Hyperpolygons and Hitchin systems

Steven Rayan
University of Toronto
stever@math.toronto.edu

Higgs Bundles and Character Varieties
SPM-EMS-AMS Joint Meeting, Porto, 10 June 2015

Hyperpolygons and Hitchin systems

Steven Rayan
University of Toronto
stever@math.toronto.edu

Higgs Bundles and Character Varieties
SPM-EMS-AMS Joint Meeting, Porto, 10 June 2015

arXiv:1410.6467

Joint w/ Jonathan Fisher, Universität Hamburg

Two Worlds

- Quiver varieties: doorway between worlds of *rigid geometry* and *holomorphic / meromorphic geometry*

Two Worlds

- Quiver varieties: doorway between worlds of *rigid geometry* and *holomorphic / meromorphic geometry*
- Moduli space of n-**gons** in Euclidean space is compact Kähler Fano variety; space of **hyper-n-gons**, its noncompact, hyperkähler analogue

Two Worlds

- Quiver varieties: doorway between worlds of *rigid geometry* and *holomorphic / meromorphic geometry*
- Moduli space of n-**gons** in Euclidean space is compact Kähler Fano variety; space of **hyper-n-gons**, its noncompact, hyperkähler analogue
- Former is a quiver variety for a **star-shaped quiver** and a moduli space of parabolic bundles

Two Worlds

- Quiver varieties: doorway between worlds of *rigid geometry* and *holomorphic / meromorphic geometry*
- Moduli space of n-**gons** in Euclidean space is compact Kähler Fano variety; space of **hyper-n-gons**, its noncompact, hyperkähler analogue
- Former is a quiver variety for a **star-shaped quiver** and a moduli space of parabolic bundles
- Latter is a quiver variety for a **doubled star-shaped quiver** and a moduli space of parabolic Higgs bundles

Plan of Talk

1. Overview of moduli space for a star (polygon space)

Plan of Talk

1. Overview of moduli space for a star (polygon space)
2. Overview of moduli space for a doubled star (hyperpolygon space)

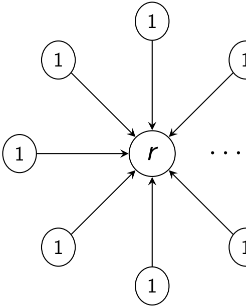
Plan of Talk

1. Overview of moduli space for a star (polygon space)
2. Overview of moduli space for a doubled star (hyperpolygon space)
3. Betti numbers of hyperpolygon space for all ranks

Plan of Talk

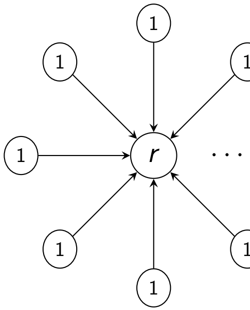
1. Overview of moduli space for a star (polygon space)
2. Overview of moduli space for a doubled star (hyperpolygon space)
3. Betti numbers of hyperpolygon space for all ranks
4. Hitchin map and integrable system on hyperpolygon space

Star Quiver $\mathcal{Q}$



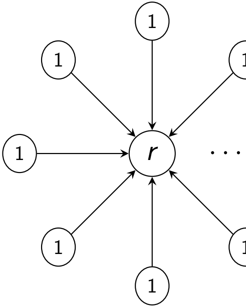
n sources, 1 sink, $r \geq 2$, $n \geq r+1$, labels $\underline{\mathbf{d}} = (r, \underbrace{1, \dots, 1}_{n})$

Star Quiver $\mathcal{Q}$



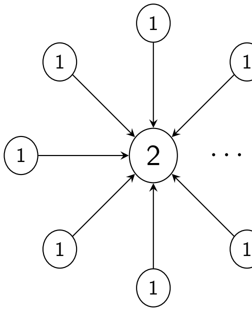
- Representation of $\mathcal{Q}$ is choice of $x_i \in \mathrm{Hom}(\mathbb{C}^1, \mathbb{C}^r) = \mathbb{C}^r$ for each arrow

Star Quiver $\mathcal{Q}$



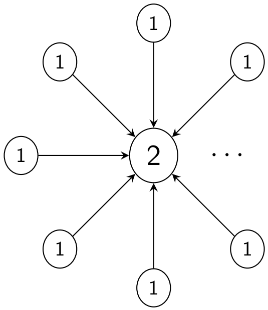
- Representation of $\mathcal{Q}$ is choice of $x_i \in \mathrm{Hom}(\mathbb{C}^1, \mathbb{C}^r) = \mathbb{C}^r$ for each arrow
- Equivalently, a matrix
$$\phi = [x_1 \cdots x_n] \in \bigoplus_{i=1}^{n} \mathrm{Hom}(\mathbb{C}, \mathbb{C}^r) \cong \mathbb{C}^{n \times r} = \mathrm{Rep}(\mathcal{Q})$$

$r = 2, n \geq 3$, Klyachko[94], Kapovitch-Millson[96]



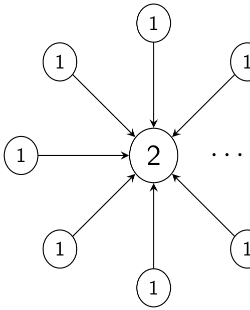
- Representation of $\mathcal{Q}$ is choice of $x_i \in \text{Hom}(\mathbb{C}^1, \mathbb{C}^2) = \mathbb{C}^2$ for each arrow
- Equivalently, a matrix
  $$[x_1 \cdots x_n] \in \bigoplus_{i=1}^{n} \text{Hom}(\mathbb{C}, \mathbb{C}^2) \cong \mathbb{C}^{2n} = \text{Rep}(\mathcal{Q})$$

$r = 2$, $n \geq 3$, Klyachko[94], Kapovitch-Millson[96]



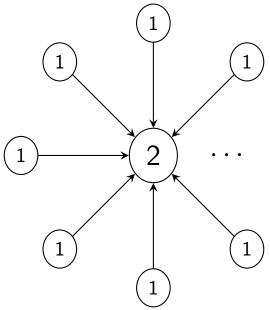
- $G = SU(2) \times U(1)^n$ acts on $(\mathbb{C}^2)^n = \mathrm{Rep}(\mathcal{Q})$

