Groups and Automata: a Perfect Match

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Abstract. We present a personal perspective, inspired by our own research experience, of the interaction between group theory and automata theory: from Benois' Theorem to Stallings' automata, from hyperbolic to automatic groups, not forgetting the exotic automaton groups.

1 Introduction

Among abstract structures, it is groups which model the idea of symmetry in Mathematics. Moreover, the existence of inverses makes them a natural model for reversibility in theoretical computer science (see [37] for a model for *partial reversibility*). At the present time, when quantum computation gives its first steps (note than in quantum mechanics transformations are always assumed reversible), it is appropriate to make the history of the interaction between group theory and automata theory, undoubtedly the branch of theoretical computer science which has been playing the major role in the development of combinatorial and geometric group theory.

We intend this text to be a brief and light account of these interactions, under a personal perspective which emerged from our own work on the subject, and relating to our talk at DCFS 2012. We therefore chose to leave out finite groups (and the connections with group languages), being out of our own experience. Anyway, such connections are well known in theoretical computer science and can be easily found in the literature on finite automata [8, 32].

A deeper and more extended survey on the interactions groups/automata can be found out in two Handbook chapters written by Bartholdi and the author [5, 4].

We shall pay special attention to free groups: we introduce them in Section 2, discuss language-theoretic concepts in Section 3 and the representation of finitely generated subgroups by automata in Section 4. We shall also explain the role played by automata in the study of three important classes of groups: hyperbolic groups in Section 5, automatic groups in Section 6 and automaton groups (also known as self-similar groups) in Section 7. In Section 8, we present an example of our recent research combining automata-theoretic and group-theoretic results.

We assume the reader to be familiar with the basic concepts of language theory and automata theory, and to know the most basic definitions of group theory. Throughout the whole paper, we assume alphabets to be *finite*.

2 Free Groups

We start by introducing free groups. Informally, the free group on A is supposed to be the most general group F_A we can generate from a given set A, in the sense that every other group generated by A turns out to be a quotient of F_A. Hence free groups play pretty much in the context of groups the same role that free monoids play in the context of monoids.

We present now the formal definition. Given an alphabet A, we denote by A^{-1} a set of *formal inverses* of A. We write $\widetilde{A} = A \cup A^{-1}$ and $(a^{-1})^{-1} = a$ for every $a \in A$. The *free group on* A, denoted by F_A, is the quotient of $\widetilde{A}^*$ by the congruence generated by the relation

$$\mathcal{R}_A = \{(aa^{-1}, 1) \mid a \in \widetilde{A}\}.$$

Thus two words $u, v \in \widetilde{A}^*$ are equivalent in F_A if and only if one can be transformed into the other by successively inserting/deleting factors of the form aa^{-1} ($a \in \widetilde{A}$). We denote by $\theta : \widetilde{A}^* \to F_A$ the canonical morphism.

We recall that a (finite) *rewriting system* on A is a (finite) subset $\mathcal{R}$ of $A^* \times A^*$. Given $u, v \in A^*$, we write $u \longrightarrow_{\mathcal{R}} v$ if there exist $(r, s) \in \mathcal{R}$ and $x, y \in A^*$ such that $u = xry$ and $v = xsy$. The reflexive and transitive closure of $\longrightarrow_{\mathcal{R}}$ is denoted by $\longrightarrow^*_{\mathcal{R}}$.

We say that $\mathcal{R}$ is:

- *length-reducing* if $|r| > |s|$ for every $(r, s) \in \mathcal{R}$;
- *confluent* if, whenever $u \longrightarrow^*_{\mathcal{R}} v$ and $u \longrightarrow^*_{\mathcal{R}} w$, there exists some $z \in A^*$ such that $v \longrightarrow^*_{\mathcal{R}} z$ and $w \longrightarrow^*_{\mathcal{R}} z$.

A word $u \in A^*$ is an *irreducible* if no $v \in A^*$ satisfies $u \longrightarrow_{\mathcal{R}} v$. We denote by $\mathrm{Irr}\,\mathcal{R}$ the set of all irreducible words in A^* with respect to $\mathcal{R}$.

If $\mathcal{R}$ is symmetric, then $\tau = \longrightarrow^*_{\mathcal{R}}$ is a congruence on A^* and we can say that the pair $\langle A \mid \mathcal{R} \rangle$ constitutes a (monoid) *presentation*, defining the monoid A^*/τ.

