E-polynomials of $\mathrm{SL}(2, \mathbb{C})$-character varieties

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Character varieties

Given a finitely generated group Γ

$$\Gamma = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_s \rangle$$

and a complex reductive Lie group G, we can consider the representation space

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Definition (Character variety)

The G-character variety is the GIT quotient

$$\mathcal{M}_G(\Gamma) := R_G(\Gamma)//G$$

We will focus on the following set of character varieties: Let X be a compact Riemann surface of genus $g \geq 1$, fix $p \in X$, take $C \in G$, $C \in [\mathcal{C}]$ and let γ be a small loop around p.
Then

$$\begin{aligned}\mathcal{M}_C &= \{\rho : \pi_1(X \setminus \{p\}) \to G \mid \rho(\gamma) \in \mathcal{C}\}//G \\ &= \{(A_1, B_1, \ldots, A_g, B_g) \in G^{2g} \mid \prod [A_i, B_i] \in \mathcal{C}\}//G \\ &= \{(A_1, B_1, \ldots, A_g, B_g) \in G^{2g} \mid \prod [A_i, B_i] = C\}//\mathrm{Stab}(C)\end{aligned}$$

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The E-polynomial

$$e_X \in \mathbb{Z}[u, v]$$

encodes topological, algebraic and arithmetic information of $\mathcal{M}_C(G)$.

Hodge decomposition

Let X be a complex compact Kähler manifold of dimension n.
A classic result is

Theorem(Hodge decomposition)

For each $0 \leq i \leq 2n$,

$$H^i(X, \mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}$$

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Theorem (Hodge decomposition)

The k-th cohomology group of a complex compact Kähler manifold has a pure Hodge structure of weight k

Define the *Hodge numbers*:

$$h^{p,q} := \dim_{\mathbb{C}} H^{p,q}$$

Mixed Hodge structures

Question

Can this setup be extended to other situations (non-compact or singular spaces)?

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The k-th cohomology group $H^k(X, \mathbb{C})$ *of a quasi-projective algebraic variety is endowed with a mixed Hodge structure.*

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Hodge numbers

$H^k(X, \mathbb{C})$ is endowed with two filtrations, $W^\bullet, F_\bullet$ that allow to define

$$h^{k,p,q} := \dim H^{p,q}(H^k(X, \mathbb{C})) = Gr_F^p Gr_{p+q}^W (H^k(X, \mathbb{C}))$$

Observation: $h^{2,0,0}, h^{2,1,1}, h^{2,3,0}$ make sense, $p + q \neq k$.

E-polynomials

Definition

Define $\chi_c^{p,q}(X) = \sum_k (-1)^k h_c^{k,p,q}$. The E-polynomial is defined as

$$e(X)(u,v) = \sum \chi_c^{p,q}(X) u^p v^q \in \mathbb{Z}[u,v]$$

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Properties

- Easier to compute.
- Additivity: If $Z = \sqcup Z_i$, Z_i locally closed, $e(Z) = \sum e(Z_i)$.
- Topological information:
  - $e(1,1) = \chi(X)$.
  - *leadcoef* (e) equals the number of irreducible components.
- Arithmetic information.

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When X is of balanced type /Hodge-Tate type ($H^{k,p,q} = 0$ for $p \neq q$),

$$e(X)(u,v) = \sum \chi_c^{p,q} u^p v^q = \sum \chi_c^{p,p} q^p \in \mathbb{Z}[q]$$

Motivation

Take $G = GL(n, \mathbb{C}), SL(n, \mathbb{C})$, the non-abelian Hodge correspondence

$$\mathcal{M}_H(G) \overset{\cong}{\longleftrightarrow} \mathcal{M}_{dR}(G) \overset{\cong}{\longleftrightarrow} \mathcal{M}_B(G)$$

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Take $G = GL(n,\mathbb{C}), SL(n,\mathbb{C})$, the non-abelian Hodge correspondence

$$\mathcal{M}_H(G) \xleftrightarrow{\cong} \mathcal{M}_{dR}(G) \xleftrightarrow{\cong} \mathcal{M}_B(G)$$

stablishes:

- Homeomorphisms $\mathcal{M}_H(G) \cong \mathcal{M}_{dR}(G) \cong \mathcal{M}_B(G)$ (same Betti numbers)
- Not algebraic (different Mixed Hodge numbers)
  - $\mathcal{M}_H(G), \mathcal{M}_{dR}(G)$. Pure HS, not Hodge-Tate ($h^{k,p,q} = 0$ if $p + q \neq k$.)
  - $\mathcal{M}_B(G)$. Mixed HS, Hodge-Tate ($h^{k,p,q} = 0$ if $p \neq q$).
- Several conjectures (relation with Mirror Symmetry).