$r = 2$, $n \geq 3$, Klyachko[94], Kapovitch-Millson[96]



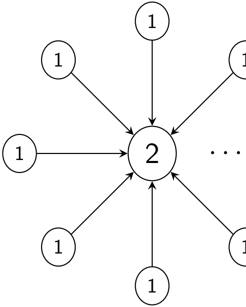
- $G = SU(2) \times U(1)^n$ acts on $(\mathbb{C}^2)^n = \mathrm{Rep}(\mathcal{Q})$
- $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathbb{R}^n$, $Z(\mathfrak{g}) \cong \mathbb{R}^n$

$r = 2$, $n \geq 3$, Klyachko[94], Kapovitch-Millson[96]



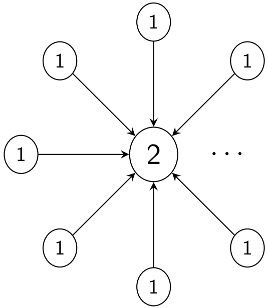
- $G = SU(2) \times U(1)^n$ acts on $(\mathbb{C}^2)^n = \mathrm{Rep}(\mathcal{Q})$
- $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathbb{R}^n$, $Z(\mathfrak{g}) \cong \mathbb{R}^n$
- $Z(\mathfrak{g}) \cong \mathbb{R}^n$ is *space of length vectors*

$r = 2$, $n \geq 3$, Klyachko[94], Kapovitch-Millson[96]



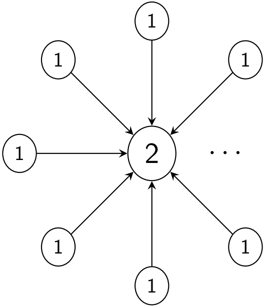
- $G = SU(2) \times U(1)^n$ acts on $(\mathbb{C}^2)^n = \mathrm{Rep}(\mathcal{Q})$
- $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathbb{R}^n$, $Z(\mathfrak{g}) \cong \mathbb{R}^n$
- $Z(\mathfrak{g}) \cong \mathbb{R}^n$ is *space of length vectors*
- $\underline{\alpha} \in \mathbb{R}^n$ is *generic* if $\alpha_i \neq 0\ \forall i$ and $\sum_{i=1}^n \alpha_i \neq 2 \sum_{i \in S} \alpha_i$ for each $S \subseteq \{1, \ldots, n\}$

$r = 2$, $n \geq 3$



- $\mu : \mathrm{Rep}(\mathcal{Q}) \to \mathfrak{g}^*$ given by
  $[x_1 \cdots x_n] \mapsto (\sum (x_i x_i^*)_0, |x_1|^2, \ldots, |x_n|^2)$

$r = 2,\ n \geq 3$



- $\mu : \mathrm{Rep}(\mathcal{Q}) \to \mathfrak{g}^*$ given by $[x_1 \cdots x_n] \mapsto (\sum (x_i x_i^*)_0, |x_1|^2, \ldots, |x_n|^2)$
- **Polygon space:** $\mathcal{P}_n^2(\underline{\alpha}) = \mathrm{Rep}(\mathcal{Q})/\!/_{\underline{\alpha}} G := \mu^{-1}(0, \underline{\alpha})/G$, where $\underline{\alpha}$ is generic

$r = 2$, $n \geq 3$

- Regard matrices $(x_1 x_1^*)_0, \ldots, (x_n x_n^*)_0 \in \mathfrak{su}(2)^*$ as vectors in $\mathbb{R}^3$

$r = 2,\ n \geq 3$

- Regard matrices $(x_1 x_1^*)_0, \ldots, (x_n x_n^*)_0 \in \mathfrak{su}(2)^*$ as vectors in $\mathbb{R}^3$
- Moment map condition $\sum (x_i x_i^*)_0 = 0$ says the vectors form a closed n-gon in $\mathbb{R}^3$

$r = 2,\ n \geq 3$

- Regard matrices $(x_1 x_1^*)_0, \ldots, (x_n x_n^*)_0 \in \mathfrak{su}(2)^*$ as vectors in $\mathbb{R}^3$
- Moment map condition $\sum (x_i x_i^*)_0 = 0$ says the vectors form a closed n-gon in $\mathbb{R}^3$
- Condition $|x_i|^2 = \alpha_i \neq 0 \implies |(x_i x_i^*)_0|^2 = \alpha_i^2/2 > 0$

$r = 2,\ n \geq 3$

- Regard matrices $(x_1 x_1^*)_0, \ldots, (x_n x_n^*)_0 \in \mathfrak{su}(2)^*$ as vectors in $\mathbb{R}^3$
- Moment map condition $\sum (x_i x_i^*)_0 = 0$ says the vectors form a closed n-gon in $\mathbb{R}^3$
- Condition $|x_i|^2 = \alpha_i \neq 0 \implies |(x_i x_i^*)_0|^2 = \alpha_i^2/2 > 0$
- Generic $\underline{\alpha} \implies n$-gon nondegenerate

$r = 2$, $n \geq 3$

- Regard matrices $(x_1 x_1^*)_0, \ldots, (x_n x_n^*)_0 \in \mathfrak{su}(2)^*$ as vectors in $\mathbb{R}^3$
- Moment map condition $\sum (x_i x_i^*)_0 = 0$ says the vectors form a closed n-gon in $\mathbb{R}^3$
- Condition $|x_i|^2 = \alpha_i \neq 0 \implies |(x_i x_i^*)_0|^2 = \alpha_i^2/2 > 0$
- Generic $\underline{\alpha} \implies n$-gon nondegenerate
- $\dim \mathcal{P}_3^2(\underline{\alpha}) = 0$

$r = 2, n = 5$, Point in $\mathcal{P}_5^2(\sqrt{2}\,|v_1|\,, \ldots, \sqrt{2}\,|v_5|)$

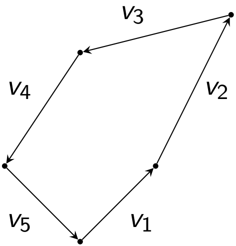


Figure : Euclidean polygon in $\mathfrak{su}(2)^* \cong \mathbb{R}^3$ w/ generic $\underline{\alpha}$

