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Bohr-Sommerfeld Lagrangians for the moduli spaces of Higgs bundles

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Motivation: Geometric Quantization

- **Quantization**

  (M, ω) classical mechanics $\longrightarrow$ $Q(M)$ wave functions

- Geometric quantization
  - **Prequantum line bundle** $(\mathcal{L}, D)$ s.t. $F_D = \omega$.
- Polarizations
  - Kahler polarization, Real polarization, …
  - **Real polarization**: a foliation of M into Lagrangian sub manifolds.
- **Theorem: (Sniaticky)** (1975) $p : M^{2n} \longrightarrow B^n$ with compact fibres

  $\mathcal{J}$ sheaf of leafwise flat sections of $\mathcal{L}$

$$Q(M) = H^n(M, \mathcal{J}) \cong \bigoplus_{\sharp BS} \mathbb{C}$$

Higgs bundles over curves

- X compact connected Riemann surface of genus $g \geq 2$
- A Higgs bundle over X is (E, ϕ) where $E \longrightarrow X$ is a holomorphic vector bundle and the so called Higgs field is

$$\phi \in H^0(X, End(E) \otimes K_X)$$

- The moduli space of Higgs bundles was introduced by Hitchin in 1987 as moduli space for solutions to a dimensional reduction of Yang-Mills equations
- GIT: A (semi) stable Higgs bundle is (E, ϕ) such that for any sub-Higgs bundle $(F, \phi) \subset (E, \phi)$ then

$$\mu(F) < (\leq) \mu(E)$$

Classic mechanical system

- Higgs bundles appear as dimensional reduction of Yang Mills equations

$$S(A) := \int_X F_A \wedge \star F_A$$

- $M_H(r,d)$ moduli space of **semistable** Higgs bundles,
  - irreducible normal complex projective variety of complex dimension

$$\dim M_H(r,d) = 2(r^2(g-1)+1)$$

- The moduli space has a natural symplectic structure as cotangent space
  - $M^s(r,d)$ moduli space of stable vector bundles

$$T^* M^s(r,d) \subset M_H(r,d)$$

- Correspondences, (Jeffrey-Weitsman: BS for Flat SU(2) connections)

$$M_H(r,d) \cong \{A \,|\, F_A = 0\}/\mathcal{G} \cong R(\pi_1(X), GL(r,\mathbb{C}))$$

BS-Lagrangians

- compact Lagrangian is a reduced irreducible compact complex analytic subset $\mathbb{L} \subset M_H(r,d)$
  - $\dim(\mathbb{L}) = \dim M_H(r,d)/2$
  - $\mathcal{L}|_{\mathbb{L}}$ flat line bundle, i.e. $F_D|_{\mathbb{L}} = 0$
- compact Bohr-Sommerfeld Lagrangian is a compact Lagrangian such that
  - $\mathcal{L}$ admits a non-zero flat section over $\mathbb{L}$ i.e.

$$F_D|_{\mathbb{L}}(s) = 0 \quad s \neq 0$$

Prequantization bundle

- $(M, \omega) = M_H(r, d)$

is equipped with an algebraic symplectic form $\omega = d\alpha$ where α is a canonical one-form. Define a **prequantization bundle**

$$\mathcal{L} := \mathcal{O}_{M_H(r,d)} = M_H(r, d) \times \mathbb{C}$$
$$D := d + \alpha$$
$$F_D = d\alpha$$

- in this context, a Lagrangian is $i^*\omega = i^*(d\alpha) = 0$
- and a BS Lagrangian $i^*\alpha = 0$ where $i : \mathbb{L} \hookrightarrow M_H(r, d)$

Hitchin map

- $M_H(r,d)$ is provided with a fibration with compact fibres

$$H : M_H(r,d) \longrightarrow \mathbb{A}^{\dim M_H(r,d)/2} = \bigoplus_{i=1}^{r} H^0(X, K_X^{\otimes i})$$

- For any $t \in \mathbb{A}^{\dim M_H(r,d)/2}$, $H^{-1}(t)$ is a compact abelian algebraic variety.
- Each $H^{-1}(t)$ is a Lagrangian for the natural symplectic form on $M_H(r,d)$.
- The fibre $H^{-1}(0)$ is known as the **nilpotent cone**.

