Homotopy Groups of the Moduli Space of Higgs Bundles

Ronald Alberto

ZÚÑIGA-ROJAS

Departamento de Matemática

Faculdade de Ciências da Universidade do Porto

Porto, Portugal

February 27, 2015

Ronald Alberto Zúñiga-Rojas

HOMOTOPY GROUPS OF THE MODULI SPACE OF HIGGS BUNDLES

Thesis submitted for the degree of Ph.D.
in the University of Porto, 2014.

Departamento de Matemática
Faculdade de Ciências da Universidade do Porto
Porto, Portugal

U.PORTO
FACULDADE DE CIÊNCIAS
UNIVERSIDADE DO PORTO

to Miguel Ángel,
to Santiago,
to Wendy.

Acknowledgments

First of all, I want to thank to my supervisor, Professor Peter B. Gothen, for teaching me so much during the last four years, for introducing me to the fascinating subject of the Higgs bundles and Hitchin pairs, for all his constant enthusiasm, patience and kindness when attending my silly questions. For all his virtues, this work has been a real pleasure. Mange tak!

I am very grateful to Professor Steven Bradlow and Professor Óscar García-Prada, for teaching me a lot in this subject, during the Spring of 2012 attending the "Third International School on Geometry and Physics", held in Barcelona, during the Spring of 2013 attending the "IST Courses in Algebraic Geometry", held here in Porto, and during the Spring of 2014, attending the "IV International School on Geometry and Physics", and attending also the "Workshop on the Geometry and Physics of the Moduli Spaces", both held in Madrid. Their talks, ideas and discussions have been truly illuminating for me. Thank you very much! ¡Muchas gracias!

I am also very grateful to Professor Richard A. Wentworth, for his talks and his minicourse during the "Third School on Geometry and Physics", and also for his notes on "Higgs Bundles and Local Systems on Riemann Surfaces", which have been a very useful tool to better understand this amazing topic. Thank you!

I want to thank to those people who shared their time with me, listening to my work. Of course, is impossible for me to mention all of them here, but I am truly grateful to André G. Oliveira, Carlos Florentino, Vicente Muñoz, Ana Peón-Nieto, Alfonso Zamora, Brian Collier, Graeme Wilkin, and Indranil Biswas, who kindly listened to my particular subject and gave me very useful hints and suggestions. Muito obrigado!

¡Muchas gracias! Thank you so much!

I am very grateful to my whole family. I want to thank to my wife Wendy, to our little kids Santiago and Miguel Ángel, for all their love, patience, and support during these years. Every single moment shared with my family has been invaluable for me. I want to thank to my parents, Francisco and Ruth, for all their best wishes, love, and faith putted on me and my brothers. ¡Son los mejores papás del Mundo! Espero ser la mitad de bueno que ustedes en la labor de padre. Of course, I want to thank to my brothers Manuel Andrés and Sebastián who, together with my parents, always have been there for me. I also want to thank to my aunts, Rosario and Estela, my uncle Carlos, for all the moral support that they send me during these years. I am deeply grateful with my grandparents Rodrigo (R.I.P.) and Flora, Alcides (R.I.P.) and Doris, with special acknowledge to Abuelita Doris, every one of you taught me that the most important thing here on Earth is family. I want to thank also my parents in law, Don Miguel y Doña Berny, who have been like a second dad and mom for me. ¡A todos los mencionados en este párrafo los amo y los extraño de corazón! ¡Gracias por su apoyo incondicional!

I am so grateful to those people in Universidad de Costa Rica, who have always trusted on me. Special thanks to Professor William Alvarado, for all the things he taught me when I was an undergraduate student, and also for showing me that poster, between second and third floor, with the announcement of the PhD Program here in Portugal. I am truly grateful to Professor José A. Ramírez, Professor Santiago Cambronero, and Professor Pedro Méndez for all those interesting topics that they taught me, and for have trusted on me from the very beginning. I also want to thank to Professor Joe Várilly and Professor Michael Josephy for their recommendation letters and all the support that gave me four years ago. I want to thank to one of my very close best friends, Ólger Pérez, ¡Chinito! Mi hermano de San Carlos, for his unconditional support, and all his best wishes for me and my family. I want to thank to Greivin Hernández, another one in the circle of best friends, ¡Petu! Mi hermano de Nicoya, for all those favors that I needed. I am deeply grateful to Oficina de Asuntos Internacionales y Cooperación Externa, OAICE, specially grateful to Vivian Madrigal, for following and joining my case from the very beginning. I would like to thank to every single person there in UCR that put part of his energy for me to be here, but is impossible for me to mention all of them here. Without all of them, I would not be here. A todos, muchas gracias. ¡Pura Vida!

Last, but no least, I thank to all those people who made my time here in Porto so invaluable. Special thanks to Thomas Baier and Irene Carvalho, for so enjoyable dinners and those good football matches no Dragão! Danke! Muito obrigado! I want to thank also to Chris Young, Alberto Hernández, and Gastón Pancrazio, for sharing all those beers here at home, or there in Casa Agrícola. Thank you so much! ¡Pura vida! Sería grato poder reunirnos los cuatro alguna vez. ¡Un gran abrazo! ¡Gracias, totales!

Finally, I gratefully acknowledge the financial support from Fundação para a Ciência e a Tecnologia, FCT, here in Portugal with the reference SFRH/BD/51174/2010. Muito obrigado!

