Geometric methods for character varieties

Vicente Muñoz

Universidad Complutense de Madrid

2015 International Meeting AMS-EMS-SPM
Special session on “Higgs bundles and character varieties”
8-12 June 2015

Character varieties

$\Gamma = \langle x_1, \ldots, x_k | r_1, \ldots, r_s \rangle$ finitely presented group

$G = \mathrm{SL}(r, \mathbb{C})$, $\mathrm{GL}(r, \mathbb{C})$, $\mathrm{PGL}(r, \mathbb{C})$ complex Lie group

$$\begin{aligned} R(\Gamma, G) &= \mathrm{Hom}(\Gamma, G) \\ &= \{(A_1, \ldots, A_k) \in G^k | r_j(A_1, \ldots, A_k) = \mathrm{Id}, 1 \leq j \leq s\} \end{aligned}$$

Character variety or moduli space of representations:
$M(\Gamma, G) = R(\Gamma, G)//G$

Character varieties

Important cases:

Knot group

Knot $K \subset S^3$

$\Gamma_K = \pi_1(S^3 - K)$ fundamental group of the knot exterior

Surface groups

X compact oriented surface of genus $g \geq 1$, $p \in X$, γ loop around p.
$C \in G$, $\mathcal{C} = [C]$ conjugacy class.

$$\begin{aligned} M_C^g(G) &= \{\rho : \pi_1(X - \{p\}) \to G | \rho(\gamma) \in \mathcal{C}\}//G \\ &= \{(A_1, B_1, \ldots, A_g, B_g) \in G^{2g} | \prod [A_i, B_i] \in \mathcal{C}\}//G \\ &= \{(A_1, B_1, \ldots, A_g, B_g) \in G^{2g} | \prod [A_i, B_i] = C\}//\mathrm{stab}(C) \end{aligned}$$

Character varieties

Important cases:

Knot group

Knot $K \subset S^3$
$\Gamma_K = \pi_1(S^3 - K)$ fundamental group of the knot exterior

Surface groups

X compact oriented surface of genus $g \geq 1$, $p \in X$, γ loop around p.
$C \in G$, $\mathcal{C} = [C]$ conjugacy class.

$$\begin{aligned} M^g_C(G) &= \{\rho : \pi_1(X - \{p\}) \to G | \rho(\gamma) \in \mathcal{C}\}//G \\ &= \{(A_1, B_1, \ldots, A_g, B_g) \in G^{2g} | \prod [A_i, B_i] \in \mathcal{C}\}//G \\ &= \{(A_1, B_1, \ldots, A_g, B_g) \in G^{2g} | \prod [A_i, B_i] = C\}//\mathrm{stab}(C) \end{aligned}$$

Character varieties

Technique

Geometric analysis by explicit use of coordinates in G^k.

Questions on character varieties

- E-polynomials
- K-theory class in Grothendieck ring $K(\mathcal{V}ar)$
- Explicit description of character varieties (components, dimensions, intersections, etc)
- Defining equations, e.g. in terms of characters (traces of matrices)

Torus knots

1. V. Muñoz, The $\mathrm{SL}(2,\mathbb{C})$-character varieties of torus knots, Rev. Mat. Complut. 22, 2009, 489-497.
2. J. Martínez and V. Muñoz, The $\mathrm{SU}(2)$-character varieties of torus knots, Rocky Mountain J. Math. 2015.
3. V. Muñoz and J. Porti, Geometry of the $\mathrm{SL}(3,\mathbb{C})$-character variety of torus knots, Algebraic & Geometric Topol.
4. V. Muñoz and J. Sánchez, Equivariant motive of the $\mathrm{SL}(3,\mathbb{C})$-character variety of torus knots, Volume in honour of J.M. Montesiones, Publ. UCM.

$K_{m,n}$ torus knot of type (m,n)
$\Gamma_{m,n} = \pi_1(S^3 - K_{m,n}) = \langle x, y \,|\, x^n = y^m \rangle$
$M(\Gamma_{m,n}, G) = \{(A,B) \in G^2 | A^n = B^m\}//G$

Torus knots

1. V. Muñoz, The $\mathrm{SL}(2,\mathbb{C})$-character varieties of torus knots, Rev. Mat. Complut. 22, 2009, 489-497.
2. J. Martínez and V. Muñoz, The $\mathrm{SU}(2)$-character varieties of torus knots, Rocky Mountain J. Math. 2015.
3. V. Muñoz and J. Porti, Geometry of the $\mathrm{SL}(3,\mathbb{C})$-character variety of torus knots, Algebraic & Geometric Topol.
4. V. Muñoz and J. Sánchez, Equivariant motive of the $\mathrm{SL}(3,\mathbb{C})$-character variety of torus knots, Volume in honour of J.M. Montesiones, Publ. UCM.

