Title

Quasi Euclidean Rings

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Plan

A Classical commutative Euclidean rings.

- Euclidean domains, Principal ideal domains, UFD.
- The transfinite case.
- The k-stage euclidean rings.
- Continuous fractions.

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- Some examples.
- Continuant polynomials.
- Some properties of continuant polynomials.
- Characterizations of Euclidean rings.
- More Properties.

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C Euclidean Modules
- Definition
- Some properties.
- Morita Context

Layout

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Euclidean domains

Definition

A commutative domain R is an euclidean domain if there exists a map $\delta : R \setminus 0 \longrightarrow \mathbb{N}$ such that for any $a, b \in R$ with $b \neq 0$, there exist $q, r \in R$with $a = bq + r$ and either $r = 0$ or $\delta(r) < \delta(b)$

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Examples

(a) $\mathbb{Z}$ $(\delta(n) = |n|)$, $\mathbb{Z}[i]$ $(\delta(a + bi) = a^2 + b^2)$, $\mathbb{Z}[\omega]$ where $\omega = e^{2i\pi/3}$ $(\delta(a + bw) = a^2 - ab + b^2)$, $k[x]$ where k is a field (δ is the degree)

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(b) Sometimes δ is multiplicative, the ring R is then said to be normed Euclidean. The norm of the quadratic field restricts sometimes to a norm for the ring of integers of that field. Even if this is not the case this ring can still be Euclidean.

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    - Those that are norm-Euclidean, such as Gaussian integers .
2. The norm-Euclidean quadratic fields have been fully classified, they are $\mathbb{Q}(\sqrt{d})$ where d takes the values $-11, -7, -3, -2, -1, 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, \ldots, 73$.

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3. Every Euclidean imaginary quadratic field is norm-Euclidean and is one of the five first fields in the preceding list.

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Theorem

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2. *If R is a Euclidean domain but not a field, then R has a universal side divisor.*

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- The last property was used by Motzkin to show that the ring $\mathbb{Z}(\alpha)$ where $\alpha = \frac{1}{2}(1+\sqrt{-19})$ is a PID but not a Euclidean domain.

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- The ring $\mathbb{R}[X, Y]/(X^2 + Y^2 + 1)$ is also a principal ideal domain that is not Euclidean.

Layout

The transfinite Euclidean Rings

Definition

A commutative domain R such that the Euclidean function φ takes its value in the class of ordinals. The Euclidean order type of such a Euclidean domain R is $min_{\varphi}\{\alpha \mid \varphi(R \setminus \{0\}) \subseteq \alpha\}$, where φ ranges among all possible Euclidean norms on R.

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Theorem {Conidis, Nielsen and Tombs}

For every ordinal α, there exists a Euclidean domain whose Euclidean order type is ω^α. Moreover, these are the only possible Euclidean order types for Euclidean domains.

Layout

The k-stage euclidean rings.

Definition

A commutative domain R is 2 stage Euclidean if there exists a map $N : R \setminus \{0\} \longrightarrow \mathbb{N}$ such that for any pair $(a, b) \in R^2$, $(b \neq 0)$, there exist $q, q', r, r' \in R$ such that $a = bq + r$, $b = rq' + r'$ and $N(r') < N(b)$.

Remark

Since, if $r' \neq 0$, we can use the same process for the couple (r, r'), getting a r'' with $N(r'') < N(r')$, hence the process will stop.

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With respect to the usual norms other fields have been shown to be 2-stage Euclidean. For instance if $d \in \{14, 22, 23, 31, 38, \dots\}$.

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If we have a sequence of divisions
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- when the r_i's are invertible,
$b^{-1}a = q_1 + (q_2 + (q_3 + \cdots + (q_{n+1})^{-1} \dots)^{-1})^{-1}$ pause
- The continuous fractions are usually used for approximations of real numbers. The successive approximations of a real number by the sequences of rational numbers obtained by stopping the division process are usualy much quicker than the successive decimal approximations.

Layout

Definitions

1. A pair $(a, b) \in R^2$ is a *right Euclidean pair* if there exist elements $(q_1, r_1), \ldots, (q_{n+1}, r_{n+1}) \in R^2$ (for some $n \geq 0$) such that $a = bq_1 + r_1$, $b = r_1 q_2 + r_2$, and

   $$(*) \qquad r_{i-1} = r_i q_{i+1} + r_{i+1} \text{ for } 1 < i \leq n, \text{ with } r_{n+1} = 0.$$

   The notion of a left Euclidean pair is defined similarly. A ring R is right quasi-euclidean if every pair (a, b) is a right Euclidean pair.
2. A ring R is of stable range 1 if for any $(a, b) \in R^2$ such that $aR + bR = R$ there exists $x \in R$ such that $a + bx$ is invertible.