Geometric technique

The geometric approach uses:

- Set of coordinates in G^k.
- Stratifications of $R_X(G)$ given by trace maps.
- Equivariant E-polynomials (to compute quotients under finite groups $e(X/G)$)
- Analytic locally trivial fibrations

$$F \longrightarrow E \longrightarrow B$$

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$$F \longrightarrow E \longrightarrow B$$

If there is no monodromy,

$$e(E) = e(F)e(B)$$

Fibrations with monodromy

When the action of $\pi_1(B)$ on $H^*(F)$ is not trivial, there is a representation:

$$\rho : \pi_1(B) \longrightarrow GL(H_c^k(F, \mathbb{C}))$$

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There is a diagram, for each p, k

$$\begin{tikzcd}
\rho : \pi_1(B) \arrow[rr, "\rho"] \arrow[dr] & & GL(H_c^{k,p,p}(F)) \\
& \Gamma \arrow[ur] &
\end{tikzcd}$$

such that $H_c^{k,p,p}(F)$ are modules over Γ.

The Hodge monodromy representation

Let $R(\Gamma)$ be the representation ring of Γ.

Definition

The Hodge monodromy representation is given by the polynomial,

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Proposition

If

$$R(E) = a_1(q)S_1 + \ldots + a_n(q)S_q \in R(\Gamma)[q]$$

there exist E-polynomials $s_1, \ldots s(q) \in \mathbb{Z}[q]$ such that

$$e(E) = a_1(q)s_1 + \ldots + a_n(q)s_q \in \mathbb{Z}[q]$$

Step 1. Building blocks $g = 1, 2, 3$.

Write $\mathrm{Id}, -\mathrm{Id}, J_+, J_-, \xi_\lambda$ for the conjugacy types in $\mathrm{SL}(2, \mathbb{C})$, where

$$\xi_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

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The building blocks correspond to the representation spaces of curves of genus 1 and 2,

$$X_C^{g=1} = \{(A, B, C) \in \mathrm{SL}(2,\mathbb{C})^3 \mid [A, B] = C\}$$

and the Hodge monodromy representation corresponding to the fibration

$$X_{\xi_\lambda}^{g=1} \longrightarrow \mathbb{C} \setminus \{0, \pm 1\}$$

and the quotient by the $\mathbb{Z}_2$-action taking $\lambda \to \lambda^{-1}$,

$$X_{\xi_\lambda}^{g=1}/\mathbb{Z}_2 \longrightarrow \mathbb{C} \setminus \{\pm 2\}$$

An example: $\mathcal{M}_{J_+}$ for $g = 1$

- $R(X_{\xi_\lambda}) \in R(\mathbb{Z}_2)[q]$, $R(X_{\xi_\lambda}/\mathbb{Z}_2) \in R(\mathbb{Z}_2 \times \mathbb{Z}_2)[q]$.

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$g = 1$, $C = J_+$

If we take explicit equations

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad B = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

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X_{J_+} is a bundle Y of complex lines

$$\mathbb{C} \longrightarrow X_{J_+}^{g=1} \longrightarrow \mathbb{C}^* \times \mathbb{C}^* - \{(\pm 1, \pm 1)\}$$

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$$e(\mathcal{M}_{J_+}^{g=1}) = e(X_{J_+}^{g=1}/\mathrm{Stab}\ J_+) = (q-1)^2 - 4 = q^2 + 2q - 3$$

Step 2. The induction $g > 3$.

The information for genus g is codified in 8 polynomials,

$$e^g_{\mathrm{Id}}, e^g_{-\mathrm{Id}}, e^g_{J_+}, e^g_{J_-}$$

where $e^g_C = e(\mathcal{M}^g_C)$ and a^g, b^g, c^g, d^g

$$R(X_{\xi_\lambda}/\mathbb{Z}_2) = a^g T + b^g N_1 + c^g N_2 + d^g N_{12} \in R(\mathbb{Z}_2 \times \mathbb{Z}_2)[q]$$

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The idea is to decompose the surface of genus g Σ^g, as

$$\Sigma^g = \Sigma^{g-1} \# \Sigma^1 = \Sigma^{g-2} \# \Sigma^2 = \dots$$

and carry this decomposition to the representation spaces X^g_C.