If we view $\mathcal{R}_A$ as a rewriting system on $\widetilde{A}$, then it turns out to be both length-reducing and confluent, and so, for every $g \in F_A$, $g\theta^{-1}$ contains a unique irreducible word, denoted by $\overline{g}$ (see [9]). We write also $\overline{u} = \overline{u\theta}$ for every $u \in \widetilde{A}^*$. Note that the equivalence $u\theta = v\theta \;\Leftrightarrow\; \overline{u} = \overline{v}$ holds for all $u, v \in \widetilde{A}^*$, providing the usual solution for the word problem of a free group (deciding whether two words on the generators represent the same element of the group). Thus the elements of a free group can be efficiently described as irreducible words.

We denote by

$$R_A = \widetilde{A}^* \setminus (\cup_{a \in \widetilde{A}} \widetilde{A}^* aa^{-1} \widetilde{A}^*)$$

the set of all *irreducible words* in $\widetilde{A}^*$ for the rewriting system $\mathcal{R}_A$. Clearly, R_A is a rational language.

3 Language Theory for Groups

If we follow Berstel's general approach to language theory [8], *rational* and *recognizable* emerge as two of the most important basic concepts. In this context, recognizable refers to finite syntactic monoids or recognizability by finite monoids. Of course, both concepts coincide for free monoids (Kleene's Thorem [8, Theorem I.4.1]) but nor for arbitrary monoids. Given a monoid M, we denote by $\mathrm{Rat}\, M$ (respectively $\mathrm{Rec}\, M$) the set of all rational (respectively recognizable) subsets of M.

To understand the situation in the context of groups, we need the following classical result of Anisimov and Seifert:

Proposition 31 [8, Theorem III.2.7] *Let H be a subgroup of a group G. Then $H \in \mathrm{Rat}\, G$ if and only if H is finitely generated.*

The analogous result for recognizable is part of the folklore of the theory. We recall that a subgroup H of G has *finite index* if G is a finite union of cosets Hg $(g \in G)$.

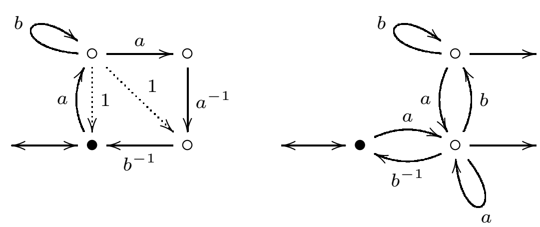
Proposition 32 *Let H be a subgroup of a group G. Then $H \in \mathrm{Rec}\, G$ if and only if H has finite index in G.*

Since the trivial subgroup has finite index in G if and only if G is finite, it follows that $\mathrm{Rat}\, G = \mathrm{Rec}\, G$ if and only if G is finite. In general, these two classes fail most nontrivial closure properties. However, free groups present a much better case, due to the seminal *Benois' Theorem*:

Theorem 33 [7]

(i) If $L \in \mathrm{Rat}\, \widetilde{A}^*$, then $\overline{L} \in \mathrm{Rat}\, \widetilde{A}^*$ and can be effectively constructed from L.
(ii) If $X \subseteq F_A$, then $X \in \mathrm{Rat}\, G$ if and only if $\overline{X} \in \mathrm{Rat}\, \widetilde{A}^*$.

The proof consists essentially on successively adding edges labelled by the empty word to an automaton recognizing L (whenever a path is labelled by aa^{-1} $(a \in \widetilde{A})$) and intersecting in the end the corresponding language with the rational language R_A.



L $\qquad\qquad$ $\overline{L}$

The following result summarizes some of the most direct consequences of Benois' Theorem:

Corollary 34 *(i) Every* $X \in \operatorname{Rat} F_A$ *is recursive.*
(ii) $\operatorname{Rat} F_A$ *is closed under the boolean operations.*

We remark that Theorem 33 has been successively adapted to groups/monoids defined by more general classes of rewriting systems, the most general versions being due to Sénizergues [33, 34].

Since F_A is a finitely generated monoid, it follows that every recognizable subset of F_A is rational [8, Proposition III.2.4]. The problem of deciding which rational subsets of F_A are recognizable was first solved by Sénizergues [34]. A shorter alternative proof was presented by the author in [36], where a third alternative proof, of a more combinatorial nature, was also given.