$K_{m,n}$ torus knot of type (m,n)
$\Gamma_{m,n} = \pi_1(S^3 - K_{m,n}) = \langle x, y \,|\, x^n = y^m \rangle$
$M(\Gamma_{m,n}, G) = \{(A,B) \in G^2 | A^n = B^m\}//G$

Theorem (M-Porti)

The character variety of the torus knot of type (m,n) (with m odd) for $G=\mathrm{SL}(3,\mathbb{C})$ has the following components:

- *One component consisting of totally reducible representations, isomorphic to $\mathbb{C}^2$.*
- $[\frac{n-1}{2}][\frac{m-1}{2}]$ *components consisting of partially reducible representations, each isomorphic to* $(\mathbb{C}-\{0,1\})\times\mathbb{C}^*$.
- *If n is even, there are $(m-1)/2$ extra components consisting of partially reducible representations, each isomorphic to* $\{(u,v)\in\mathbb{C}^2 | v\neq 0, v\neq u^2\}$.
- $\frac{1}{12}(n-1)(n-2)(m-1)(m-2)$ *componens of dimension* 4, *consisting of irreducible representations, all isomorphic to* $GL(3,\mathbb{C})//T\times_D T$.
- $\frac{1}{2}(n-1)(m-1)(n+m-4)$ *components consisting of irreducible representations, each isomorphic to* $(\mathbb{C}^*)^2-\{x+y=1\}$.

$\Longrightarrow$ K-theory class of $M(\Gamma_{m,n},\mathrm{SL}(3,\mathbb{C}))$, which recovers m,n

Theorem (M-Porti)

The character variety of the torus knot of type (m,n) (with m odd) for $G = \mathrm{SL}(3,\mathbb{C})$ has the following components:

- *One component consisting of totally reducible representations, isomorphic to $\mathbb{C}^2$.*
- $[\frac{n-1}{2}][\frac{m-1}{2}]$ *components consisting of partially reducible representations, each isomorphic to $(\mathbb{C}-\{0,1\})\times\mathbb{C}^*$.*
- *If n is even, there are $(m-1)/2$ extra components consisting of partially reducible representations, each isomorphic to $\{(u,v)\in\mathbb{C}^2 | v\neq 0, v\neq u^2\}$.*
- $\frac{1}{12}(n-1)(n-2)(m-1)(m-2)$ *componens of dimension 4, consisting of irreducible representations, all isomorphic to $GL(3,\mathbb{C})//T\times_D T$.*
- $\frac{1}{2}(n-1)(m-1)(n+m-4)$ *components consisting of irreducible representations, each isomorphic to $(\mathbb{C}^*)^2-\{x+y=1\}$.*

$\Longrightarrow$ K-theory class of $M(\Gamma_{m,n},\mathrm{SL}(3,\mathbb{C}))$, which recovers m,n

Theorem (M-Porti)

The character variety $M(\Gamma_{m,n}, \mathrm{SL}(r,\mathbb{C}))$ has dimension $\leq (r-1)^2$. For $r \geq 3$, the number of irreducible components of this dimension is

$$\frac{1}{r}\binom{n-1}{r-1}\binom{m-1}{r-1}$$

Demostración.

Take (A, B) irreducible representation, $A^n = B^m$

A, B diagonalize in different basis

$A \sim \mathrm{diag}(\epsilon_1, \ldots, \epsilon_r)$, $B \sim \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_r)$

$M \in \mathrm{GL}(r, \mathbb{C})$ matrix comparing the two basis

Count: $\epsilon_1 \cdots \epsilon_r = 1$, ϵ_i distinct,

$\varepsilon_1 \cdots \varepsilon_r = 1$, ε_j distinct,

$\epsilon_i^n = \varepsilon_j^m = \varpi$, $\varpi^r = 1$.