Let us now give some examples;

Layout

Examples

1. For a, b, q in any ring R, both (bq, b) and $(a, 0)$ are Euclidean pairs as $bq = b \cdot q + 0$, and $a = 0 \cdot 1 + a$ along with $0 = a \cdot 0 + 0$. If b has a right inverse c, then (a, b) is a Euclidean pair for all $a \in R$ since $a = b(ca) + 0$.

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2. If (a, b) is a Euclidean pair and $q \in R$, we see easily that (b, a), $(a + bq, b)$, and $(b + aq, a)$ are also Euclidean pairs. Over a right chain ring R, all pairs in R^2 are Euclidean, so R is a right quasi-Euclidean ring.

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2. If (a, b) is a Euclidean pair and $q \in R$, we see easily that (b, a), $(a + bq, b)$, and $(b + aq, a)$ are also Euclidean pairs. Over a right chain ring R, all pairs in R^2 are Euclidean, so R is a right quasi-Euclidean ring.
3. If $a, b \in R$ are such that $a + bq$ is right-invertible for some q, then (a, b) is a Euclidean pair. In particular, if R is any ring of stable range one then every pair (a, b) with $aR + bR = R$ is Euclidean.

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2. If $e = e^2$ is such that $eR = eRe$ then for any $r \in R$ (e, r) is an Euclidean pair.

Suppose (a, b) is a Euclidean pair with $a = bq_1 + r_1$, $b = r_1 q_2 + r_2$, and

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- In matrix form we get the following

$$(a, b) = (r_n, 0)\, P(q_{n+1}) \cdots P(q_1).$$

where $P(q)$ is the invertible matrix $\begin{pmatrix} q & 1 \\ 1 & 0 \end{pmatrix}$.

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- Let us develop the right handside product of matrices:

$$\begin{pmatrix} q_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} q_1 q_2 + 1 & q_1 \\ q_2 & 1 \end{pmatrix}$$

Layout

Definition

Let $X = \{x_1, x_2, \dots\}$ be a countable set of noncommuting variables, and let $\mathbb{Z}\langle X \rangle$ be the free $\mathbb{Z}$-algebra generated by X. We define the n-th *right continuant polynomials*

$$p_n(x_1, \dots, x_n) \in \mathbb{Z}\langle x_1, \dots, x_n \rangle \subseteq \mathbb{Z}\langle X \rangle$$

by $p_0 \equiv 1$, $p_1(x_1) = x_1$, and inductively for $i \geq 2$ by

$$p_i(x_1, \dots, x_i) = p_{i-1}(x_1, \dots, x_{i-1})\, x_i + p_{i-2}(x_1, \dots, x_{i-2}).$$

Thus, $p_2(x_1, x_2) = x_1 x_2 + 1$, $p_3(x_1, x_2, x_3) = x_1 x_2 x_3 + x_3 + x_1$, etc.

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(4) *For some* $n \geq 0$ $\exists\, q_1, \ldots, q_{n+1} \in R$ *and* $\exists\, r_n \in R$ *with* $a = r_n p_{n+1}(q_{n+1}, \ldots, q_1)$ *and* $b = r_n p_n(q_{n+1}, \ldots, q_2)$.

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Statement (2) *above shows, in particular, that every right quasi-Euclidean ring is right K-Hermite (cf. later).*

Layout

It is possible to see the continuant polynomials as particular cases of other more general polynomials called Fibonacci polynomials. These ones are defined by the following relations:

$$\begin{aligned} &f_{-1} = 0, \quad f_0 = 1, \\ &f_n(x_1, \ldots, x_n, y_1, \ldots, y_n) = f_{n-1}(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1})x_n + \\ &\qquad + f_{n-2}(x_1, \ldots, x_{n-2}, y_1, \ldots, y_{n-2})y_n. \end{aligned} \tag{1}$$

Putting $y_i = 1$ we get back the relations used to define the contnuant polynomials. These Fibonacci polynomials are related to paving $1 \times n$ rectangles with black and white squares.

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- *Every semilocal ring is of stable range* 1 *(Bass's theorem).*
- *If a pair* (a, b) *is Euclidean the ideal* $aR + bR$ *is principal.*
- *Let* R *be of stable range* 1. *Then* $(a, b) \in R^2$ *is a Euclidean pair if and only if* $aR + bR$ *is principal.*

Definition

$\mathrm{GE}_n(R)$ is the subgroup of $\mathrm{GL}_n(R)$ generated by $E_n(R)$ and invertible diagonal matrices. R is a **GE$_n$-ring** if $\mathrm{GL}_n(R) = \mathrm{GE}_n(R)$. If R is a GE$_n$-ring $\forall n \geq 2$, then R is a **GE-ring.** Rings with stable range one (e.g. semilocal rings, unit-regular rings) are GE-ring.