Abstract

Let X be a closed and connected Riemann surface of genus $g \geqslant 2$. The main object of study in this thesis is the moduli space $\mathcal{M}^k$ of k-Higgs bundles. These are a generalization of the usual Higgs bundles, where the Higgs field is twisted by $\mathcal{O}(k \cdot p)$, for $p \in X$. There are natural inclusions $\mathcal{M}^k \to \mathcal{M}^{k+1}$.

Here, we study the stabilization with respect to k of the homotopy groups of $\mathcal{M}^k$ using the natural $\mathbb{C}^*$-action on the moduli space. We prove results on freeness and stabilization of homology groups in rank two and three. This conjecturely implies stabilization for homotopy groups. However, we do not obtain precise numerical estimates for the range of the stabilization of the homology and homotopy indices. This work partially generalizes the result by Hausel in rank two.

Moreover, we study the inclusion of the fixed loci of the $\mathbb{C}^*$-action, where the most important case is the one that corresponds to holomorphic triples. The moduli spaces of triples depend on a stability parameter σ, and we investigate the relation of the various stability conditions, finding in particular natural inclusions of triples moduli spaces corresponding to the inclusions $\mathcal{M}^k \to \mathcal{M}^{k+1}$. An essential ingredient is the study of the flips relating moduli spaces of triples for different values of the parameter σ.

The moduli space $\mathcal{M}$ is stratified by the Harder-Narasimhan type of the underlying vector bundle of a Higgs bundle. This stratification is called the Shatz stratification. We study the relationship between the Shatz stratification and the Bialynicki-Birula stratification on $\mathcal{M}$, coming from the limit $z \to 0$ of the $\mathbb{C}^*$-action, for rank two and three. Our results should produce a more refined stratification for rank three, which we expect to be useful in generalizing Hausel's results for rank two to rank three. We present a

different proof for the rank two case stratifications equivalence obtained by Hausel, and we give a description, for the rank three case, of how the Shatz stratification relates to the Bialynicki-Birula stratification and also the other way around.

The Nilpotent Cone in $\mathcal{M}$ is the pre-image of zero under the Hitchin map. It has another Bialynicki-Birula stratification, using the limit $z \to \infty$ for $z \in \mathbb{C}^*$. Finally, we study this stratification of the Nilpotent Cone of $\mathcal{M}$. These results complement those of the relationship between the Shatz stratification and the Bialynicki-Birula stratification mentioned above.

Key Words

Algebraic Geometry, Algebraic Topology, Differential Geometry, Moduli Spaces, Gauge Theory, Morse Theory, Higgs Bundles, Hitchin Pairs, Homotopy, Homology, Cohomology, Connections, Holomorphic Structures, Vector Bundles.

Resumo

Seja X uma superfície de Riemann fechada e conexa de género $g \geqslant 2$. O principal objeto de estudo desta tese é o espaço móduli $\mathcal{M}^k$ de k-fibrados de Higgs. Estes são uma generalização dos fibrados de Higgs habituais, onde o campo de Higgs é torcido por $\mathcal{O}(k \cdot p)$, para $p \in X$. Existem mergulhos naturais $\mathcal{M}^k \to \mathcal{M}^{k+1}$.

Aqui estuda-se a estabilização com respeito a k dos grupos de homotopia de $\mathcal{M}^k$ utilizando a acção natural de $\mathbb{C}^*$ sobre o espaço de móduli. Provamos resultados de torção livre e de estabilização de grupos de homologia em posto dois e três. Isto implica, como uma conjectura, a estabilização para grupos de homotopia. Contudo, não é possível obter estimativas numéricas precisas para a estabilização dos índices de homologia e de homotopia. Este trabalho generaliza parcialmente o resultado de Hausel para posto dois.

Além disso, é estudado o mergulho dos lugares geométricos de pontos fixos da acção de $\mathbb{C}^*$, onde o caso mais importante é aquele que corresponde a triplos holomorfos. Os espaços móduli de triplos dependem dum parâmetro de estabilidade σ, e investiga-se a relação das distintas condições de estabilidade, encontrando em particular mergulhos naturais de espaços móduli de triplos correspondentes às inclusões $\mathcal{M}^k \to \mathcal{M}^{k+1}$. Um ingrediente essencial é o estudo dos lugares geométricos de salto relacionando espaços móduli de triplos para diferentes valores do parâmetro σ.

O espaço móduli $\mathcal{M}$ é estratificado pelo tipo Harder-Narasimhan do fibrado vectorial subjacente dum fibrado de Higgs. Esta estratificação chama-se a estratificação de Shatz. Estuda-se a relação entre a estratificação de Shatz e a estratificação de Bialynicki-Birula em $\mathcal{M}$, associada ao limite de $z \to 0$ da acção de $\mathbb{C}^*$, para posto dois e três. Os nossos resultados devem produzir uma estratificação mais refinada para posto três, que

esperamos seja útil na generalização dos resultados de Hausel de posto dois para posto três. Apresentamos uma prova diferente para a equivalência dessas duas estratificações no caso de posto dois, obtida por Hausel, e damos uma descrição, para o caso de posto três, de como se relaciona a estratificação de Shatz com a estratificação de Bialynicki-Birula e também reciprocamente.