These results are also related to the Sakarovitch conjecture [32], solved in [34] (see also [36]), which states that every rational subset of F_A must be either recognizable or *disjunctive* (it has trivial syntactic congruence).

The quest for groups G such that $\operatorname{Rat} G$ enjoys good properties has spread over the years to wider classes of groups. An important case is given by *virtually free groups*, i.e. groups having a free subgroup of finite index, as remarked by Grunschlag [21]. In fact, in view of Nielsen's Theorem, this free subgroup can be assumed to be *normal* [27]. Virtually free groups will keep making unexpected appearances throughout this paper.

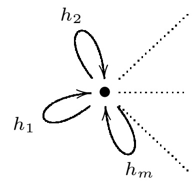
Another important case is given by free partially abelian groups (the group-theoretic version of trace monoids). Lohrey and Steinberg proved in [26] that the recursiveness of the rational subsets depends on the independence graph being a *transitive forest*.

A different idea of relating groups and language theory involves the classification of the set $1\pi^{-1} \subseteq \widetilde{A}^*$ which collects all the words representing the identity for a given matched surjective homomorphism $\pi : \widetilde{A}^* \to G$ (matched in the sense that $a^{-1}\pi = (a\pi)^{-1}$ for every $a \in A$). Clearly, $1\pi^{-1}$ determines the structure of G, and it is a simple exercise to show that $1\pi^{-1}$ is rational if and only if G is finite. What about higher classes in the Chomsky's hierarchy? The celebrated theorem proved by Muller and Schupp (with a contribution from Dunwoody) states the following:

Theorem 35 [30, 10] *Let* $\pi : \widetilde{A}^* \to G$ *be a matched homomorphism onto a group* G*. Then* $1\pi^{-1}$ *is a context-free language if and only if* G *is virtually free.*

4 Stallings Automata

Finite automata became over the years the standard representation of finitely generated subgroups H of a free group F_A. The *Stallings construction* constitutes a simple and efficient algorithm for building an automaton $\mathcal{S}(H)$ which can be used for solving the membership problem for H in F_A and many other applications. Many features of $\mathcal{S}(H)$, which has a geometric interpretation (the core of the Schreier graph of H) were (re)discovered over the years and were known to Reidemeister, Schreier, and particularly Serre [35]. One of the greatest contributions of Stallings [41] is certainly the algorithm to construct $\mathcal{S}(H)$: taking a finite set of generators $h_1, \ldots, h_m$ of H in reduced form, we start with the so-called flower automaton $\mathcal{F}(H)$, where *petals* labelled by the words h_i (and their inverse edges) are glued to a basepoint q_0 (both initial and terminal vertex):



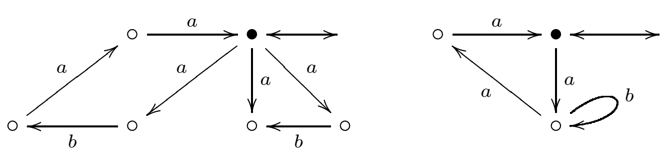
Then we proceed by successively folding pairs of edges of the form $q \xleftarrow{a} p \xrightarrow{a} r$ until reaching a deterministic automaton. And we will have just built $\mathcal{S}(H)$. For details and applications of the Stallings construction, see [5, 24, 29].

The geometric interpretation of $\mathcal{S}(H)$ shows that its construction is independent of the finite set of generators of H chosen at the beginning, and of the particular sequence of foldings followed. And the membership problem is a consequence of the following result:

Theorem 41 [41] *Let H be a finitely generated subgroup of F_A and let $u \in R_A$. Then u represents an element of H if and only if $u \in L(\mathcal{S}(H))$.*

The main reason for this is that any irreducible word representing an element of H can be obtained by successively cancelling factors aa^{-1} in a word accepted by the flower automaton of H, and folding edges is a geometric realization of such cancellations.

For instance, taking $H = \langle aba^{-1}, aba^2 \rangle$, we get



$\mathcal{F}(H)$ $\mathcal{S}(H)$

We can then deduce that a^3 represents an element of H but a^4 does not.