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$\mathrm{(C)}$ *R is a GE_2-ring that is right K-Hermite.*
$\mathrm{(D)}$ *For any $a, b \in R$, $(a, b) = (r, 0)\,Q$ for some $r \in R$ and $Q \in \mathrm{GE}_2(R)$.*

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For any ring R, T.F.A.E.: (A) R is right quasi-Euclidean.
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(C) *R is a GE_2-ring that is right K-Hermite.*
(D) *For any $a, b \in R$, $(a, b) = (r, 0)\,Q$ for some $r \in R$ and $Q \in \mathrm{GE}_2(R)$.*
(E) *For any $a, b \in R$, $(a, b) = (r, 0)\,Q$ for some $r \in R$ and $Q \in \mathrm{E}_2(R)$.*

Layout

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Theorem

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There exist right Euclidean rings which are not Dedekind finite i.e. $ab = 1$ but $ba \neq 1$ (Bergmann example).

Layout

definitions

Definitions

1. An ordered pair (x, y) over a module M_R is said to be a right Euclidean pair if there exists elements $(q_1, p_1) \dots (q_{n+1}, p_{n+1}) \in R \times M$ (for some $n \geq 0$) such that $x = yq_1 + p_1, y = p_1q_2 + p_2$ and $p_{i-1} = p_iq_i + p_{i+1}$ for $1 < i \leq n$ with $p_{n+1} = 0$. If all pairs $(x, y) \in M^2$ are Euclidean then M is called right quasi-Euclidean module. (e.g rational number Q as a $\mathbb{Z}$ module)
2. A right R-module M is said to be a right K-Hermite if for any pair $(x, y) \in M^2$ there exists an invertible matrix $P \in M_2(R)$ and $c \in M$ such that $(x, y) = (c, 0)P$.

Some properties

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3. *If any R module M is quasi-Euclidean then R is clearly quasi-Euclidean but the converse does not hold (e.g. any ideal of a quasi-Euclidean non Noetherian ring)*
4. *A simple right module M over a right quasi-Euclidean ring R is right quasi-Euclidean*

Proposition

Let x, y be elements in a module M. Then the following are equivalent:

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2. *For some* $n \geq 0$ *there exists* $q_1, \ldots q_{n+1} \in R$ *and* $p_n \in M$ *such that* $(x, y) = (p_n, 0)P(q_{n+1}) \ldots P(q_1)$ *where* $P(q_i) = \begin{pmatrix} q_i & 1 \\ 1 & 0 \end{pmatrix}$ *where* $i = 1, \ldots, n+1$. *In particular it shows every right quasi-Euclidean module is right K-Hermite.*

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1. *The module M_R is right quasi-Euclidean.*
2. *The module M_R is right K-Hermite.*
3. *The module M_R is right Bézout.*

Morita context

Definition

A Morita context is a 4-tuple $\begin{pmatrix} R & M \\ N & S \end{pmatrix}$, where R, S are rings, ${}_RM_S$ and ${}_SN_R$ are bimodules, and there exist context products $M \times N \longrightarrow R$ and $N \times M \longrightarrow S$ written multiplicatively as $(m, n) \mapsto mn$ and $(n, m) \mapsto nm$, such that $\begin{pmatrix} R & M \\ N & S \end{pmatrix}$ is an associative ring with the obvious matrix operations.

Definition {Chen, W.K.Nicholson, Stable modules and a theorem of Camillo and Yu, J. Pure and App. Algebra (2014)}

If $C = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ is a Morita context, an element $m \in M$ is called regular if $mnm = m$ for some $n \in N$. Similarly, $n \in N$ is called regular if $nmn = n$ for some $m \in M$.

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Theorem

Let $C = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ *be a Morita context. Then C is right (left) Bézout if and only if* R_R *and* $(M \oplus S)_S$ $({}_S S$ *and* ${}_R(R \oplus M)$ $)$ *are right (left) Bézout.*

Theorem

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Let $C = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ be a Morita context.

(i) C is regular if and only if R, S are regular and $x \in xNx$, $y \in yMy$ for all $x \in M$ and $y \in N$.

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(i) C is regular if and only if R, S are regular and $x \in xNx$, $y \in yMy$ for all $x \in M$ and $y \in N$.

(ii) C is unit-regular if and only if R, S are unit-regular and $x \in xNx$, $y \in yMy$ for all $x \in M$ and $y \in N$.

Corollary

Let $C = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ *be a Morita context. If* C *is right Bézout and both* R *and* S *have stable range 1 then* C *is a right quasi-Euclidean ring.*

THANK YOU