O Cone Nilpotente em $\mathcal{M}$ é a imagem inversa de zero sob o mapeo de Hitchin. O Cone Nilpotente, tem outra estratificação de Bialynicki-Birula, usando o limite de $z \to \infty$ para $z \in \mathbb{C}^*$. Finalmente, estudamos esta estratificação do Cone Nilpotente de $\mathcal{M}$. Estes resultados complementam aqueles da relação entre a estratificação de Shatz e a de Bialynicki-Birula mencionados anteriormente.

Palavras-Chave

Geometria Algébrica, Topologia Algébrica, Geometria Diferencial, Espaços Moduli, Teoria de Gauge, Teoria de Morse, Fibrados de Higgs, Pares de Hitchin, Homotopia, Homologia, Cohomologia, Conexões, Estructuras Holomorfas, Fibrados Vectoriais.

Resumen

Sea X una superficie de Riemann cerrada y conexa de género $g \geqslant 2$. El principal objeto de estudio de esta tesis es el espacio móduli $\mathcal{M}^k$ de k-fibrados de Higgs. Estos son una generalización de los fibrados de Higgs habituales, donde el campo de Higgs es torcido por $\mathcal{O}(k \cdot p)$, para $p \in X$. Existen inclusiones naturales $\mathcal{M}^k \to \mathcal{M}^{k+1}$.

Aquí, se estudia la estabilización con respecto a k de los grupos de homotopía de $\mathcal{M}^k$ utilizando la acción natural de $\mathbb{C}^*$ sobre el espacio de móduli. Demostramos resultados de torsión libre y de estabilización de grupos de homología en rango dos y tres. Esto implica, a modo de conjetura, la estabilización para grupos de homotopía. Sin embargo, no obtenemos estimaciones numéricas precisas para la estabilización de los índices de homología y de homotopía. Este trabajo generaliza parcialmente el resultado de Hausel para rango dos.

Por otra parte, se estudia la inclusión de los lugares geométricos de puntos fijos de la acción de $\mathbb{C}^*$, donde el caso más importante es el que corresponde con triples holomorfos. Los espacios móduli de triples dependen de un parámetro de estabilidad σ, y se investiga la relación de las distintas condiciones de estabilidad, encontrando en particular inclusiones naturales de espacios móduli de triples correspondientes a las inclusiones $\mathcal{M}^k \to \mathcal{M}^{k+1}$. Un ingrediente esencial es el estudio de los lugares geométricos de salto relacionando espacios móduli de triples para diferentes valores del parámetro σ.

El espacio móduli $\mathcal{M}$ es estratificado por el tipo Harder-Narasimhan del fibrado vectorial subyacente de un fibrado de Higgs. Esta estratificación se llama la estratificación de Shatz. Se estudia la relación entre la estratificación de Shatz y la estratificación de Bialynicki-Birula en $\mathcal{M}$, procedente del límite de $z \to 0$ de la acción de $\mathbb{C}^*$, para rango

dos y tres. Nuestros resultados deben producir una estratificación más refinada para rango tres, que esperamos sea útil en la generalización de los resultados de Hausel de rango dos para rango tres. Presentamos una prueba diferente para la equivalencia de estas dos estratificaciones en el caso de rango dos, obtenida por Hausel, y damos una descripción, para el caso de rango tres, de cómo la estratificación de Shatz se relaciona con la estratificación de Bialynicki-Birula y también a la inversa.

El Cono Nilpotente en $\mathcal{M}$ es la imagen inversa de cero bajo el mapeo de Hitchin. El Cono Nilpotente, tiene otra estratificación de Bialynicki-Birula, usando el límite de $z \to \infty$ para $z \in \mathbb{C}^*$. Finalmente, estudiamos esta estratificación del Cono Nilpotente de $\mathcal{M}$. Estos resultados complementan los de la relación entre la estratificación de Shatz y la de Bialynicki-Birula mencionados anteriormente.

Palabras Clave

Geometría Algebraica, Topología Algebraica, Geometría Diferencial, Espacios Moduli, Teoría de Gauge, Teoría de Morse, Fibrados de Higgs, Pares de Hitchin, Homotopía, Homología, Cohomología, Conexiones, Estructuras Holomorfas, Fibrados Vectoriales.

Contents

Introduction

Higgs bundles appeared in the work of Hitchin [24] and they are of interest for a lot of reasons in a lot of mathematical fields like: Algebraic Geometry, Algebraic Topology, Differential Geometry, Mathematical Physics, Quantum Field Theory, among others. Even so, this thesis is concerned only with Mathematics, especifically with Algebraic Geometry, Algebraic Topology and Differential Geometry, but is not concerned directly with Physics.

Let X be a closed and connected Riemann surface of genus $g \geqslant 2$. Let $K = K_X \cong (TX)^*$ be the canonical line bundle over X.

From the point of view of Algebraic Geometry, a ***Higgs bundle*** is a pair (E, Φ) where $E \to X$ is a holomorphic vector bundle over X and $\Phi \in H^0(X, \mathrm{End}(E) \otimes K)$ is a holomorphic section of $\mathrm{End}(E)$, the endomorphism bundle of E, called as a ***Higgs field***.

