$ and $(0, \cdots, q_{n-1}, 0)$, the metric solving Hitchin equation is diagonal on the line bundles

$$K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus \cdots \oplus K^{-\frac{n-3}{2}} \oplus K^{-\frac{n-1}{2}}$$

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Since the metric is diagonal, for $(0,\cdots,q_n)$, the equations

$$(\star)\qquad \begin{cases} F_{A_h}+[\phi,\phi^{*_h}]=0\\ \nabla^{0,1}_{A_h}\phi=0\end{cases}$$

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$$\begin{cases} F_{A_1} + t^2 h_1^2 q_n \wedge \bar{q}_n - h_1^{-1} h_2 = 0 \\ F_{A_j} + h_{j-1}^{-1} h_j - h_j^{-1} h_{j+1} = 0 \qquad 1 < j < \dfrac{n}{2} \\ F_{A_{\frac{n}{2}}} + h_{\frac{n}{2}-1}^{-1} h_{\frac{n}{2}} - h_{\frac{n}{2}}^{-2} = 0 \end{cases}$$

Metric Asymptotics

Theorem (Collier-L)

For $(0, \cdots, tq_n) \in \bigoplus_{j=2}^{n} H^0(K^j)$, *at any point* $p \in \Sigma$ *away from the zeros of* q_n, *as* $t \to \infty$, *the metric* $h_j(t)$ *on* $K^{\frac{n+1-2j}{2}}$ *admits the expansion*

$$h_j(t) = (t|q_n|)^{-\frac{n+1-2j}{n}} \left(1 + O\left(t^{-\frac{2}{n}}\right)\right) \qquad \text{for all } j$$

The analogous result is also true for $(0, \cdots, tq_{n-1}, 0)$.

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- Main Tool: the maximum principle (numerous times)

Parallel Transport Asymptotics

Given a path $\gamma : [0, L] \to \widetilde{\Sigma}$, let $T_\gamma(t) : E_{\gamma(0)} \to E_{\gamma(L)}$ be parallel transport operators along γ for flat connections ∇_t.

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- Both theorems are proved in $(0, q_3)$ case by J.Loftin.

Work in Progress (Quiver bundles)

Instead of thinking Higgs bundle

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Application: Understanding asymptotics of maximal representation in $Sp(4, \mathbb{R})$.

Thank You!