The applications of Stallings automata to the algorithmics of finitely generated subgroups of a free group are immense. One of the most important is the construction of a *basis* for H (a free group itself by Nielsen's Theorem) using a *spanning tree* of $\mathcal{S}(H)$.

The following result illustrates how automata-theoretic properties of $\mathcal{S}(H)$ can determine group-theoretic properties of H:

Proposition 42 [41] *Let H be a finitely generated subgroup of F_A. Then H is a finite index subgroup of F_A if and only if $\mathcal{S}(H)$ is a complete automaton.*

Note that Stallings automata constitute examples of *inverse* automata: they are deterministic, trim and (p, a, q) is an edge if and only if (q, a^{-1}, p) is an edge. Inverse automata play a major role in the geometric theories of groups and, more generally, inverse monoids [42].

The Stallings construction invites naturally generalizations for further classes of groups. For instance, an elegant geometric construction of Stallings type automata was achieved for amalgams of finite groups by Markus-Epstein [28]. On the other hand, the most general results were obtained by Kapovich, Weidmann and Miasnikov [25], but the complex algorithms were designed essentially to solve the generalized word problem, and it seems very hard to extend other features of the free group case, either geometric or algorithmic. In joint work with Soler-Escrivà and Ventura [39], the author developed a new idea: restricting the type of irreducible words used to represent elements (leading to the concept of *Stallings section*), find out which groups admit a representation of finitely generated subgroups by finite automata obtained through edge folding from some sort of flower automaton. It turned out that the groups admitting a Stallings section are precisely the virtually free groups! And many of the geometric/algorithmic features of the classical free group case can then be generalized to the virtually free case.

5 Hyperbolic Groups

Automata also play an important role in the beautiful geometric theory of hyperbolic groups, introduced by Gromov in the eighties [20]. For details on this class of groups, the reader is referred to [12].

Let $\pi : \widetilde{A}^* \to G$ be a matched epimorphism onto a group G. The *Cayley graph* $\Gamma_A(G)$ of G with respect to π has vertex set G and edges $g \xrightarrow{a} g(a\pi)$ for all $g \in G$ and $a \in \widetilde{A}$. If we fix the identity as basepoint, we get an inverse automaton (which is precisely the minimal automaton of the language $1\pi^{-1}$).

If $G = F_A$ and π is canonical, then $\Gamma_A(F_A)$ is an infinite tree. In particular, the local structure of $\Gamma_A(F_A)$ determines the global structure... and if we understand the global structure of the Cayley graph, then we understand the group.

So the aim is to consider geometric conditions on the structure of $\Gamma_A(G)$ that can lead to a global understanding of the Cayley graph through the local structure (taking finitely many finite subgraphs of $\Gamma_A(G)$ as local charts, actually). But which conditions? The answer came in the form of *hyperbolic geometry*. What does this mean and how does it relate to automata or theoretical computer science in general?

We say that a path $p \xrightarrow{u} q$ in $\Gamma_A(G)$ is a *geodesic* if it has shortest length among all the paths connecting p to q in $\Gamma_A(G)$. We denote by $\mathrm{Geo}_A(G)$ the set of labels of all geodesics in $\Gamma_A(G)$. Note that, since $\Gamma_A(G)$ is vertex-transitive (the left action of G on itself produces enough automorphisms of $\Gamma_A(G)$ to make it completely symmetric), it is irrelevant whether or not we fix a basepoint for this purpose.

The *geodesic distance* d on G is defined by taking $d(g,h)$ to be the length of a geodesic from g to h. Given $X \subseteq G$ nonempty and $g \in G$, we define

$$d(g,X) = \min\{d(g,x) \mid x \in X\}.$$

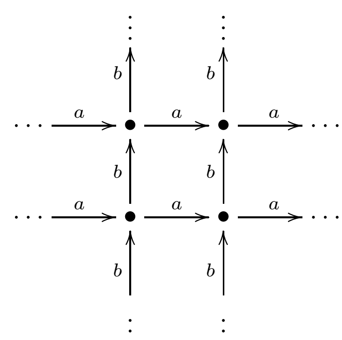
A *geodesic triangle* in $\Gamma_A(G)$ is a collection of three geodesics

$$P_1 : g_1 \longrightarrow g_2, \quad P_2 : g_2 \longrightarrow g_3, \quad P_3 : g_3 \longrightarrow g_1$$