Theorem

- The **compact Bohr-Sommerfeld Lagrangians** in $M_H(r,d)$ are precisely the irreducible components of the nilpotent cone.
  - It generalises to **parabolic Higgs bundles** and **principal G-Higgs** bundles for G reductive complex algebraic Lie group.

Natural algebraic one-form (I)

- deformation complex $C^\bullet$: $\quad C^0 := End(E) \longrightarrow C^1 := End(E) \otimes K_X$
- and the natural morphism of complexes

$$\begin{array}{ccc} End(E) & \to & End(E) \otimes K_X \\ \downarrow & & \downarrow \\ End(E) & \to & 0 \end{array}$$

induces $\quad q : \mathbb{H}^1(C^\bullet) \longrightarrow H^1(X, End(E))$

- Serre duality pairing

$$SD : H^1(X, End(E)) \otimes H^0(X, End(E) \otimes K_X) \longrightarrow \mathbb{C}$$

- gives a homomorphism $\quad \alpha : \mathbb{H}^1(C^\bullet) \longrightarrow \mathbb{C}$

$$\alpha(E, \phi)(v) = SD(q(v) \otimes \phi)$$

Natural algebraic one-form (II)

- α is the Liouville form on $M_H(r,d)$, so $d\alpha = \omega$
- The $\mathbb{C}^*$ action $$T_{\lambda\in\mathbb{C}^*} : M_H(r,d) \longrightarrow M_H(r,d)$$ $$(E,\phi) \mapsto (E,\lambda\phi)$$
- **Fact:** for any $\xi \in H^0(M_H(r,d), T\,M_H(r,d))$ then $\imath_\xi\, d\alpha = \alpha$
  - from definitions $\alpha(\xi) = 0$
  - $T_\lambda^*\alpha = \lambda\,\alpha$ therefore $\alpha = L_\xi\alpha$
  - $\alpha = L_\xi\alpha = \imath_\xi d\alpha + d\,\imath_\xi\alpha = \imath_\xi d\alpha\,.$

The $\mathbb{C}^*$ action on $M_H(r,d)$.

- **Proposition:** For a BS Lagrangian for all $z \in \mathbb{L}$ then $\xi(z) \in T_z\mathbb{L}$
  - $\alpha(v) = 0$,
  - $\omega(\xi(z), v) = d\alpha(\xi(z), v) = \imath_\xi d\alpha(v) = \alpha(v) = 0$.
- **Proposition:** Any BS Lagrangian is contained in $H^{-1}(0)$.
  - consider the $\mathbb{C}^*$ action on the Hitchin base
  - H is $\mathbb{C}^*$ equivariant
  - $\mathbb{L}$ does not admit non constant holomorphic maps, in particular , $H \circ i$ is constant so $i(\mathbb{L}) \subset H^{-1}(t)$ for some t.
  - $i(\mathbb{L})$ is preserved by the $\mathbb{C}^*$ action so $t = 0$.

- **Theorem (Biswas, Gammergaard, L.)**
  Bohr-Sommerfeld Lagrangians are precisely the irreducible components of the nilpotent cone.
  - take $\mathbb{L}$ an irreducible component we need to prove that for any $z \in \mathbb{L}$ and $v \in T_z\mathbb{L}$ then $\alpha(v) = 0$
  - $\alpha(v) = \imath_\xi d\alpha(v) = d\alpha(\xi(z), v)$ and $\xi(z) \in T_z\mathbb{L}$ since $\mathbb{L}$ is closed for the $\mathbb{C}^*$action
  - since $\mathbb{L}$ is lagrangian $\omega(\xi(z), v) = d\alpha(\xi(z), v) = 0$ so then $\alpha(v) = 0$

THANK YOU!