The maximal dimensional component is isomorphic to $\mathrm{GL}(r, \mathbb{C})//(T \times_D T)$, where $T \cong (\mathbb{C}^*)^r$ are the diagonal matrices, acting on the left and the right, and $D = \mathbb{C}^* \cdot \mathrm{Id}$ □

Theorem (M-Porti)

The character variety $M(\Gamma_{m,n}, \mathrm{SL}(r,\mathbb{C}))$ has dimension $\leq (r-1)^2$. For $r \geq 3$, the number of irreducible components of this dimension is

$$\frac{1}{r}\binom{n-1}{r-1}\binom{m-1}{r-1}$$

Demostración.

Take (A, B) irreducible representation, $A^n = B^m$
A, B diagonalize in different basis
$A \sim \mathrm{diag}(\epsilon_1, \ldots, \epsilon_r)$, $B \sim \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_r)$
$M \in \mathrm{GL}(r, \mathbb{C})$ matrix comparing the two basis
Count: $\epsilon_1 \cdots \epsilon_r = 1$, ϵ_i distinct,
$\varepsilon_1 \cdots \varepsilon_r = 1$, ε_j distinct,
$\epsilon_i^n = \varepsilon_j^m = \varpi$, $\varpi^r = 1$.
The maximal dimensional component is isomorphic to $\mathrm{GL}(r, \mathbb{C})//(T \times_D T)$, where $T \cong (\mathbb{C}^*)^r$ are the diagonal matrices, acting on the left and the right, and $D = \mathbb{C}^* \cdot \mathrm{Id}$ $\square$

Theorem (M-Porti)

The character variety $M(\Gamma_{m,n}, \mathrm{SL}(r, \mathbb{C}))$ has dimension $\leq (r-1)^2$. For $r \geq 3$, the number of irreducible components of this dimension is

$$\frac{1}{r}\binom{n-1}{r-1}\binom{m-1}{r-1}$$

Demostración.

Take (A, B) irreducible representation, $A^n = B^m$
A, B diagonalize in different basis
$A \sim \mathrm{diag}(\epsilon_1, \ldots, \epsilon_r)$, $B \sim \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_r)$
$M \in \mathrm{GL}(r, \mathbb{C})$ matrix comparing the two basis
Count: $\epsilon_1 \cdots \epsilon_r = 1$, ϵ_i distinct,
$\varepsilon_1 \cdots \varepsilon_r = 1$, ε_j distinct,
$\epsilon_i^n = \varepsilon_j^m = \varpi$, $\varpi^r = 1$.
The maximal dimensional component is isomorphic to $\mathrm{GL}(r, \mathbb{C})//(T \times_D T)$, where $T \cong (\mathbb{C}^*)^r$ are the diagonal matrices, acting on the left and the right, and $D = \mathbb{C}^* \cdot \mathrm{Id}$ $\square$

Figure eight knot

Character varieties for knot groups

M. Heusener, V. Muñoz and J. Porti, The $\mathrm{SL}(3,\mathbb{C})$-character variety of the figure eight knot, arXiv:1505.04451

Figure eight knot K_8

$\Gamma_8 = \langle a, b, t \mid tat^{-1} = ab, tbt^{-1} = bab \rangle$
$M(\Gamma_8, G) = \{(A, B, T) | TA = ABT, TB = BABT\} // G$

Coordinates for $G = \mathrm{SL}(3,\mathbb{C})$:

$$\alpha = \mathrm{tr}(A), \quad \beta = \mathrm{tr}(B), \quad y = \mathrm{tr}(T), \quad z = \mathrm{tr}(TA^{-1}TA),$$
$$\bar\alpha = \mathrm{tr}(A^{-1}), \quad \bar\beta = \mathrm{tr}(B^{-1}), \quad \bar y = \mathrm{tr}(T^{-1}), \quad \bar z = \mathrm{tr}(A^{-1}T^{-1}AT^{-1})$$

Character varieties for knot groups

M. Heusener, V. Muñoz and J. Porti, The $\mathrm{SL}(3,\mathbb{C})$-character variety of the figure eight knot, arXiv:1505.04451

Figure eight knot K_8

$\Gamma_8 = \langle a, b, t \mid tat^{-1} = ab, tbt^{-1} = bab \rangle$
$M(\Gamma_8, G) = \{(A, B, T) | TA = ABT, TB = BABT\} // G$