connecting three vertices $g_1, g_2, g_3 \in G$. Let $V(P_i)$ denote the set of vertices occurring in the path P_i. We say that $\Gamma_A(G)$ is *δ-hyperbolic* for some $\delta \geq 0$ if

$$\forall g \in V(P_1) \qquad d(g, V(P_2) \cup V(P_3)) < \delta$$

holds for every geodesic triangle $\{P_1, P_2, P_3\}$ in $\Gamma_A(G)$. If this happens for some δ, we say that G is *hyperbolic*. It is well known that the concept is independent from both alphabet and matched epimorphism, but the hyperbolicity constant δ may change. Virtually free groups are among the most important examples of hyperbolic groups (in fact, they can be characterized by strengthening the geometric condition in the definition of hyperbolicity, replacing geodesic triangles by geodesic polygons). However, the free Abelian group $\mathbb{Z} \times \mathbb{Z}$, whose Cayley graph (for the canonical generators) is the infinite grid



is not hyperbolic. However, there exist plenty of hyperbolic groups: Gromov remarked that, under some reasonable assumptions, the probability of a finitely presented group being hyperbolic is 1.

One of the extraordinary geometric properties of hyperbolic groups is closure under quasi-isometry, being thus one of the few examples where algebra deals well with the concept of *deformation*.

From an algorithmic viewpoint, hyperbolic groups enjoy excellent properties: they have solvable word problem, solvable conjugacy problem and many other positive features. We shall enhance three, which relate to theoretic computer science.

The first result states that geodesics constitute a rational language.

Theorem 51 [11, Theorem 3.4.5] *Let $\pi : \widetilde{A}^* \to G$ be a matched homomorphism onto a hyperbolic group G. Then the set of geodesics* $\mathrm{Geo}_A(G)$ *is a rational language.*

The second one shows how $1\pi^{-1}$ can be described by means of a suitable rewriting system:

Theorem 52 [2] *Let $\pi : \widetilde{A}^* \to G$ be a matched homomorphism onto a group G. Then the following conditions are equivalent:*

(i) G is hyperbolic;
(ii) there exists a finite length-reducing rewriting system $\mathcal{R}$ such that

$$\forall u \in \widetilde{A}^* \qquad u \in 1\pi^{-1} \Leftrightarrow u \longrightarrow^*_{\mathcal{R}} 1.$$

It follows easily that $1\pi^{-1}$ is a context-sensitive language if G is hyperbolic. However, the converse fails, $\mathbb{Z} \times \mathbb{Z}$ being a counter-example.

In connection with the preceding theorem, it is interesting to recall a result by Gilman, Hermiller, Holt and Rees [14, Theorem 1], which states that a group G is virtually free if and only if there exists a matched homomorphism $\pi : \widetilde{A}^* \to G$ and a finite length-reducing rewriting system $\mathcal{R} \subseteq \mathrm{Ker}\,\pi$ such that $\mathrm{Irr}\,\mathcal{R} = \mathrm{Geo}_A(G)$.

The third property is possibly the most intriguing. To present it, we need to introduce the concept of isoperimetric function.

Suppose that G is a group defined by a finite presentation $\mathcal{P} = \langle \widetilde{A} \mid \mathcal{R} \rangle$, and let $\pi : \widetilde{A}^* \to G$ be the respective matched homomorphism. We say that $\delta : \mathbb{N} \to \mathbb{N}$ is an *isoperimetric function* for $\mathcal{P}$ if, whenever $u \in 1\pi^{-1}$, we need at most $\delta(|u|)$ transitions $\longrightarrow_{\mathcal{R}}$ to transform u into the empty word 1. In other words, an isoperimetric function bounds the number of elementary transitions we need to transform a word of a certain length into the empty word.

It is easy to see that the existence of an isoperimetric function belonging to a certain complexity class depends only on the group and not on the finite presentation considered. We note also that every hyperbolic group is finitely presented.

Theorem 53 [20] *Let G be a finitely presented group. Then the following conditions are equivalent:*

(i) G is hyperbolic;
(ii) G admits a linear isoperimetric function;
(iii) G admits a subquadratic isoperimetric function.

In this extraordinary result, geometry unexpectedly meets complexity theory.

6 Automatic Groups

Also in the eighties, another very interesting idea germinated in geometric group theory, and automata were to play the leading role. The new concept was due to Cannon, Epstein, Holt, Levy, Paterson and Thurston [11] (see also [6]).