On the other hand, if we fix a Hermitian metric on X, compatible with its Riemann surface structure, since $\dim_{\mathbb{C}} X = 1$, this metric will be Kähler, and so, there is a Kähler form ω that we can choose such that:

$$\int_X \omega = 2\pi, \tag{1}$$

and so, from the gauge theory point of view, a Higgs bundle is defined as a pair (d_A, Φ) where d_A is a unitary connection on a smooth complex vector bundle $E \to X$ and

$\Phi \in \Omega^{1,0}(X, End(E))$, satisfying Hitchin's equations:

$$\begin{cases} F_A + [\Phi, \Phi^*] &= -i \cdot \mu \cdot I_E \cdot \omega \\ \\ \bar{\partial}_A \Phi &= 0 \end{cases} \tag{2}$$

a set of non-linear differential equations for d_A and Φ, related through the curvature F_A, where Φ^* is the adjoint of Φ with respect to a hermitian metric on E (see Theorem 1.3.7), where $I_E \in \mathrm{End}(E)$ is the identity and $\mu = \mu(E)$ is the slope of E, and one consequence is that Φ is holomorphic with respect to the holomorphic structure of E induced by d_A:

$$\text{i.e.} \quad \bar{\partial}_E \Phi = 0$$

where $\bar{\partial}_E = \bar{\partial}_A$ comes from the Chern-correspondence:

$$d_A = d + A = d + A^{0,1} d\bar{z} - A^{1,0} dz \longmapsto \bar{\partial} + A^{0,1} d\bar{z} = \bar{\partial}_A.$$

A solution to Hitchin's equations gives us a holomorphic Higgs bundle (E, Φ) by giving E the holomorphic structure induced by the unitary connection d_A, and this Higgs bundle will be polystable. Stability can be introduced as follows:

A holomorphic vector bundle $E \to X$, is called *semistable* if $\mu(F) \leqslant \mu(E)$ for any F such that $0 \subsetneq F \subseteq E$. Similarly, a holomorphic vector bundle $E \to X$ is called *stable* if $\mu(F) < \mu(E)$ for any non-zero proper subbundle $0 \subsetneq F \subsetneq E$. Finally, E is called *polystable* if it is the direct sum of stable subbundles, all of the same slope.

We can then generalize the notion of stability to Higgs bundles applying it only to Φ-invariant subbundles of E: for a Higgs bundle (E, Φ), a subbundle $F \subset E$ is said to be Φ-*invariant* if $\Phi(F) \subset F \otimes K$. A Higgs bundle is said to be *semistable* (respectively *stable*) if $\mu(F) \leqslant \mu(E)$ (respectively$\mu(F) < \mu(E)$) for any non-zero, Φ-invariant subbundle $F \subseteq E$ (respectively$F \subsetneq E$). Similarly, (E, Φ) is called *polystable* if E is the direct sum of stable Φ-invariant subbundles, all of the same slope.

The converse is quite hard to prove, but also true: any polystable Higgs bundle (E, Φ) admits a hermitian metric on it such that (d_A, Φ) solves the Hitchin's equations (2), where d_A is the Chern connection (see Theorem 1.3.7).

A *gauge transformation* is an automorphism of E. Locally, a gauge transformation $g \in Aut(E)$ is a $C^\infty(E)$-function with values in $GL_r(\mathbb{C})$. A gauge transformation g is called *unitary* if g preserves the hermitian inner product. We will denote $\mathcal{G}$ as the group of unitary gauge transformations. Atiyah and Bott [2] denote $\bar{\mathcal{G}}$ as the quotient of $\mathcal{G}$ by its constant central $U(1)$-subgroup. We will follow this notation too. Moreover, denote $B\mathcal{G}$ and $B\bar{\mathcal{G}}$ as the classifying spaces of $\mathcal{G}$ and $\bar{\mathcal{G}}$, respectively.

A *Hitchin pair* is a generalization of a Higgs bundle. Instead of consider K, the canonical line bundle of X, if we consider a general line bundle $L \to X$, we get a *Hitchin pair* where now $\Phi \in H^0(X, \mathrm{End}(E) \otimes L)$. The stability condition for Hitchin pairs is the obvious generalization of the one for Higgs bundles.

For $k \geqslant 0$, a *k-Higgs bundle* or *Higgs bundle with poles of order* k is the particular case of a Hitchin pair where $L = K \otimes L_p^{\otimes k}$. More clearly, if we consider a fixed point $p \in X$ as a divisor $p \in \mathrm{Sym}^1(X) = X$, and L_p the line bundle that corresponds to that divisor p, we get a complex of the form

$$E \xrightarrow{\Phi^k} E \otimes K \otimes L_p^{\otimes k}$$

where $\Phi^k \in H^0(X, \mathrm{End}(E) \otimes K \otimes L_p^{\otimes k})$ is a *Higgs field with poles of order* k. So, we call such a complex as a k-Higgs bundle and Φ^k as its k-Higgs field. A k-Higgs bundle (E, Φ^k) is stable (respectively semistable) if the slope of any Φ^k-invariant subbundle of E is strictly less (respectively less or equal) than the slope of $E:\ \mu(E)$. Finally, (E, Φ^k) is called *polystable* if E is the direct sum of stable Φ^k-invariant subbundles, all of the same slope.