$\mathcal{P}_n^r(\underline{\alpha})$ for General r

- Picture generalizes to arbitrary r w/ appropriate generalization of genericity condition on $\underline{\alpha}$

$\mathcal{P}_n^r(\underline{\alpha})$ for General r

- Picture generalizes to arbitrary r w/ appropriate generalization of genericity condition on $\underline{\alpha}$
- $\mathrm{Rep}(\mathcal{Q}) = \mathbb{C}^{nr}$, $x_i \in \mathbb{C}^r$, $G = SU(r) \times U(1)^n$, $\mathcal{P}_r^n(\underline{\alpha}) := \mathrm{Rep}(Q)/\!/_{\underline{\alpha}}\, G$

$\mathcal{P}_n^r(\underline{\alpha})$ for General r

- Picture generalizes to arbitrary r w/ appropriate generalization of genericity condition on $\underline{\alpha}$
- $\mathrm{Rep}(\mathcal{Q}) = \mathbb{C}^{nr}$, $x_i \in \mathbb{C}^r$, $G = SU(r) \times U(1)^n$, $\mathcal{P}_r^n(\underline{\alpha}) := \mathrm{Rep}(Q) /\!/_{\underline{\alpha}} G$
- For generic $\underline{\alpha}$, a representation $[x_1 \cdots x_n]$ determines a closed, nondegenerate n-gon in $\mathfrak{su}(r)^* \cong \mathbb{R}^{r^2-1}$ with vertices $(x_i x_i^*)_0$

$\mathcal{P}_n^r(\underline{\alpha})$ for General r

- $\mathcal{P}_n^r$ can also be thought of as space of polygons in $\mathbf{P}^{r-1}$ (ordered n-tuples of points; King[94], Khesin-Soloviev[11])

$\mathcal{P}_n^r(\underline{\alpha})$ for General r

- $\mathcal{P}_n^r$ can also be thought of as space of polygons in $\mathbf{P}^{r-1}$ (ordered n-tuples of points; King[94], Khesin-Soloviev[11])
- Comes from $\mathrm{Rep}(\mathcal{Q})\underset{\underline{\alpha}}{/\!/}(U(1))^n = \mathbb{C}^{nr}\underset{\underline{\alpha}}{/\!/}U(1)^n = (\mathbf{P}^{r-1})^n$, where each factor has symplectic form $\alpha_i\omega_{FS}$

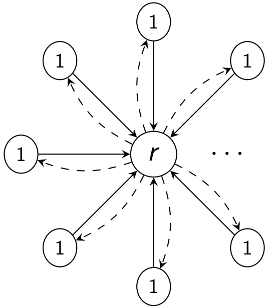
$\mathcal{P}_n^r(\underline{\alpha})$ for General r

- $\mathcal{P}_n^r$ can also be thought of as space of polygons in $\mathbf{P}^{r-1}$ (ordered n-tuples of points; King[94], Khesin-Soloviev[11])
- Comes from $\mathrm{Rep}(\mathcal{Q}) \underset{\underline{\alpha}}{/\!/} (U(1))^n = \mathbb{C}^{nr} \underset{\underline{\alpha}}{/\!/} U(1)^n = (\mathbf{P}^{r-1})^n$, where each factor has symplectic form $\alpha_i \omega_{FS}$
- $\dim_{\mathbb{C}} \mathcal{P}_n^r(\underline{\alpha}) = (r-1)(n-r-1)$

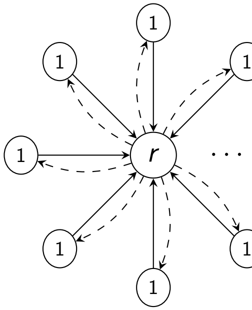
$\mathcal{P}_n^r(\underline{\alpha})$ for General r

- $\mathcal{P}_n^r$ can also be thought of as space of polygons in $\mathbf{P}^{r-1}$ (ordered n-tuples of points; King[94], Khesin-Soloviev[11])
- Comes from $\mathrm{Rep}(\mathcal{Q})\underset{\underline{\alpha}}{/\!/}(U(1))^n = \mathbb{C}^{nr}\underset{\underline{\alpha}}{/\!/}U(1)^n = (\mathbf{P}^{r-1})^n$, where each factor has symplectic form $\alpha_i\omega_{FS}$
- $\dim_{\mathbb{C}} \mathcal{P}_n^r(\underline{\alpha}) = (r-1)(n-r-1)$
- Cohomology of $\mathcal{P}_n^r(\underline{\alpha})$ can be determined via standard techniques for compact symplectic quotients: Jeffrey-Kirwan[95], Tolman-Weitsman[03]

Doubled Quivers



Doubled Quivers



$$[\underbrace{x_1 \cdots x_n}_{\text{incoming}};\ \underbrace{y_1 \cdots y_n}_{\text{outgoing}}] = [\phi_x \mid \phi_y] \in T^*\mathrm{Rep}(\mathcal{Q})$$

Doubled Quivers and Hyperpolygons

- $\mu_{\mathbb{R}}(x_1, \ldots, x_n; y_1, \ldots, y_n) =$
  $(\sum (x_i x_i^*)_0 - (y_i y_i^*)_0, |x_1|^2 - |y_1|^2, \ldots, |x_n|^2 - |y_n|^2)$

Doubled Quivers and Hyperpolygons

- $\mu_{\mathbb{R}}(x_1, \ldots, x_n; y_1, \ldots, y_n) =$
  $(\sum (x_i x_i^*)_0 - (y_i y_i^*)_0, |x_1|^2 - |y_1|^2, \ldots, |x_n|^2 - |y_n|^2)$
- $\mu_{\mathbb{C}}(x_1, \ldots, x_n; y_1, \ldots, y_n) = ((\phi_x \phi_y)_0, y_1 x_1, \ldots, y_n x_n)$