Step 3. The induction.

The different decompositions correspond to the algebraic identities

$$\prod_{i=1}^{k+h}[A_i, B_i] = C \Longleftrightarrow \prod_{i=1}^{k}[A_i, B_i] = C \prod_{j=k+1}^{k+h}[B_i, A_i]$$

which stratify the representation spaces X_C^g in terms of the lower genus cases X_C^k, X_C^h.

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For example,

$$\begin{aligned} e_{\mathrm{Id}}^g = &(q^4 + 4q^3 - q^2 - 4q)e_{\mathrm{Id}}^{g-1} + (q^3 - q)e_{-\mathrm{Id}}^{g-1} \\ &+ (q^5 - 2q^4 - 4q^3 + 2q^2 + 3q)e_{J_+}^{g-1} + (q^5 + 3q^4 - q^3 - 3q^2)e_{J_-}^{g-1} \\ &+ (q^6 - 2q^5 - 4q^4 + 3q^2 + 2q)a^{g-1} + (-q^5 - 4q^4 + 4q^2 + q)b^{g-1} \\ &+ (2q^5 - 7q^4 - 3q^3 + 7q^2 - q)c^{g-1} + (-5q^4 - q^3 + 5q^2 - q)d^{g-1}. \end{aligned}$$

Step 3. The induction

This procedure works for

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For a^g, b^g, c^g, d^g, we can use the previous equations for $g+1$ and obtain

$$\begin{aligned} e^{g+1}_{\mathrm{Id}} &= e^{g+1}_{\mathrm{Id}}(e^g_{\mathrm{Id}}, e^g_{-\mathrm{Id}}, e^g_{J_+}, e^g_{J_-}, a^g, b^g, c^g, d^g) \\ &= e^{g+1}_{\mathrm{Id}}(e^{g-1}_{\mathrm{Id}}, e^{g-1}_{-\mathrm{Id}}, e^{g-1}_{J_+}, e^{g-1}_{J_-}, a^{g-1}, b^{g-1}, c^{g-1}, d^{g-1}) \end{aligned}$$

equations for a_g, b_g, c_g, c_g in terms of the data of genus $g-1$.

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equations for a_g, b_g, c_g, c_g in terms of the data of genus $g-1$.
Two more equations for $a^g + d^g$ and $b^g + c^g$ come from the analysis of the fibration given by $X^g_{\xi_\lambda}$

Step 3. The induction

If we codify the information of the E-polynomials for genus g in a vector

$$L^g = (e^g_{\mathrm{Id}}, e^g_{Id}, e^g_{J_+}, e^g_{J_-}, a^g, b^g, c^g, d^g)$$

there is an 8×8 matrix Q with entries in $\mathbb{Z}[q]$ such that:

$$L^g = QL^{g-1}$$

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This relation allows to:

- Reproduce the topological handle attachment by a linear map of E-polynomials.
- Obtain closed formulas for L^g.

The glueing matrix

$$\begin{pmatrix}
\begin{matrix}q^4+4q^3\\-q^2-4q\end{matrix} & q^3-q & \begin{matrix}q^5-2q^4-4q^3\\+2q^2+3q\end{matrix} & \begin{matrix}q^5+3q^4\\-q^3-3q^2\end{matrix} & \begin{matrix}q^6-2q^5-4q^4\\+3q^2+2q\end{matrix} & \begin{matrix}-q^5-4q^4\\+4q^2+q\end{matrix} & \begin{matrix}2q^5-7q^4-3q^3\\+7q^2+q\end{matrix} & \begin{matrix}-5q^4-q^3\\+5q^2+q\end{matrix} \\
q^3-q & \begin{matrix}q^4+4q^3\\-q^2-4q\end{matrix} & \begin{matrix}q^5+3q^4\\-q^3-3q^2\end{matrix} & \begin{matrix}q^5-2q^4-4q^3\\+2q^2+3q\end{matrix} & \begin{matrix}q^6-2q^5-4q^4\\+3q^2+2q\end{matrix} & \begin{matrix}2q^5-7q^4-3q^3\\+7q^2+q\end{matrix} & \begin{matrix}-q^5-4q^4\\+4q^2+q\end{matrix} & \begin{matrix}-5q^4-q^3\\+5q^2+q\end{matrix} \\
\begin{matrix}q^3-2q^2\\-3q\end{matrix} & q^3+3q^2 & \begin{matrix}q^5+q^4\\+3q^2+3q\end{matrix} & \begin{matrix}q^5-3q^3\\-6q^2\end{matrix} & \begin{matrix}q^6-2q^5-3q^4\\+q^3+3q^2\end{matrix} & \begin{matrix}-q^5+2q^4\\-4q^3+3q^2\end{matrix} & \begin{matrix}-q^5-q^4\\-4q^3+6q^2\end{matrix} & \begin{matrix}-2q^4-q^3\\+3q^2\end{matrix} \\
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q^3 & q^3 & q^5-3q^3 & q^5-3q^3 & \begin{matrix}q^6-2q^5-2q^4\\+4q^3+q^2\end{matrix} & \begin{matrix}-q^5-q^4\\+2q^3\end{matrix} & \begin{matrix}-q^5-q^4\\+2q^3\end{matrix} & -2q^4 \\
-3q & 3q^2 & 3q^2+3 & -6q^2 & -3q^3+3q^2 & 4q^4-6q^3+4q^2 & -8q^3+6q^2 & -3q^3+3q^2 \\
3q^2 & -3q & -6q^2 & 3q^3+3q & -3q^3+3q^2 & -8q^3+6q^2 & 4q^4-6q^3+4q^2 & -3q^3+3q^2 \\
-1 & -1 & 2q^2 & 2q^2 & -4q^2+2 & -2q^2+q+1 & -2q^2+q+1 & \begin{matrix}q^4-2q^2\\+2q+1\end{matrix}
\end{pmatrix}$$