The moduli space of stable Hitchin pairs $\mathcal{M}_L(r,d)$, can be constructed either analytically:

$$\mathcal{M}_L(r,d) = \mathcal{M}_L := \mathcal{B}^s(r,d)/\mathcal{G}^{\mathbb{C}}$$

with

$$\mathcal{B}^s(r,d) = \big\{(\bar{\partial}_A, \Phi):\ \bar{\partial}_A(\Phi) = 0 \text{ and } (E,\Phi) \text{ is stable}\big\} \subset \Big(\mathcal{A}^{0,1}(r,d) \times \Omega^0(X; \mathrm{End}(E) \otimes L\Big),$$

and where, by abuse of notation, we denote the $\bar{\partial}$-operator on $\mathrm{End}(E) \otimes L$ comming from $\bar{\partial}_A$ on E and the fixed holomorphic structure on L; or using Geometric Invariant Theory, considering Φ as a 0-section:

$$\Phi \in H^0(X; End(E) \otimes L).$$

This construction is carried out by Nitsure [34]:

Theorem (Nitsure [34, Proposition 7.4.]). *The space $\mathcal{M}_L(r,d)$ is a quasi-projective smooth variety of complex dimension*

$$\dim_{\mathbb{C}}(\mathcal{M}_L(r,d)) = (r^2-1)\deg(L).$$

In particular:

$$\dim_{\mathbb{C}}(\mathcal{M}^k(r,d)) = (r^2-1)\deg(K \otimes L_p^{\otimes k}) = (r^2-1)(2g-2+k).$$

An important feature of $\mathcal{M}_L(r,d)$ is that it carries an action of $\mathbb{C}^*$: $z \cdot (E,\Phi) = (E, z \cdot \Phi)$. According to Hitchin [24], $(\mathcal{M}, I, \Omega)$ is a Kähler manifold, where I is its complex structure and Ω its corresponding Kähler form. Furthermore, $\mathbb{C}^*$ acts on $\mathcal{M}$ biholomorphically with respect to the complex structure I by the action mentioned above, where the Kähler form Ω is invariant under the induced action $e^{i\theta} \cdot (E,\Phi) = (E, e^{i\theta} \cdot \Phi)$ of the circle $\mathbb{S}^1 \subset \mathbb{C}^*$. Besides, this circle action is Hamiltonian with proper momentum map

$$f : \mathcal{M} \longrightarrow \mathbb{R}$$

defined by:

$$f(E,\Phi) = \frac{1}{2\pi}\|\Phi\|^2_{L^2} = \frac{i}{2\pi}\int_X tr(\Phi\Phi^*). \tag{3}$$

where Φ^* is again the adjoint of Φ with respect to the hermitian metric on E given by Theorem 1.3.7, and f has finitely many critical values.

There is another important fact mentioned by Hitchin [24](see the original version in Frankel [10], and its application to Higgs bundles in Hitchin [24]): the critical points of f are exactly the fixed points of the circle action on $\mathcal{M}$.

If $(E,\Phi) = (E, e^{i\theta}\Phi)$ then $\Phi = 0$ with critical value $c_0 = 0$. The corresponding critical submanifold is $F_0 = f^{-1}(c_0) = f^{-1}(0) = \mathcal{N}$, the moduli space of stable bundles. On the other hand, when $\Phi \neq 0$, there is a type of algebraic structure for Higgs bundles introduced by Simpson [36]: a *Variation of Hodge Structure*, or simply a *VHS*, for a Higgs bundle (E,Φ) is a decomposition:

$$E = \bigoplus_{j=1}^{n} E_j \text{ such that } \Phi : E_j \to E_{j+1} \otimes K \text{ for } 1 \leqslant j \leqslant n-1. \tag{4}$$

Has been proved by Simpson [37] that the fixed points of the circle action on $\mathcal{M}(r,d)$, and so, the critical points of f, are these Variations of the Hodge Structure, VHS, where the critical values $c_\lambda = f(E,\Phi)$ will depend on the degrees d_j of the components $E_j \subset E$. By Morse theory, we can stratify $\mathcal{M}$ in such a way that there is a non-zero critical submanifold $F_\lambda := f^{-1}(c_\lambda)$ for each non-zero critical value $0 \neq c_\lambda = f(E,\Phi)$ where (E,Φ) represents a fixed point of the circle action, or equivalently, a VHS. We said then that (E,Φ) is a $(\mathrm{rk}(E_1),\ ...,\ \mathrm{rk}(E_n))$-VHS.

The calculation of the Betti numbers of the moduli space of stable Higgs bundles has been done by Hitchin [24] for the rank two case, by Gothen [14] for the rank three case, and by García-Prada, Heinloth and Schmitt [13] for the rank four case. Hitchin [24] and Gothen [14] work using the proper momentum map (3) mentioned above as a Morse-Bott function. Gothen follows an approach quite similar to the one that Hitchin does, but with the main difference that in the determination of the critical submanifolds, Gothen uses the vortex pairs from the work of Bradlow [4] and their generalization to stable triples from the work of Bradlow and García-Prada [5]. These vortex pairs (V,φ) consist of a bundle together with a section, and there are stability conditions studied by Bradlow [4] and the moduli space of vortex pairs has been widely studied by Thaddeus [38]. On the other hand, triples of the form (V_1, V_2, Φ) consisting of two vector bundles $V_1 \to X$, $V_2 \to X$ and a map $\Phi : V_2 \to V_1$ between them, were introduced by Bradlow and García-Prada as a generalization of the vortex pairs, and these structures have been widely worked by Bradlow, García-Prada, Gothen [6], by Muñoz, Ortega and Vásquez-Gallo [32], by Muñoz, Oliveira and Sánchez [31], among others. The work of García-Prada, Heinloth and Schmitt [13] is a little bit different: their computation is done in the dimensional completion of the Grothendieck ring of varieties and starts by describing the classes of moduli stacks of chains rather than their coarse moduli space.