Doubled Quivers and Hyperpolygons

- $\mu_{\mathbb{R}}(x_1, \ldots, x_n; y_1, \ldots, y_n) = (\sum (x_i x_i^*)_0 - (y_i y_i^*)_0, |x_1|^2 - |y_1|^2, \ldots, |x_n|^2 - |y_n|^2)$
- $\mu_{\mathbb{C}}(x_1, \ldots, x_n; y_1, \ldots, y_n) = ((\phi_x \phi_y)_0, y_1 x_1, \ldots, y_n x_n)$
- **Hyperpolygon space:** $\mathcal{X}_n^r(\underline{\alpha}) = T^*\mathrm{Rep}(\mathcal{Q})/\!/\!/_{\underline{\alpha}} G := \mu_{\mathbb{R}}^{-1}((0, \underline{\alpha})) \cap \mu_{\mathbb{C}}^{-1}(\underline{0})/G$

Doubled Quivers and Hyperpolygons

- $\mu_{\mathbb{R}}(x_1, \dots, x_n; y_1, \dots, y_n) = (\sum (x_i x_i^*)_0 - (y_i y_i^*)_0, |x_1|^2 - |y_1|^2, \dots, |x_n|^2 - |y_n|^2)$
- $\mu_{\mathbb{C}}(x_1, \dots, x_n; y_1, \dots, y_n) = ((\phi_x \phi_y)_0, y_1 x_1, \dots, y_n x_n)$
- **Hyperpolygon space:** $\mathcal{X}_n^r(\underline{\alpha}) = T^*\mathrm{Rep}(\mathcal{Q})\underset{\underline{\alpha}}{/\!/\!/} G := \mu_{\mathbb{R}}^{-1}((0, \underline{\alpha})) \cap \mu_{\mathbb{C}}^{-1}(\underline{0})/G$
- Nakajima[95]: G acts freely on $\mu_{\mathbb{R}}^{-1}((0, \underline{\alpha})) \cap \mu_{\mathbb{C}}^{-1}(\underline{0})$ when $\underline{\alpha}$ is generic, and $\mathcal{X}_n^r(\underline{\alpha})$ is noncompact, complete hyperkähler manifold of dimension $2(r-1)(n-r-1) = 2\dim_{\mathbb{C}} \mathcal{P}_n^r(\underline{\alpha})$

Doubled Quivers and Hyperpolygons

- $\mu_{\mathbb{R}}(x_1, \ldots, x_n; y_1, \ldots, y_n) =$
  $(\sum(x_i x_i^*)_0 - (y_i y_i^*)_0, |x_1|^2 - |y_1|^2, \ldots, |x_n|^2 - |y_n|^2)$
- $\mu_{\mathbb{C}}(x_1, \ldots, x_n; y_1, \ldots, y_n) = ((\phi_x \phi_y)_0, y_1 x_1, \ldots, y_n x_n)$
- **Hyperpolygon space:**
  $\mathcal{X}_n^r(\underline{\alpha}) = T^*\mathrm{Rep}(\mathcal{Q}) /\!/\!/_{\underline{\alpha}} G := \mu_{\mathbb{R}}^{-1}((0, \underline{\alpha})) \cap \mu_{\mathbb{C}}^{-1}(\underline{0}) / G$
- Nakajima[95]: G acts freely on $\mu_{\mathbb{R}}^{-1}((0, \underline{\alpha})) \cap \mu_{\mathbb{C}}^{-1}(\underline{0})$ when $\underline{\alpha}$ is generic, and $\mathcal{X}_n^r(\underline{\alpha})$ is noncompact, complete hyperkähler manifold of dimension $2(r-1)(n-r-1) = 2 \dim_{\mathbb{C}} \mathcal{P}_n^r(\underline{\alpha})$
- **What the heck is a hyperpolygon?**

Hyperpolygon...or Superpolygon??

Hyperpolygon...or Superpolygon??

- Stable hyper-n-gon of rank r is a pair of n-gons, each in $\mathbb{C}^r$, but first has bounded perimeter and other does not (Harada-Proudfoot[03])

Cohomology

Theorem (Fisher, R.)

The hyperkähler Kirwan map

$$\kappa : H^*(B(S(U(r) \times U(1)^n))) \to H^*(\mathcal{X}_n^r(\underline{\alpha}))$$

is surjective, and the Poincaré polynomial $P_t(\mathcal{X}_n^r(\underline{\alpha}))$ is independent of generic $\underline{\alpha}$ and satisfies

$$\frac{P_t(\mathbf{Gr}(r,n))}{(1-t^2)^{n-1}} = \sum_{\lambda} \frac{1}{m(\lambda)!} \sum_{\rho \geq \lambda} \frac{t^{2\beta(\lambda,\rho)}}{(1-t^2)^{s(\lambda,\rho)}} \binom{n}{\rho} \prod_{j=1}^{\ell(\lambda)} P_t(\mathcal{X}_{\rho_j}^{\lambda_j}).$$

Cohomology

Theorem (Fisher, R.)

The hyperkähler Kirwan map

$$\kappa : H^*(B(S(U(r) \times U(1)^n))) \to H^*(\mathcal{X}_n^r(\underline{\alpha}))$$

is surjective, and the Poincaré polynomial $P_t(\mathcal{X}_n^r(\underline{\alpha}))$ *is independent of generic* $\underline{\alpha}$ *and satisfies*

$$\frac{P_t(\mathbf{Gr}(r,n))}{(1-t^2)^{n-1}} = \sum_{\lambda} \frac{1}{m(\lambda)!} \sum_{\rho \geq \lambda} \frac{t^{2\beta(\lambda,\rho)}}{(1-t^2)^{s(\lambda,\rho)}} \binom{n}{\rho} \prod_{j=1}^{\ell(\lambda)} P_t(\mathcal{X}_{\rho_j}^{\lambda_j}).$$

- Natural $\mathbb{C}^*$ action on $\mathcal{X}_n^r(\underline{\alpha})$: $[\phi_x \,|\, \phi_y] \overset{\eta}{\mapsto} [\phi_x \,|\, \eta \cdot \phi_y]$

Cohomology

Theorem (Fisher, R.)