In view of Theorem 51, it is easy to see that every hyperbolic group admits a rational set of normal forms. But this is by no means an exclusive of hyperbolic groups, and rational normal forms are not enough to understand the structure of a group. We need to understand the product, or at least the action of generators on the set of normal forms. Can automata help?

There are different ways of encoding mappings as languages, synchronously or asynchronously. We shall mention only the most popular way of doing it, through *convolution*.

Given an alphabet A, we assume that \$ is a new symbol (called the *padding symbol*) and define a new alphabet

$$A_{\$} = (A \times A) \cup (A \times \{\$\}) \cup (\{\$\} \times A).$$

For all $u, v \in A^*$, $u \diamond v$ is the unique word in $A_{\$}^*$ whose projection to the first (respectively second) components yields a word in $u\$^*$ (respectively $v\$^*$). For instance, $a \diamond ba = (a, b)(\$, a)$.

Let $\pi : A^* \to G$ be a homomorphism onto a group G. We say that $L \in \operatorname{Rat} A^*$ is a *section* for π if $L\pi = G$. For every $u \in A^*$, write

$$L_u = \{v \diamond w \mid v, w \in L,\ (vu)\pi = w\pi\}.$$

We say that $L \in \operatorname{Rat} A^*$ is an *automatic structure* for π if:

- L is a section for π;
- $L_a \in \operatorname{Rat} A_{\$}^*$ for every $a \in A \cup \{1\}$.

It can be shown that the existence of an automatic structure is independent from the alphabet A or the homomorphism π, and implies the existence of an *automatic structure with uniqueness* (where $\pi|_L$ is injective). A group is said to be *automatic* if it admits an automatic structure.

The class of automatic groups contains all hyperbolic groups (in fact, $\operatorname{Geo}_A(G)$ is then an automatic structure!) and is closed under such operators as free products, finite extensions or direct products. As a consequence, it contains all free abelian groups of finite rank and so automatic groups need not be hyperbolic. By the following result of Gilman, hyperbolic groups can be characterized within automatic groups by a language-theoretic criterion:

Theorem 61 [13] *Let G be a group. Then the following conditions are equivalent:*

(i) G is hyperbolic;
(ii) G admits an automatic structure with uniqueness L such that the language $\{u\$v\$w \mid u, v, w \in L,\ uvw =_G 1\}$ is context-free.

Among many other good algorithmic properties, automatic groups are finitely presented, have decidable word problem (in quadratic time) and admit a quadratic isoperimetric function (but the converse is false, unlike Theorem 53). The reader is referred to [6, 11] for details.

Geometry also plays an important part in the theory of automatic groups, through the *fellow traveller property*. Given a word $u \in A^*$, let $u^{[n]}$ denote the prefix of u of length n (or u itself if $n > |u|$). Let $\pi : \widetilde{A}^* \to G$ be a matched homomorphism and recall the geodesic distance d on G introduced in Section 5 in connection with the Cayley graph $\Gamma_A(G)$. We say that a section L for π satisfies the fellow traveller property if there exists some constant $K > 0$ such that

$$\forall u, v \in L \; (d(u\pi, v\pi) \leq 1 \Rightarrow \forall n \in \mathbb{N} \; d(u^{[n]}\pi, v^{[n]}\pi) \leq K).$$

Intuitively, this expresses the fact that two paths in $\Gamma_A(G)$ labelled by words $u, v \in L$ which start at the same vertex and end up in neighbouring (or equal) vertices *stay close all the way through*.

This geometric property provides an alternative characterization of automatic groups which avoids convolution:

Theorem 62 [11, Theorem 2.3.5] *Let $\pi : \widetilde{A}^* \to G$ be a matched homomorphism onto a group G and let L be a rational section for π. Then the following conditions are equivalent:*

(i) L is an automatic structure for π;
(ii) L satisfies the fellow traveller property.

The combination of automata-theoretic and geometric techniques is typical of the theory of automatic groups.

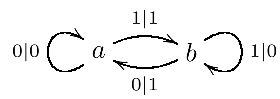
7 Automaton Groups

Automaton groups, also known as self-similar groups, were introduced in the sixties by Glushkov [15] (see also [1]) but it was through the leading work of Grigorchuk in the eighties [18] that they became a main research subject in geometric group theory. Here automata play a very different role compared with previous sections.