Main theorem

Theorem

Let X be a complex curve of genus $g \geq 1$. Let $\mathcal{M}_C = \mathcal{M}_C(\mathrm{SL}(2,\mathbb{C}))$ be the character variety corresponding to $C \in \mathrm{SL}(2,\mathbb{C})$. Then:

$$\begin{aligned}
e(\mathcal{M}_{\mathrm{Id}}) = {} & (q^3-q)^{2g-2} + (q^2-1)^{2g-2} - q(q^2-q)^{2g-2} - 2^{2g}q^{2g-2} \\
& + \frac{1}{2}q^{2g-2}(q+2^{2g}-1)((q+1)^{2g-2}+(q-1)^{2g-2}) \\
& + \frac{1}{2}q((q+1)^{2g-1}+(q-1)^{2g-1}), \\
e(\mathcal{M}_{-\mathrm{Id}}) = {} & (q^3-q)^{2g-2} + (q^2-1)^{2g-2} - 2^{2g-1}(q^2+q)^{2g-2} + (2^{2g-1}-1)(q^2-q)^{2g-2}, \\
e(\mathcal{M}_{J_+}) = {} & (q^3-q)^{2g-2}(q^2-1) + (2^{2g-1}-1)(q-1)(q^2-q)^{2g-2} \\
& - 2^{2g-1}(q+1)(q^2+q)^{2g-2} + \frac{1}{2}q^{2g-2}(q-1)\left((q-1)^{2g-1}-(q+1)^{2g-1}\right), \\
e(\mathcal{M}_{J_-}) = {} & (q^3-q)^{2g-2}(q^2-1) + (2^{2g-1}-1)(q-1)(q^2-q)^{2g-2} \\
& + 2^{2g-1}(q+1)(q^2+q)^{2g-2}, \\
e(\mathcal{M}_{\xi_\lambda}) = {} & (q^3-q)^{2g-2}(q^2+q) + (q^2-1)^{2g-2}(q+1) + (2^{2g}-2)(q^2-q)^{2g-2}q.
\end{aligned}$$

Corollaries

- All character varieties are of balanced/Hodge-Tate type.
- Euler characteristics of $\mathrm{SL}(2,\mathbb{C})$-character varieties.
- Number of irreducible components.
- Relations between E-polynomials

$$e(\mathcal{M}_{J_-}) + (q+1)e(\mathcal{M}_{-\mathrm{Id}}) = e(\mathcal{M}_{\xi_\lambda})$$

- Behaviour of the parabolic family $\mathcal{M}_{\xi_\lambda} \to \mathbb{C} - \{0, \pm 1\}$,

$$R(\mathcal{M}_{par}) = \left((q^3-q)^{2g-2}(q^2+q) + (q+1)(q^2-1)^{2g-2} - q(q^2-q)^{2g-2}\right) T \\ + \left((2^{2g}-1)q(q^2-q)^{2g-2}\right) N.$$

- Study of other situations ($PGL(2,\mathbb{C})$, higher ranks, non-orientable surfaces, other Γ).