The hyperkähler Kirwan map

$$\kappa : H^*(B(S(U(r) \times U(1)^n))) \to H^*(\mathcal{X}_n^r(\underline{\alpha}))$$

is surjective, and the Poincaré polynomial $P_t(\mathcal{X}_n^r(\underline{\alpha}))$ is independent of generic $\underline{\alpha}$ and satisfies

$$\frac{P_t(\mathbf{Gr}(r,n))}{(1-t^2)^{n-1}} = \sum_{\lambda} \frac{1}{m(\lambda)!} \sum_{\rho \geq \lambda} \frac{t^{2\beta(\lambda,\rho)}}{(1-t^2)^{s(\lambda,\rho)}} \binom{n}{\rho} \prod_{j=1}^{\ell(\lambda)} P_t(\mathcal{X}_{\rho_j}^{\lambda_j}).$$

- Natural $\mathbb{C}^*$ action on $\mathcal{X}_n^r(\underline{\alpha})$: $[\phi_x \,|\, \phi_y] \overset{\eta}{\mapsto} [\phi_x \,|\, \eta \cdot \phi_y]$
- Betti numbers follow from equivariant Morse theory on orbifold compactification (study fixed points of $\mathbb{C}^*$ action at infinity)

Cohomology

Theorem (Fisher, R.)

The hyperkähler Kirwan map

$$\kappa : H^*(B(S(U(r) \times U(1)^n))) \to H^*(\mathcal{X}_n^r(\underline{\alpha}))$$

is surjective, and the Poincaré polynomial $P_t(\mathcal{X}_n^r(\underline{\alpha}))$ is independent of generic $\underline{\alpha}$ and satisfies

$$\frac{P_t(\mathbf{Gr}(r,n))}{(1-t^2)^{n-1}} = \sum_{\lambda} \frac{1}{m(\lambda)!} \sum_{\rho \geq \lambda} \frac{t^{2\beta(\lambda,\rho)}}{(1-t^2)^{s(\lambda,\rho)}} \binom{n}{\rho} \prod_{j=1}^{\ell(\lambda)} P_t(\mathcal{X}_{\rho_j}^{\lambda_j}).$$

- Natural $\mathbb{C}^*$ action on $\mathcal{X}_n^r(\underline{\alpha})$: $[\phi_x \,|\, \phi_y] \overset{\eta}{\mapsto} [\phi_x \,|\, \eta \cdot \phi_y]$
- Betti numbers follow from equivariant Morse theory on orbifold compactification (study fixed points of $\mathbb{C}^*$ action at infinity)
- Recover Konno[00] (computed $b_k(\mathcal{X}_n^2(\underline{\alpha}))$)

Parabolic Bundles

- Take trivial bundle $\mathbf{E} = \mathbb{C}^r \times \mathbf{P}^1$

Parabolic Bundles

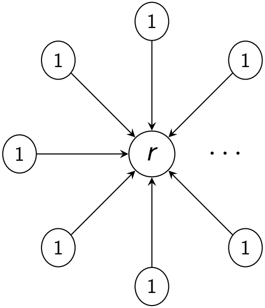
- Take trivial bundle $\mathbf{E} = \mathbb{C}^r \times \mathbf{P}^1$
- Choose $D = \sum_{i=1}^{n} p_i$ in $\mathbf{P}^1$, and fix a flag of type $(1, r)$ in $\mathbf{E}$ at each p_i (i.e. $\mathbf{E}_{p_i}$ has a distinguished complex line L_i)

Parabolic Bundles

- Take trivial bundle $\mathbf{E} = \mathbb{C}^r \times \mathbf{P}^1$
- Choose $D = \sum_{i=1}^{n} p_i$ in $\mathbf{P}^1$, and fix a flag of type $(1, r)$ in $\mathbf{E}$ at each p_i (i.e. $\mathbf{E}_{p_i}$ has a distinguished complex line L_i)
- This data is a *(minimally) parabolic bundle*

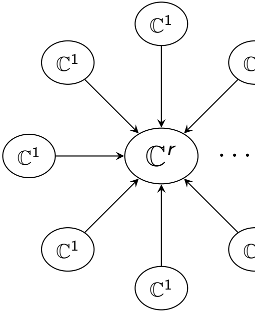
Star Quivers and Parabolic Bundles

- Embedding $L_i \to \mathbf{E}_{p_i}$ is choice of map $x_i : \mathbb{C} \to \mathbb{C}^r$
- Parabolic structure on $\mathbf{E} = \mathbb{C}^r \times \mathbf{P}^1$ is $[x_1 \ldots, x_n] \in \mathrm{Rep}(\mathcal{Q})$ for a star quiver $\mathcal{Q}$



Star Quivers and Parabolic Bundles

- Embedding $L_i \to \mathbf{E}_{p_i}$ is choice of map $x_i : \mathbb{C} \to \mathbb{C}^r$
- Parabolic structure on $\mathbf{E} = \mathbb{C}^r \times \mathbf{P}^1$ is $[x_1 \ldots, x_n] \in \mathrm{Rep}(\mathcal{Q})$ for a star quiver $\mathcal{Q}$



*Can get other flag types by having different lengths of arms with label values descending away from the sink

$r = 2$

- $\mathcal{P}_n^2(\underline{\alpha}) = \mathbf{P}^{n-3}$ is a moduli space of minimal parabolic structures on $\mathbb{C}^2 \times \mathbf{P}^1$ with n marked points on $X = \mathbf{P}^1$

$r = 2$

- $\mathcal{P}_n^2(\underline{\alpha}) = \mathbf{P}^{n-3}$ is a moduli space of minimal parabolic structures on $\mathbb{C}^2 \times \mathbf{P}^1$ with n marked points on $X = \mathbf{P}^1$
- Notions of equivalence (resp. polygons and parabolic bundles) are compatible

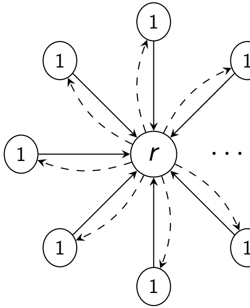
$r = 2$

- $\mathcal{P}_n^2(\underline{\alpha}) = \mathbf{P}^{n-3}$ is a moduli space of minimal parabolic structures on $\mathbb{C}^2 \times \mathbf{P}^1$ with n marked points on $X = \mathbf{P}^1$
- Notions of equivalence (resp. polygons and parabolic bundles) are compatible
- Stability condition induced on parabolic structures by $\underline{\alpha}$ is identical to GIT stability for parabolic Higgs ($\underline{\alpha}$ fixes the appropriate parabolic weights; Godinho-Mandini[11])