Coordinates for $G = \mathrm{SL}(3,\mathbb{C})$:

$$\begin{array}{llll} \alpha = \mathrm{tr}(A), & \beta = \mathrm{tr}(B), & y = \mathrm{tr}(T), & z = \mathrm{tr}(TA^{-1}TA), \\ \bar{\alpha} = \mathrm{tr}(A^{-1}), & \bar{\beta} = \mathrm{tr}(B^{-1}), & \bar{y} = \mathrm{tr}(T^{-1}), & \bar{z} = \mathrm{tr}(A^{-1}T^{-1}AT^{-1}) \end{array}$$

Figure eight knot

Theorem (Heusener-M-Porti)

The character variety $M(\Gamma, \mathrm{SL}(3,\mathbb{C})) \subset \mathbb{C}^8$ has five algebraic components.

- *The component of totally reducible representations:* $\alpha = \bar{\alpha} = \beta = \bar{\beta} = 3, z = y^2 - 2\bar{y}, \bar{z} = \bar{y}^2 - 2y$.
- *The component of partially reducible representations parametrized by:* $\alpha = \bar{\alpha} = x_1 + 1$, $\beta = \bar{\beta} = \frac{x_1}{x_1 - 1} + 1$, $y = v + \frac{1}{w}$, $\bar{y} = w + \frac{v}{w}$, $z = w\alpha + \frac{1}{w^2}$, $\bar{z} = \frac{\alpha}{w} + w^2$, *where* $(x_1^2 + x_1 - 1)w = (x_1 - 1)v^2$.
- *The first non distinguished component* V_1*:* $\alpha = \bar{\alpha} = 1$, $y\bar{y} = \beta + \bar{\beta} + 2$, $y^3 + \bar{y}^3 = \beta\bar{\beta} + 5\beta + 5\bar{\beta} + 5$, $\bar{z} = y, z = \bar{y}$.

Theorem (Heusener-M-Porti)

- *The second non distinguished component V_2:* $\beta=\bar\beta=1$, $y\bar y=\alpha+\bar\alpha+2$, $y^3+\bar y^3=\alpha\bar\alpha+5\alpha+5\bar\alpha+5$, $z=y^2-\bar y$, $\bar z=\bar y^2-y$.
- *The distinguished component V_0 (irreducible representations):*

$$
\begin{aligned}
\alpha=\bar\alpha,\quad \beta&=\bar\beta,\\
y\bar y&=(\alpha+1)(\beta+1),\\
z\bar z&=2\alpha^2\beta+\alpha^2+1,\\
y^3+\bar y^3&=\alpha^2\beta+\alpha\beta^2+6\alpha\beta+3\alpha+3\beta+2,\\
z^3+\bar z^3&=\alpha^4\beta^2+10\alpha^2\beta+9\alpha^2-2\alpha^3-2,\\
yz+\bar y\bar z&=\alpha^2\beta+3\alpha\beta+3\alpha+1,\\
\bar y^2z+y^2\bar z&=\alpha^2\beta^2+4\alpha^2\beta+2\alpha^2+4\alpha\beta+2\alpha+2\beta+1,\\
\bar y z^2+y\bar z^2&=\alpha^3\beta^2+\alpha^3\beta+4\alpha^2\beta+3\alpha^2+5\alpha\beta+3\alpha-1\,.
\end{aligned}
$$

Figure eight knot

Use of Lawton's coordinates for $M(F_2, \mathrm{SL}(3,\mathbb{C}))$
and the map $M(\Gamma_8, \mathrm{SL}(3,\mathbb{C})) \to M(F_2, \mathrm{SL}(3,\mathbb{C}))$, $(A, B, T) \mapsto (A, B)$
prove that the image $\{(A, B) | A \sim AB, B \sim BAB\}$ satisfies:

- $\alpha = \beta$, $\bar{\alpha} = \bar{\beta}$ (distinguished component)
- $\alpha = \bar{\alpha} = 1$
- $\beta = \bar{\beta} = 1$

Geometric analysis of character variety leads to

- Intersection patterns
- Singular points
- Character varieties for $\mathrm{GL}(3,\mathbb{C})$ and $\mathrm{PGL}(3,\mathbb{C})$
- Explicit matrices parametrizing the character varieties
- Action of $\mathrm{Out}(K_8) \cong D_4$

