GIT characterizations of Harder-Narasimhan filtrations

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Solution of moduli problem $(\mathcal{A}, \sim, \mathcal{F})$ is existence of a moduli space M *representing* or *corepresenting* the functor $\mathcal{F}$

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Correspondence between 2 notions of maximal unstability which appear on GIT constructions of moduli spaces

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E unstable $\Rightarrow \exists\ E'$ of rank $r' < r$ and degree d', $0 \subsetneq E' \subsetneq E$ such that $\mu(E') = \frac{d'}{r'} > \mu(E) = \frac{\deg E}{\operatorname{rk} E}$

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Choose unique E_1 with maximal slope $\mu(E_1) > \mu(E)$ and maximal rank among those of same slope

- Call it *the maximal destabilizing subbundle of E*

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$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E$$

verifying:

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$$\sum n_i (r \dim V_i - r_i \dim V) > 0 \text{ (measure of unstability)}$$

where $r_i = \mathrm{rk}\, E_i$, $V_i \otimes \mathcal{O}_X \twoheadrightarrow E_i(m)$

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Denominator is $\|\Gamma\|$, norm in the space of 1-parameter subgroups measuring its velocity

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Is Kempf filtration = Harder-Narasimhan filtration?

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A pair is δ-semistable if for every subpair $(F, \varphi|_F) \subsetneq (E, \varphi)$

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Take (E, φ) unstable $\Rightarrow q$ GIT unstable, for each $m \in \mathbb{Z}$, $\Rightarrow \exists!$ m-Kempf filtration $(V_\bullet, n_\bullet)$ GIT maximally destabilizing $\overset{\text{evaluating}}{\Longrightarrow} (E_\bullet, \varphi|_{E_\bullet}) \subset (E, \varphi)$, m-Kempf filtration of E

- A holomorphic pair is $(E, \varphi : E \to \mathcal{O}_X)$

Definition [Bradlow - García-Prada, Huybrechts - Lehn]

A pair is δ-semistable if for every subpair $(F, \varphi|_F) \subsetneq (E, \varphi)$

$$\frac{P_F - \delta\varepsilon(F)}{\operatorname{rk} F} \leq \frac{P_E - \delta}{\operatorname{rk} E}\ , \quad \varepsilon(F) = 1 \text{ si } \varphi|_F \neq 0 \text{ and } 0 \text{ otherwise}$$

Take (E, φ) unstable $\Rightarrow$ q GIT unstable, for each $m \in \mathbb{Z}$, $\Rightarrow \exists!$ m-Kempf filtration $(V_\bullet, n_\bullet)$ GIT maximally destabilizing $\overset{\text{evaluating}}{\Longrightarrow}$ $(E_\bullet, \varphi|_{E_\bullet}) \subset (E, \varphi)$, m-Kempf filtration of E

Theorem [Gómez-Sols-Z.]

The m-Kempf filtrations do not depend on m for $m \geq m'$ and coincide with the Harder-Narasimhan filtration $0 \subset (E_1, \varphi|_{E_1}) \subset (E_2, \varphi|_{E_2}) \subset \ldots \subset (E_t, \varphi|_{E_t}) \subset (E, \varphi)$ of a δ-unstable holomorphic pair (E, φ).

Higgs sheaves

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Rank 2 tensors

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Definition

The Kempf filtration $0\subset(L,\varphi|_L)\subset(E,\varphi)$ defines, by uniqueness, the Harder-Narasimhan filtration of (E,φ)

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M is (Θ, σ)-semistable if $\forall\, M' \subsetneq M$, $\mu_{(\Theta,\sigma)}(M') \leq \mu_{(\Theta,\sigma)}(M)$, where $\mu_{(\Theta,\sigma)}(M) = \frac{\Theta(M)}{\sigma(M)}$ and $\Theta, \sigma : \mathbb{Z}Q_0 \to \mathbb{Z}$, $\sigma(v) > 0$

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1. $\mu_{(\Theta,\sigma)}(M^1) > \mu_{(\Theta,\sigma)}(M^2) > \ldots > \mu_{(\Theta,\sigma)}(M^t) > \mu_{(\Theta,\sigma)}(M^{t+1})$
2. $M^i := M_i / M_{i-1}$ are (Θ, σ)-semistable

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Theorem [Terpereau, Z.]

Given a θ-unstable (G, h)-constellation $\mathcal{F}$, the D-filtrations converge to its Harder-Narasimhan filtration when $D \to \operatorname{Irr} G$.

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Non-abelian categories

- Harder-Narasimhan filtrations in non-abelian categories?