$r = 2$

- $\mathcal{P}^2_n(\underline{\alpha}) = \mathbf{P}^{n-3}$ is a moduli space of minimal parabolic structures on $\mathbb{C}^2 \times \mathbf{P}^1$ with n marked points on $X = \mathbf{P}^1$
- Notions of equivalence (resp. polygons and parabolic bundles) are compatible
- Stability condition induced on parabolic structures by $\underline{\alpha}$ is identical to GIT stability for parabolic Higgs ($\underline{\alpha}$ fixes the appropriate parabolic weights; Godinho-Mandini[11])
- We conjecture equivalence of stability for all r

Doubled Star Quivers



$x_i \in \mathrm{Hom}(\mathbb{C}^1, \mathbb{C}^r)$, $y_i \in \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^1)$, representations are in $T^*\mathrm{Rep}(\mathcal{Q})$, where $\mathcal{Q}$ is underlying star

Parabolic Higgs bundles

- Let $\mathbf{E} = \mathbb{C}^r \times \mathbf{P}^1$ have the minimal parabolic structure along D induced by a choice of representation of $\mathcal{Q}$, and let z be a coord on $\mathbf{P}^1$ such that $\infty \neq p_i$ for any $p_i \in D$

Parabolic Higgs bundles

- Let $\mathbf{E} = \mathbb{C}^r \times \mathbf{P}^1$ have the minimal parabolic structure along D induced by a choice of representation of $\mathcal{Q}$, and let z be a coord on $\mathbf{P}^1$ such that $\infty \neq p_i$ for any $p_i \in D$
- Construct a "Higgs field" for $\mathbf{E}$ by setting $\phi(z) = \sum_{i=1}^{n} \frac{\phi_i}{z - p_i} dz$, where residue ϕ_i is $x_i y_i$

Parabolic Higgs bundles

- Let $\mathbf{E} = \mathbb{C}^r \times \mathbf{P}^1$ have the minimal parabolic structure along D induced by a choice of representation of $\mathcal{Q}$, and let z be a coord on $\mathbf{P}^1$ such that $\infty \neq p_i$ for any $p_i \in D$
- Construct a "Higgs field" for $\mathbf{E}$ by setting $\phi(z) = \sum_{i=1}^{n} \frac{\phi_i}{z - p_i} dz$, where residue ϕ_i is $x_i y_i$
- $\Theta : ([x_i | y_i]) \mapsto \phi(z)$ is a natural map from rank r hyper-n-gons to rank r minimally parabolic Higgs bundles

Parabolic Higgs bundles

- Let $\mathbf{E} = \mathbb{C}^r \times \mathbf{P}^1$ have the minimal parabolic structure along D induced by a choice of representation of $\mathcal{Q}$, and let z be a coord on $\mathbf{P}^1$ such that $\infty \neq p_i$ for any $p_i \in D$
- Construct a "Higgs field" for $\mathbf{E}$ by setting $\phi(z) = \sum_{i=1}^{n} \frac{\phi_i}{z - p_i} dz$, where residue ϕ_i is $x_i y_i$
- $\Theta : ([x_i|y_i]) \mapsto \phi(z)$ is a natural map from rank r hyper-n-gons to rank r minimally parabolic Higgs bundles
- Notions of equivalence, stability compatible

Parabolic Higgs bundles

- Let $\mathbf{E} = \mathbb{C}^r \times \mathbf{P}^1$ have the minimal parabolic structure along D induced by a choice of representation of $\mathcal{Q}$, and let z be a coord on $\mathbf{P}^1$ such that $\infty \neq p_i$ for any $p_i \in D$
- Construct a "Higgs field" for $\mathbf{E}$ by setting $\phi(z) = \sum_{i=1}^{n} \frac{\phi_i}{z - p_i} dz$, where residue ϕ_i is $x_i y_i$
- $\Theta : ([x_i|y_i]) \mapsto \phi(z)$ is a natural map from rank r hyper-n-gons to rank r minimally parabolic Higgs bundles
- Notions of equivalence, stability compatible
- $\mu_{\mathbb{C}} = 0$ condition implies $y_i x_i = 0$, and so $\phi_i^2 = x_i(y_i x_i)y_i = 0$ (image of Θ lies *strict* parabolic locus)

$r = 2$

- $r = 2$ parabolic Higgs bundles automatically minimal

$r = 2$

- $r = 2$ parabolic Higgs bundles automatically minimal
- Studied by Biswas-Florentino-Godinho-Mandini[13], identified $\mathcal{X}_n^2(\underline{\alpha})$ with moduli space of rank 2 parabolic Higgs bundles, which is algebraically completely integrable Hamiltonian system (Logares-Martens[10])

Hitchin Map

- *Hitchin map* $h : \mathcal{X}_n^r(\underline{\alpha}) \longrightarrow \bigoplus_{i=2}^r H^0(\mathbf{P}^1, K^i(D)) =: B_{r,D}$
  defined by $h([x_i|y_i]) = (\mathrm{tr}(\wedge^2\Theta([x_i|y_i]), \ldots, \mathrm{tr}(\wedge^r\Theta([x_i|y_i]))$

Hitchin Map

- *Hitchin map* $h : \mathcal{X}_n^r(\underline{\alpha}) \longrightarrow \bigoplus_{i=2}^r H^0(\mathbf{P}^1, K^i(D)) =: B_{r,D}$ defined by $h([x_i|y_i]) = (\mathrm{tr}(\wedge^2\Theta([x_i|y_i]), \dots, \mathrm{tr}(\wedge^r\Theta([x_i|y_i]))$
- $\dim B_{r,D} = \dfrac{1}{2}\dim \mathcal{X}_n^r(\underline{\alpha})$