Figure eight knot

Use of Lawton's coordinates for $M(F_2, \mathrm{SL}(3,\mathbb{C}))$
and the map $M(\Gamma_8, \mathrm{SL}(3,\mathbb{C})) \to M(F_2, \mathrm{SL}(3,\mathbb{C}))$, $(A, B, T) \mapsto (A, B)$
prove that the image $\{(A, B) | A \sim AB, B \sim BAB\}$ satisfies:

- $\alpha = \beta$, $\bar{\alpha} = \bar{\beta}$ (distinguished component)
- $\alpha = \bar{\alpha} = 1$
- $\beta = \bar{\beta} = 1$

Geometric analysis of character variety leads to

- Intersection patterns
- Singular points
- Character varieties for $\mathrm{GL}(3,\mathbb{C})$ and $\mathrm{PGL}(3,\mathbb{C})$
- Explicit matrices parametrizing the character varieties
- Action of $\mathrm{Out}(K_8) \cong D_4$

Character varieties for free groups

S. Lawton and V. Muñoz, E-polynomial of the SL(3,C)-character variety of free groups, arXiv:1405.0816

$G = \mathrm{SL}(r,\mathbb{C}), \mathrm{PGL}(r,\mathbb{C})$
$M(F_k, G) = \{(A_1, \ldots, A_k) \in G^k\}//G$
$e(M) = \sum (-1)^k h^{k,p,p}(M) q^p$

Theorem (Lawton-M)

The E-polynomials $e(M(F_k, \mathrm{SL}(3,\mathbb{C}))) = e(M(F_k, \mathrm{PGL}(3,\mathbb{C})))$ and they are equal to

$$(q^8 - q^6 - q^5 + q^3)^{k-1} + (q-1)^{2k-2}(q^{3k-3} - q^k)$$
$$+ \frac{1}{6}(q-1)^{2k-2} q(q+1) + \frac{1}{2}(q^2-1)^{k-1} q(q-1)$$
$$+ \frac{1}{3}(q^2+q+1)^{k-1} q(q+1) - (q-1)^{k-1} q^{k-1}(q^2-1)^{k-1}(2q^{2k-2} - q).$$

Free groups

Character varieties for free groups

S. Lawton and V. Muñoz, E-polynomial of the SL(3,C)-character variety of free groups, arXiv:1405.0816

$G = \mathrm{SL}(r,\mathbb{C}),\ \mathrm{PGL}(r,\mathbb{C})$
$M(F_k, G) = \{(A_1,\dots,A_k) \in G^k\}//G$
$e(M) = \sum (-1)^k h^{k,p,p}(M) q^p$

Theorem (Lawton-M)

The E-polynomials $e(M(F_k, \mathrm{SL}(3,\mathbb{C}))) = e(M(F_k, \mathrm{PGL}(3,\mathbb{C})))$ *and they are equal to*

$$(q^8-q^6-q^5+q^3)^{k-1} + (q-1)^{2k-2}(q^{3k-3}-q^k)$$
$$+\frac{1}{6}(q-1)^{2k-2}q(q+1) + \frac{1}{2}(q^2-1)^{k-1}q(q-1)$$
$$+\frac{1}{3}(q^2+q+1)^{k-1}q(q+1) - (q-1)^{k-1}q^{k-1}(q^2-1)^{k-1}(2q^{2k-2}-q).$$

Surface groups

1. M. Logares, V. Muñoz and P. Newstead, Hodge-Deligne polynomials of $\mathrm{SL}(2,\mathbb{C})$-character varieties for curves of small genus, Rev. Mat. Complut. 26, 2013, 635-703.
2. M. Logares and V. Muñoz, Hodge polynomials of the $\mathrm{SL}(2,\mathbb{C})$-character variety of an elliptic curve with two marked points, Internat. J. Math.
3. J. Martínez and V. Muñoz, E-polynomial of the $\mathrm{SL}(2,\mathbb{C})$-character variety of a complex curve of genus 3, Osaka J. Math.
4. J. Martínez and V. Muñoz, E-polynomials of the $\mathrm{SL}(2,\mathbb{C})$-character varieties of surface groups, Internat. Math. Res. Not.