We are particularly interested in the homotopy groups of the moduli space of Higgs bundles. The works of Bradlow, García-Prada and Gothen [7] give an estimate of some of the homotopy groups of $\mathcal{M}(r,d)$, the moduli space of Higgs bundles of rank $\mathrm{rk}(E) = r$ and degree $\deg(E) = d$:

Theorem (Bradlow, García-Prada and Gothen [7, Theorem 4.4])**.** *Let $\mathcal{G}$ be the unitary gauge group. If $r > 1,\ g \geqslant 3$ and $GCD(r,d) = 1$, then:*

(1) $\pi_1(\mathcal{M}(r,d)) \cong H_1(X,\mathbb{Z})$;

(2) $\pi_2(\mathcal{M}(r,d)) \cong \mathbb{Z}$;

(3) $\pi_j(\mathcal{M}(r,d)) \cong \pi_{j-1}(\mathcal{G})$ *for* $2 < j \leqslant 2(g-1)(r-1) - 2$.

Let $\mathcal{M}^\infty := \lim_{k\to\infty} \mathcal{M}^k = \bigcup_{k=0}^\infty \mathcal{M}^k$ be the direct limit of the spaces $\left\{\mathcal{M}^k(r,d)\right\}_{k=0}^\infty$. Hausel [19], while estimating the homotopy groups of $\mathcal{M}^k(2,1)$ the moduli space of k-Higgs bundles of rank $rk(E) = 2$, finds that the estimate of Bradlow, García-Prada and Gothen [7, Theorem 4.4] holds for a higher homotopy index:

Theorem (Hausel [19, Theorem 7.5.7.])**.** *For $k \geqslant 0$ we have:*

$$\pi_j(\mathcal{M}^k(2,1)) \cong \pi_j(\mathcal{M}^\infty(2,1)) \cong \pi_j(B\bar{\mathcal{G}})$$

for $\quad 0 \leqslant j \leqslant 4g - 8 + k.$

The work of the present thesis is motivated by the problem of generalizing this result to higher rank. Nevertheless, Hausel uses two principal tools that can not be used in general: first, the Morse stratification of $\mathcal{M}(2,1)$ coincides with its Shatz stratification; and second, the study of the higher connectedness properties of the inclusions

$$\mathcal{M}^k(2,1) \hookrightarrow \mathcal{M}^{k+1}(2,1).$$

Before describing the Morse stratification, we will describe the Bialynicki-Birula strata: consider the set

$$U_\lambda^{BB} := \{(E,\Phi) \in \mathcal{M} | \lim_{z\to 0} z\cdot(E,\Phi) \in F_\lambda\}.$$

This set U_λ^{BB} is the *upward stratum* of the *Bialynicki-Birula stratification*:

$$\mathcal{M} = \bigcup_\lambda U_\lambda^{BB}.$$

On the other hand, let U_λ^M be the set of points $(E,\Phi) \in \mathcal{M}$ such that its path of steepest descent for the Morse function f and the Kähler metric have limit points in F_λ. This set is called the *upward Morse flow of* F_λ, and it gives another stratification of $\mathcal{M}$:

$$\mathcal{M} = \bigcup_\lambda U_\lambda^M$$

Kirwan proves that these two stratifications are always equivalent:

Theorem (Kirwan [27, (6.16.)])**.** *Bialynicki-Birula stratification and Morse stratification are smooth and diffeomorphic. In other words, using the above notation, we get:*

$$U_\lambda^{BB} = U_\lambda^M \quad \forall\lambda.$$

We will denote simply $U_\lambda^+ := U_\lambda^{BB} = U_\lambda^M$.

As a consequence of Shatz [35, Proposition 10 and Proposition 11], there is a finite stratification of $\mathcal{M}(r,d)$ by the Harder-Narasimhan type of the underlying vector bundle E of a Higgs bundle (E,Φ):

$$\mathcal{M}(r,d) = \bigcup_t U'_t$$

where $U'_t \subset \mathcal{M}(r,d)$ is the subspace of Higgs bundles (E,Φ) which associated vector bundle E has $\mathrm{HNT}(E) = t$, and where we are taking this union over the existing types in $\mathcal{M}(r,d)$. This stratification is known as the *Shatz stratification*.

Let $U'_0 \subset \mathcal{M}(2,d)$ be the locus of points $(E,\Phi) \in \mathcal{M}(2,d)$ such that E is stable, and let $U'_{d_1} \subset \mathcal{M}$ be the locus of points $(E, \Phi) \in \mathcal{M}(2,d)$ such that E is unstable and its destabilizing line bundle E_1 is of degree $d_1 > 0$. This family $\{U'_{d_1}\}_{d_1=0}^{g-1}$ gives us the Shatz stratification of $\mathcal{M}(2,d)$:

$$\mathcal{M}(2,d) = \bigcup_{d_1=0}^{g-1} U'_{d_1}.$$

Hausel proves that $U^+_{d_1} = U'_{d_1}$ for rank two, $\forall d_1$ such that $0 \leqslant d_1 \leqslant g-1$. The general rank case inclusions $\mathcal{M}^k(r,d) \hookrightarrow \mathcal{M}^{k+1}(r,d)$ are 'well behaved' some how, but the Morse stratification and the Shatz stratification do not coincide in general.