We can view a free monoid A^* as a rooted tree T with edges u — ua for all $u \in A^*$, $a \in A$ and root 1. The automorphism group of T, which is uncountable if $|A| > 1$, is self-similar in the following sense: if we restrict an automorphism φ of T to a cone uA^*, we get a mapping of the form $uA^* \to (u\varphi)A^* : uv \mapsto (u\varphi)(v\psi)$ for some automorphism ψ of T. This leads to wreath product decompositions (see [31]) and the possibility of recursion.

But Aut$T$ is huge and non finitely generated except in trivial cases, hence it is a natural idea to study subgroups G of T generated by a finite set of self-similar generators (in the above sense) to keep all the chances of effective recursion methods within a finitely generated context. It turns out that this is equivalent to define G through a finite invertible *Mealy automaton*.

A Mealy automaton on the alphabet A is a finite complete deterministic transducer where edges are labelled by pairs of letters of A. No initial/terminal vertices are assigned. It is said to be invertible if the local transformations of A (induced by the labels of the edges leaving a given vertex) are permutations. Here is a famous example of an invertible Mealy automaton:



The transformations of $A = \{0, 1\}$ induced by the vertices a and b are the identity mapping and the transposition (01), respectively.

Each vertex q of a Mealy automaton $\mathcal{A}$ defines an endomorphism φ_q of the tree T through the paths $q \xrightarrow{u|u\varphi_q} \dots$ ($u \in A^*$). If the automaton is invertible, each φ_q is indeed an automorphism and the set of all φ_q, for all vertices q of $\mathcal{A}$, satisfies the desired self-similarity condition. The (finitely generated) subgroup of Aut$T$ generated by the φ_q is the automaton group $\mathcal{G}(\mathcal{A})$ generated by $\mathcal{A}$.

For instance, the automaton group generated by the Mealy automaton in the above example is the famous *lamplighter group* [17].

Automaton groups have decidable word problem. Moreover, the recursion potential offered by their wreath product decompositions allowed successful computations which were hard to foresee with more traditional techniques and turned automaton groups into the most rich source of counterexamples in infinite group theory ever. The Grigorchuk group [18] is the most famous of the lot, but their exist many others exhibiting fascinating exotic properties [22, 19].

An interesting infinite family of Mealy automata was studied by the author in collaboration with Steinberg [40] and Kambites and Steinberg [23]: Cayley machines of finite groups G (the Cayley graph is adapted by taking edges $g \xrightarrow{a|g(a\pi)} g(a\pi)$, and all the elements of the group as generators). If G is abelian, these Cayley machines generate the wreath product $G \,\mathrm{wr}\, \mathbb{Z}$, and the lamplighter group corresponds to the case $G = \mathbb{Z}_2$.

Surprising connections with fractals were established in recent years. We shall briefly describe one instance. Given a matched homomorphism $\pi : \widetilde{A}^* \to G$ and a subgroup P of G, the *Schreier graph* $\Gamma_A(G, P)$ has the cosets Pg as vertices and edges $Pg \xrightarrow{a} Pg(a\pi)$ for all $g \in G$ and $a \in \widetilde{A}$. Note that $P = \{1\}$ yields the familiar Cayley graph $\Gamma_A(G)$. It turns out that classical fractals can be obtained as limits of the sequence of graphs $(\Gamma_A(G, P_n))_n$ for some adequate automaton group G, where P_n denotes the stabilizer of the nth level of the three T [3, 31]. Note that P_n has finite index and so the Schreier graphs $\Gamma_A(G, P_n)$ are finite.

8 Automata and Dynamics

Automata appear also as a major tool in the study of the dynamics of many families of group endomorphisms. We shall present an example taken from our own recent research work [38].

We shall call $\mathcal{T} = (Q, q_0, \delta, \lambda)$ an A-*transducer* if:

– Q is a (finite) set;
– $q_0 \in Q$;
– $\delta : Q \times A \to Q$ and $\lambda : Q \times A \to A^*$ are mappings.

We can view $\mathcal{T}$ as a directed graph with edges labelled by elements of $A \times A^*$ (represented in the form $a|w$) by identifying $(p,a)\delta = q$, $(p,a)\lambda = w$ with the edge $p \xrightarrow{a|w} q$.