Focus: compute E-polynomials of character varieties $M^g_C(\mathrm{SL}(2,\mathbb{C}))$

- Chopping
- Finite quotients
- Fibrations locally trivial in the analytic topology (with monodromy)

Surface groups

1. M. Logares, V. Muñoz and P. Newstead, Hodge-Deligne polynomials of $\mathrm{SL}(2,\mathbb{C})$-character varieties for curves of small genus, Rev. Mat. Complut. 26, 2013, 635-703.
2. M. Logares and V. Muñoz, Hodge polynomials of the $\mathrm{SL}(2,\mathbb{C})$-character variety of an elliptic curve with two marked points, Internat. J. Math.
3. J. Martínez and V. Muñoz, E-polynomial of the $\mathrm{SL}(2,\mathbb{C})$-character variety of a complex curve of genus 3, Osaka J. Math.
4. J. Martínez and V. Muñoz, E-polynomials of the $\mathrm{SL}(2,\mathbb{C})$-character varieties of surface groups, Internat. Math. Res. Not.

Focus: compute E-polynomials of character varieties $M_C^g(\mathrm{SL}(2,\mathbb{C}))$

- Chopping
- Finite quotients
- Fibrations locally trivial in the analytic topology (with monodromy)

Surface groups

For each $g \geq 1$, we have a vector $v_g = (e_g^0, e_g^1, e_g^2, e_g^3, a_g, b_g, c_g, d_g)$ encoding the E-polynomial information of a genus g surface with a puncture:

- $e_g^0 = e(R_{\mathrm{Id}}^g(\mathrm{SL}(2,\mathbb{C})))$,
- $e_g^1 = e(R_{-\mathrm{Id}}^g(\mathrm{SL}(2,\mathbb{C})))$,
- $e_g^2 = e(R_{J_+}^g(\mathrm{SL}(2,\mathbb{C})))$,
- $e_g^3 = e(R_{J_-}^g(\mathrm{SL}(2,\mathbb{C})))$.

The varieties $R_{\xi_\lambda}^g(\mathrm{SL}(2,\mathbb{C}))$, $\xi_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, form a fibration $\mathcal{R}^g \to \mathbb{C} - \{\pm 2\}$, over $s = \lambda + \lambda^{-1}$, with *Hodge monodromy representation*

$$R(\mathcal{R}^g) = a_g T + b_g S_2 + c_g S_{-2} + d_g S_0$$

$\Longrightarrow$ get a TFT picture: $v_{g+1} = M \cdot v_g$

Surface groups

For each $g \geq 1$, we have a vector $v_g = (e_g^0, e_g^1, e_g^2, e_g^3, a_g, b_g, c_g, d_g)$ encoding the E-polynomial information of a genus g surface with a puncture:

- $e_g^0 = e(R_{\mathrm{Id}}^g(\mathrm{SL}(2,\mathbb{C})))$,
- $e_g^1 = e(R_{-\mathrm{Id}}^g(\mathrm{SL}(2,\mathbb{C})))$,
- $e_g^2 = e(R_{J_+}^g(\mathrm{SL}(2,\mathbb{C})))$,
- $e_g^3 = e(R_{J_-}^g(\mathrm{SL}(2,\mathbb{C})))$.

The varieties $R_{\xi_\lambda}^g(\mathrm{SL}(2,\mathbb{C}))$, $\xi_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, form a fibration $\mathcal{R}^g \to \mathbb{C} - \{\pm 2\}$, over $s = \lambda + \lambda^{-1}$, with *Hodge monodromy representation*