Hitchin Map

- *Hitchin map* $h : \mathcal{X}_n^r(\underline{\alpha}) \longrightarrow \bigoplus_{i=2}^r H^0(\mathbf{P}^1, K^i(D)) =: B_{r,D}$ defined by $h([x_i|y_i]) = (\mathrm{tr}(\wedge^2\Theta([x_i|y_i]), \ldots, \mathrm{tr}(\wedge^r\Theta([x_i|y_i]))$
- $\dim B_{r,D} = \dfrac{1}{2}\dim \mathcal{X}_n^r(\underline{\alpha})$
- Components of h Poisson commute

Hitchin Map

- *Hitchin map* $h : \mathcal{X}_n^r(\underline{\alpha}) \longrightarrow \bigoplus_{i=2}^r H^0(\mathbf{P}^1, K^i(D)) =: B_{r,D}$ defined by $h([x_i|y_i]) = (\mathrm{tr}(\wedge^2\Theta([x_i|y_i]), \ldots, \mathrm{tr}(\wedge^r\Theta([x_i|y_i]))$
- $\dim B_{r,D} = \dfrac{1}{2}\dim \mathcal{X}_n^r(\underline{\alpha})$
- Components of h Poisson commute
- To show integrable system, need to show that components are functionally independent (i.e. show that h is surjective)

Spectral Curves

- Spectral curve of $\phi(z)$ is subvariety away from infinity in compactification of $\mathrm{tot}(K(D))$, and covers $\mathbf{P}^1$ generically $r:1$, ramified at repeated eigenvalues

Spectral Curves

- Spectral curve of $\phi(z)$ is subvariety away from infinity in compactification of $\mathrm{tot}(K(D))$, and covers $\mathbf{P}^1$ generically $r:1$, ramified at repeated eigenvalues
- Singular over D

Spectral Curves

- Spectral curve of $\phi(z)$ is subvariety away from infinity in compactification of $\mathrm{tot}(K(D))$, and covers $\mathbf{P}^1$ generically $r : 1$, ramified at repeated eigenvalues
- Singular over D
- In smooth case, *spectral correspondence*: Higgs bundle on the $\mathbf{P}^1$ is equivalent to a generically rank-1 sheaf on the spectral curve (and vice-versa)

Normalization and Lagrangians

Theorem (Fisher, R.)

For $r \geq 2$ and generic $\underline{\alpha}$, $\mathcal{X}_n^r(\underline{\alpha})$ is a completely integrable system.

Normalization and Lagrangians

Theorem (Fisher, R.)

For $r \geq 2$ and generic $\underline{\alpha}$, $\mathcal{X}_n^r(\underline{\alpha})$ is a completely integrable system.

- Extends BFGM[13]

Normalization and Lagrangians

Theorem (Fisher, R.)

For $r \geq 2$ and generic $\underline{\alpha}$, $\mathcal{X}_n^r(\underline{\alpha})$ is a completely integrable system.

- Extends BFGM[13]
- **Main idea of proof:** study rank-1 sheaves on normalizations of spectral curves, show that all Higgs fields with characteristic data in $B_{r,D}$ have the desired nilpotence and parabolic structure

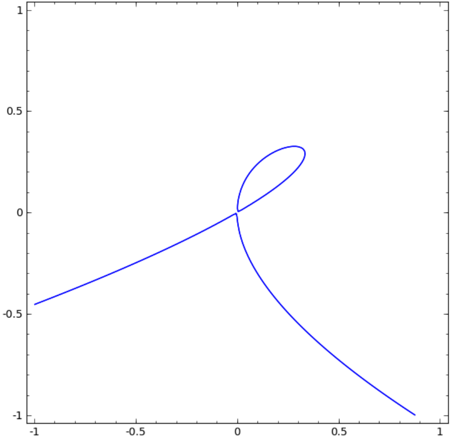
Normalization and Lagrangians

Theorem (Fisher, R.)

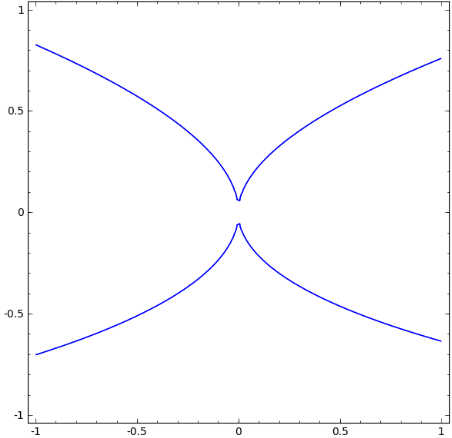
For $r \geq 2$ and generic $\underline{\alpha}$, $\mathcal{X}_n^r(\underline{\alpha})$ is a completely integrable system.

- Extends BFGM[13]
- **Main idea of proof:** study rank-1 sheaves on normalizations of spectral curves, show that all Higgs fields with characteristic data in $B_{r,D}$ have the desired nilpotence and parabolic structure
- For $r \geq 2$, $\mathcal{X}_n^r(\underline{\alpha})$ is Lagrangian fibration over $B_{r,D}$, fibres are $(r-1)(n-r-1)$-dimensional tori (Jacobians of normalized spectral curves)