This thesis is structured in five chapters. In Chapter 1 we introduce some general facts and basic definitions useful along the whole thesis.

In Chapter 2 we prove the stabilization of the homotopy groups of $\mathcal{M}^k(r,d)$ the moduli spaces of k-Higgs bundles of general rank $\mathrm{rk}(E) = r$ and degree $\deg(E) = d$ using the results from the works of Hausel and Thadeus [21] and [22], among other tools. We do not obtain precise numerical estimates for stabilization in the general case:

Theorem (Corollary 2.2.17). *If $H^n\big(\mathcal{M}^k(r,d), \mathbb{Z}\big)$ is torsion free $\forall k \in \mathbb{N}$ and $\forall n \in \mathbb{N}$, and if $\pi_1(\mathcal{M}^k)$ acts trivially on $\pi_n(\mathcal{M}^\infty, \mathcal{M}^k)$ for all $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$, then for all n exists $k_0 = k_0(n)$ such that*

$$\pi_j(\mathcal{M}^k) \xrightarrow{\cong} \pi_j(\mathcal{M}^\infty)$$

for all $k \geqslant k_0$ and for all $j \leqslant n-1$.

Note that $H^n\big(\mathcal{M}^k(2,d), \mathbb{Z}\big)$ and $H^n\big(\mathcal{M}^k(3,d), \mathbb{Z}\big)$ are torsion free $\forall k \in \mathbb{N}$ and $\forall n \in \mathbb{N}$ (see Theorem 2.2.7), while the fact that $\pi_1\big(\mathcal{M}^k(2,1)\big)$ acts trivially on $\pi_n(\mathcal{M}^\infty, \mathcal{M}^k)$, has been taken for granted in the work of Hausel [19].

In Chapter 3, motivated by the result of Hausel [19] in rank two:

$$\mathcal{M}^k(2,1) \hookrightarrow \mathcal{M}^{k+1}(2,1),$$

we study the inclusions of the fixed loci. The most important case is the one of fixed loci corresponding to holomorphic triples.

A holomorphic triple on X is a triple $T = (E_1, E_2, \phi)$ consisting of two holomorphic vector bundles $E_1 \to X$ and $E_2 \to X$ and a homomorphism $\phi : E_2 \to E_1$, i.e. an element $\phi \in H^0(Hom(E_2, E_1))$. A homomorphism from a triple $T' = (E_1', E_2', \phi')$ to another triple $T = (E_1, E_2, \phi)$ is a commutative diagram of the form:

$$\begin{array}{ccc} E_1' & \xrightarrow{\phi'} & E_2' \\ \downarrow & & \downarrow \\ E_1 & \xrightarrow{\phi} & E_2 \end{array}$$

where the vertical arrows represent holomorphic maps. $T' \subset T$ is a subtriple if the sheaf homomorphisms $E_1' \to E_1$ and $E_2' \to E_2$ are injective. As usual, a subtriple is called proper if $0 \neq T' \subsetneq T$.

For any $\sigma \in \mathbb{R}$ the σ-degree and the σ-slope of $T = (E_1, E_2, \phi)$ are defined as:

$$deg_\sigma(T) := \deg(E_1) + \deg(E_2) + \sigma \cdot \mathrm{rk}(E_2)$$

and

$$\mu_\sigma(T) := \frac{deg_\sigma(T)}{\mathrm{rk}(E_1) + \mathrm{rk}(E_2)} =$$

$$\frac{\deg(E_1) + \deg(E_2) + \sigma \cdot \mathrm{rk}(E_2)}{\mathrm{rk}(E_1) + \mathrm{rk}(E_2)} = \mu(E_1 \oplus E_2) + \sigma \frac{\mathrm{rk}(E_2)}{\mathrm{rk}(E_1) + \mathrm{rk}(E_2)}.$$

T is then called σ-semistable (respectively σ-stable) if $\mu_\sigma(T') \leqslant \mu_\sigma(T)$ (respectively $\mu_\sigma(T') < \mu_\sigma(T)$) for any subtriple $T' \subsetneq T$ (proper subtriple $0 \neq T' \subsetneq T$). A triple is called σ-polystable if it is the direct sum of σ-stable triples of the same σ-slope.

We will use the following notation for Moduli Spaces of Triples:

i. Denote $\mathbf{r} = (r_1, r_2)$ and $\mathbf{d} = (d_1, d_2)$, and then consider

$$\mathcal{N}_\sigma = \mathcal{N}_\sigma(\mathbf{r}, \mathbf{d}) = \mathcal{N}_\sigma(r_1, r_2, d_1, d_2)$$

as the moduli space of σ-polystable triples $T = (E_1, E_2, \phi)$ such that $\mathrm{rk}(E_j) = r_j$ and $\deg(E_j) = d_j$.

ii. Denote $\mathcal{N}_\sigma^s = \mathcal{N}_\sigma^s(\mathbf{r}, \mathbf{d})$ as the subspace of σ-stable triples.

iii. Refer $(\mathbf{r}, \mathbf{d}) = (r_1, r_2, d_1, d_2)$ as the type of the triple $T = (E_1, E_2, \phi)$.