$$R(\mathcal{R}^g) = a_g T + b_g S_2 + c_g S_{-2} + d_g S_0$$

$\Longrightarrow$ get a TFT picture: $v_{g+1} = M \cdot v_g$

Surface groups

$$M =$$

$$\begin{pmatrix}
q^4+4q^3-q^2-4q & q^3-q & q^5-2q^4-4q^3+2q^2+3q & q^5+3q^4-q^3-3q^2 & q^6-2q^5-4q^4+3q^2+2q & -q^5-4q^4+4q^2+q & 2q^5-7q^4-3q^3+7q^2+q & -5q^4-q^3+5q^2+q \\
q^3-q & q^4+4q^3-q^2-4q & q^5+3q^4-q^3-3q^2 & q^5-2q^4-4q^3+2q^2+3q & q^6-2q^5-4q^4+3q^2+2q & 2q^5-7q^4-3q^3+7q^2+q & -q^5-4q^4+4q^2+q & -5q^4-q^3+5q^2+q \\
q^3-2q^2-3q & q^3+3q^2 & q^5+q^4+3q^2+3q & q^5-3q^3-6q^2 & q^6-2q^5-3q^4+q^3+3q^2 & -q^5+2q^4-4q^3+3q^2 & -q^5-q^4-4q^3+6q^2 & -2q^4-q^3+3q^2 \\
q^3+3q^2 & q^3-2q^2-3q & q^5-3q^3-6q^2 & q^5+q^4+3q^2+3q & q^6-2q^5-3q^4+q^3+3q^2 & -q^5-q^4-4q^3+6q^2 & -q^5+2q^4-4q^3+3q^2 & -2q^4-q^3+3q^2 \\
q^3 & q^3 & q^5-3q^3 & q^5-3q^3 & q^6-2q^5-2q^4+4q^3+q^2 & -q^5-q^4+2q^3 & -q^5-q^4+2q^3 & -2q^4 \\
-3q & 3q^2 & 3q^2+3 & -6q^2 & -3q^3+3q^2 & 4q^4-6q^3+4q^2 & -8q^3+6q^2 & -3q^3+3q^2 \\
3q^2 & -3q & -6q^2 & 3q^3+3q & -3q^3+3q^2 & -8q^3+6q^2 & 4q^4-6q^3+4q^2 & -3q^3+3q^2 \\
-1 & -1 & 2q^2 & 2q^2 & -4q^2+2 & -2q^2+q+1 & -2q^2+q+1 & q^4-2q^2+2q+1
\end{pmatrix}$$

Theorem (Martínez-M)

The E-polynomials for $M_C^g(\mathrm{SL}(2,\mathbb{C}))$ are:

$$
\begin{aligned}
e(M^g_{\mathrm{Id}}) &= (q^3-q)^{2g-2} + (q^2-1)^{2g-2} - q(q^2-q)^{2g-2} - 2^{2g}q^{2g-2} \\
&\quad + \frac{1}{2}q^{2g-2}(q+2^{2g}-1)((q+1)^{2g-2} + (q-1)^{2g-2}) \\
&\quad + \frac{1}{2}q((q+1)^{2g-1} + (q-1)^{2g-1}) \\
e(M^g_{-\mathrm{Id}}) &= (q^3-q)^{2g-2} + (q^2-1)^{2g-2} - 2^{2g-1}(q^2+q)^{2g-2} \\
&\quad + (2^{2g-1}-1)(q^2-q)^{2g-2} \\
e(M^g_{J_+}) &= (q^3-q)^{2g-2}(q^2-1) + \frac{1}{2}q^{2g-2}(q-1)((q-1)^{2g-1} - (q+1)^{2g-1}) \\
&\quad + (2^{2g-1}-1)(q-1)(q^2-q)^{2g-2} - 2^{2g-1}(q+1)(q^2+q)^{2g-2} \\
e(M^g_{J_-}) &= (q^3-q)^{2g-2}(q^2-1) + (2^{2g-1}-1)(q-1)(q^2-q)^{2g-2} \\
&\quad + 2^{2g-1}(q+1)(q^2+q)^{2g-2} \\
e(M^g_{\xi_\lambda}) &= (q^3-q)^{2g-2}(q^2+q) + (q^2-1)^{2g-2}(q+1) \\
&\quad + (2^{2g}-2)(q^2-q)^{2g-2}q.
\end{aligned}
$$

Open research directions

Possible applications of geometric techniques for character varieties

- E-polynomials of $\mathrm{SL}(3,\mathbb{C})$-character varieties of surface groups
- E-polynomials of $\mathrm{SL}(2,\mathbb{C})$-character varieties of surfaces with many punctures
- E-polynomials for non-orientable surfaces
- E-polynomials for $\mathrm{PGL}(2,\mathbb{C})$-character varieties
- Develop TFT formalism
- E-polynomials of $\mathrm{SL}(r,\mathbb{C})$-character varieties of free groups
- $\mathrm{SL}(r,\mathbb{C})$-character varieties of torus knots, $r \geq 4$
- $\mathrm{SL}(3,\mathbb{C})$-character varieties of hyperbolic knots
- Other groups: $\mathrm{SO}(r,\mathbb{C})$, $\mathrm{Sp}(2r,\mathbb{C})$, etc