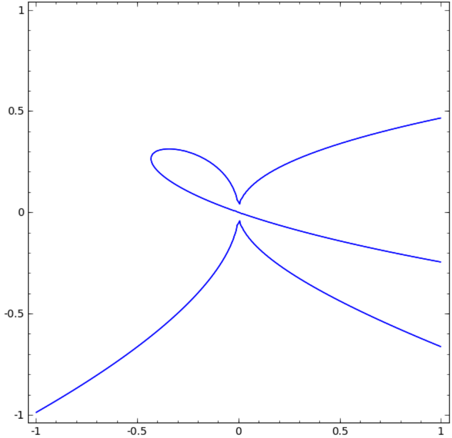
Gallery of Spectra, $r = 3$



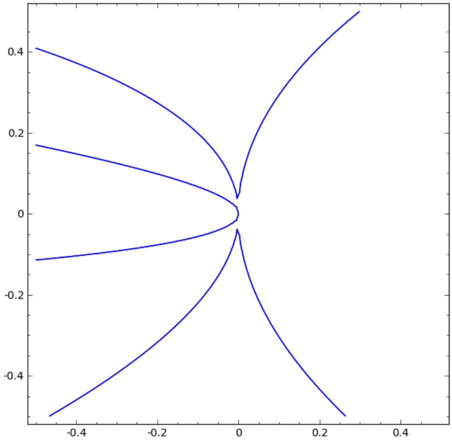
Gallery of Spectra, $r = 4$



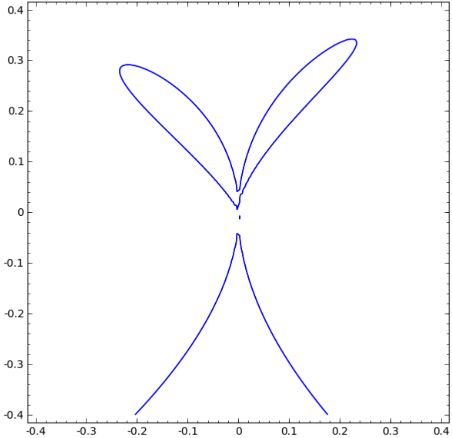
Gallery of Spectra, $r = 5$



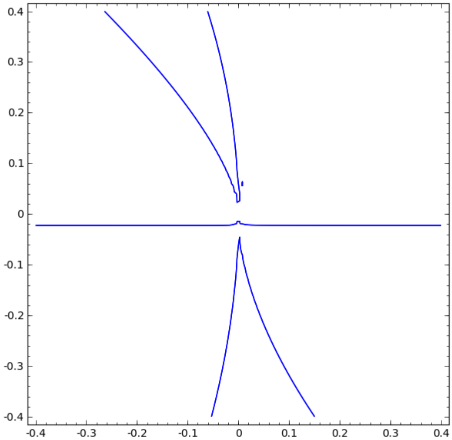
Gallery of Spectra, $r = 6$



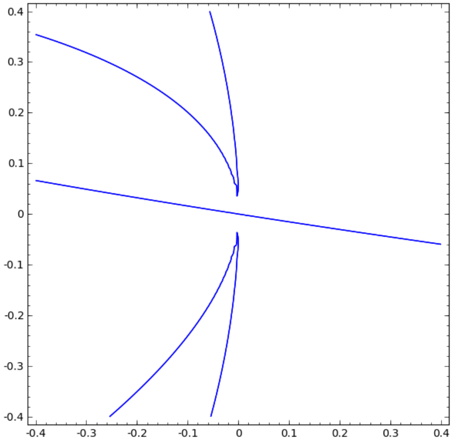
Gallery of Spectra, $r = 7$



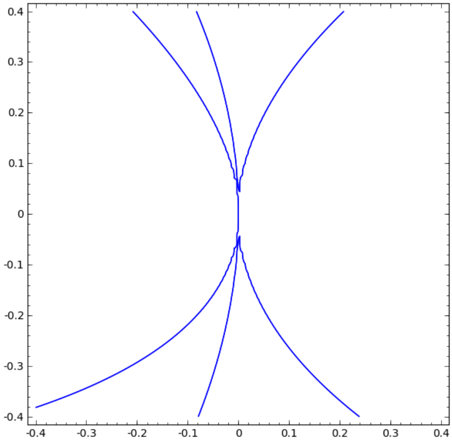
Gallery of Spectra, $r = 8$



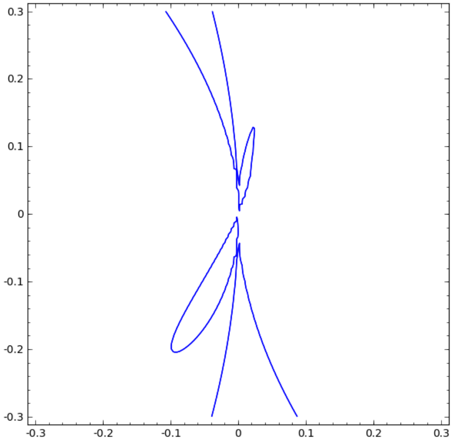
Gallery of Spectra, $r = 9$



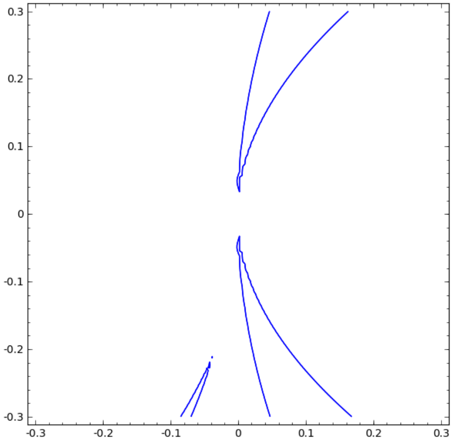
Gallery of Spectra, $r = 10$



Gallery of Spectra, $r = 11$



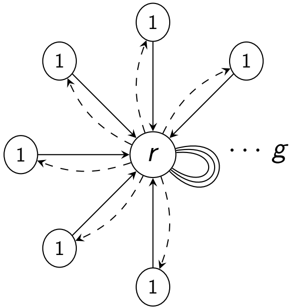
Gallery of Spectra, $r = 12$



Thank You

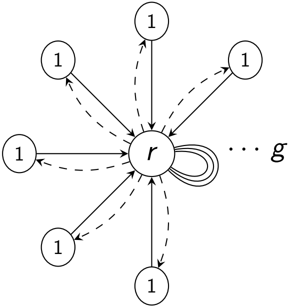
Thank You

Bonus Level: Comets



- Representations are in $GL(r, \mathbb{C})^{2g} \times T^*\mathrm{Rep}(\mathcal{Q})$

Bonus Level: Comets



- Representations are in $GL(r, \mathbb{C})^{2g} \times T^*\mathrm{Rep}(\mathcal{Q})$
- Represent incoming by $\phi_x = [x_1 \cdots x_n]$, outgoing by $\phi_y = [y_1 \cdots y_n]$, loops by $(A_1, B_1, \ldots, A_g, B_g) \in GL(r, \mathbb{C})^{2g}$