We may extend δ and λ to $Q \times A^*$ by considering the paths $q \xrightarrow{u|(q,u)\lambda} (q,u)\delta$ for all $u \in A^*$. When the transducer is clear from the context, we write $qa = (q,a)\delta$. The transformation $\widehat{\mathcal{T}} : A^* \to A^*$ is defined by $u\widehat{\mathcal{T}} = (q_0, u)\lambda$.

If $\mathcal{T} = (Q, q_0, T, \delta, \lambda)$ is an $\widetilde{A}$-transducer such that

$$p \xrightarrow{a|u} q \text{ is an edge of } \mathcal{T} \text{ if and only if } q \xrightarrow{a^{-1}|u^{-1}} p \text{ is an edge of } \mathcal{T},$$

then $\mathcal{T}$ is said to be *inverse*.

As an easy consequence of this definition, we get:

Proposition 81 [38, Proposition 3.1] *Let $\mathcal{T} = (Q, q_0, \delta, \lambda)$ be an inverse $\widetilde{A}$-transducer. Then:*

(i) $\delta : Q \times \widetilde{A}^* \to Q$ induces a mapping $\widetilde{\delta} : Q \times F_A \to Q$ by $(q, u\theta)\widetilde{\delta} = (q,u)\delta$;
(ii) $\widehat{\mathcal{T}} : \widetilde{A}^* \to \widetilde{A}^*$ induces a partial mapping $\widetilde{\mathcal{T}} : F_A \to F_A$ by $u\theta\widetilde{\mathcal{T}} = u\widehat{\mathcal{T}}\theta$.

We can prove the following result:

Theorem 82 [38, Theorem 3.2] *Let $\mathcal{T}$ be a finite inverse $\widetilde{A}$-transducer and let $z \in F_A$. Then*

$$L = \{g \in F_A \mid g\widetilde{\mathcal{T}} = gz\}$$

is rational.

The proof is inspired in Goldstein and Turner's proof [16] for endomorphisms of the free group. We give a brief sketch.

Write $\mathcal{T} = (Q, q_0, \delta, \lambda)$. For every $g \in F_A$, let $P_1(g) = g^{-1}(g\widetilde{\mathcal{T}}) \in F_A$ and write $q_0 g = (q_0, g)\widetilde{\delta}$, $P(g) = (P_1(g), q_0 g)$. Note that $g \in L$ if and only if $P_1(g) = z$. We define a deterministic $\widetilde{A}$-automaton $\mathcal{A}_\varphi = (P, (1, q_0), S, E)$ by

$P = \{P(g) \mid g \in F_A\}$;
$S = P \cap (\{z\} \times Q)$;
$E = \{(P(g), a, P(ga)) \mid g \in F_A,\ a \in \widetilde{A}\}$.

Clearly, $\mathcal{A}_\varphi$ is a possibly infinite automaton. Note that, since $\mathcal{T}$ is inverse, we have $qaa^{-1} = q$ for all $q \in Q$ and $a \in \widetilde{A}$. It follows that, whenever $(p, a, p') \in E$, then also $(p', a^{-1}, p) \in E$. We say that such edges are the *inverse* of each other.

Since every $w \in \widetilde{A}^*$ labels a unique path $P(1) \xrightarrow{w} P(w\theta)$, it follows that

$$L(\mathcal{A}_\varphi) = L\theta^{-1}.$$

To prove that L is rational, we show that only finitely many edges can occur in the successful paths of $\mathcal{A}_\varphi$ labelled by reduced words.

This is achieved by defining an appropriate subset $E' \subseteq E$ satisfying $E = E' \cup (E')^{-1}$ and showing that there are only finitely many vertices in $\mathcal{A}_\varphi$ which are starting points for more than one edge in E'.

Theorem 82 can be used to produce an alternative proof [38, Theorem 4.1] of the following Sykiotis' theorem:

Theorem 83 [43, Proposition 3.4] *Let φ be an endomorphism of a finitely generated virtually free group. Then* $\mathrm{Fix}\,\varphi$ *is finitely generated.*

Automata are also at the heart of other results in [38], concerning the *infinite fixed points* of endomorphism extensions to the boundary of virtually free groups. The boundary is a very important topological concept defined for hyperbolic groups [12], but out of the scope of this paper.

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