As mentioned by Bradlow, García-Prada and Gothen [6], there are certain necessary conditions in order for σ-polystable triples to exist. Denote $\mu_j = \mu(E_j) = \frac{d_j}{r_j}$ and define then:

$$\sigma_m := \mu_1 - \mu_2 \tag{5}$$

and

$$\sigma_M := \left(1 + \frac{r_1 + r_2}{|r_1 - r_2|}\right)(\mu_1 - \mu_2), \text{ when } r_1 \neq r_2. \tag{6}$$

Then:

Proposition (Bradlow, García-Prada and Gothen [6, Proposition 2.2.]). *The moduli space $\mathcal{N}_\sigma(r_1, r_2, d_1, d_2)$ is a complex analytic variety, which is projective when $\sigma \in \mathbb{Q}$. A necessary condition for $\mathcal{N}_\sigma(r_1, r_2, d_1, d_2) \neq \emptyset$ is:*

$$0 \leqslant \sigma_m < \sigma < \sigma_M \text{ when } r_1 \neq r_2,$$

or

$$0 \leqslant \sigma_m < \sigma \text{ when } r_1 = r_2.$$

If $\sigma_m = 0$ and $r_1 \neq r_2$ then $\sigma_m = \sigma_M = 0$ and $\mathcal{N}_\sigma^s(r_1, r_2, d_1, d_2) = \emptyset$ unless $\sigma = 0$. We denote by $I \subset \mathbb{R}$ the following interval:

$$I = \begin{cases} [\sigma_m, \sigma_M] & \text{if } r_1 \neq r_2, r_1 \neq 0, r_2 \neq 0, \\ [\sigma_m, \infty[& \text{if } r_1 = r_2 \neq 0, \\ \mathbb{R} & \text{if } r_1 = 0 \text{ or } r_2 = 0. \end{cases} \tag{7}$$

Muñoz, Ortega and Vásquez-Gallo [32] present useful results that we will use later:

Proposition (Muñoz, Ortega and Vásquez-Gallo [32, Proposition 3.7]). *Let $\sigma_0 \in I$ and let $T = (E_1, E_2, \phi) \in \mathcal{N}_{\sigma_0}(r_1, r_2, d_1, d_2)$ be a strictly σ_0-semistable triple. Then one of the following conditions holds:*

(1) For all σ_0-destabilizing subtriples $T' = (E_1', E_2', \phi')$, we have

$$\frac{r_2'}{r_1' + r_2'} = \frac{r_2}{r_1 + r_2}.$$

Then T is strictly σ-semistable for $\sigma \in \,]\sigma_0 - \varepsilon, \sigma_0 + \varepsilon[$, for some $\varepsilon > 0$ small enough.

(2) There exists a σ_0-destabilizing subtriple $T' = (E_1', E_2', \phi')$ with

$$\frac{r_2'}{r_1' + r_2'} \neq \frac{r_2}{r_1 + r_2}.$$

Then:

- *either*
$$\frac{r_2'}{r_1' + r_2'} > \frac{r_2}{r_1 + r_2},$$
and so T is σ-unstable for any $\sigma > \sigma_0$,
- *or*
$$\frac{r_2'}{r_1' + r_2'} < \frac{r_2}{r_1 + r_2},$$
and so T is σ-unstable for any $\sigma < \sigma_0$.

Those values of σ for which Case (2) in the last proposition occurs are called *critical values*.

Lemma (Muñoz, Ortega and Vásquez-Gallo [32, Lemma 3.16])**.** *(1) If $d_1 < d_2$ then $\mathcal{N}_\sigma(1, 1, d_1, d_2) = \emptyset$.*

(2) If $d_1 > d_2$ then:

- $\mathcal{N}_{\sigma_m}(1, 1, d_1, d_2) \cong \mathcal{J}^{d_1} \times \mathcal{J}^{d_2}$ *and* $\mathcal{N}^s_{\sigma_m}(1, 1, d_1, d_2) = \emptyset$.
- $\mathcal{N}_\sigma(1, 1, d_1, d_2) = \mathcal{N}^s_\sigma(1, 1, d_1, d_2) \cong \mathcal{J}^{d_1} \times \mathrm{Sym}^{d_1 - d_2}(X)\ \forall \sigma > \sigma_m$.
- $\mathcal{N}_\sigma(1, 1, d_1, d_2) = \mathcal{N}^s_\sigma(1, 1, d_1, d_2) = \emptyset$ *for* $\sigma < \sigma_m$.

Fixing the type $(\mathbf{r}, \mathbf{d}) = (r_1, r_2, d_1, d_2)$ for the moduli spaces of holomorphic triples, Muñoz, Ortega and Vásquez-Gallo [32] describe the differences between two spaces

$\mathcal{N}_{\sigma_1}$ and $\mathcal{N}_{\sigma_2}$ when σ_1 and σ_2 are separated by a critical value. For a critical value $\sigma_c \in I$ set $\sigma_c^+ = \sigma + \varepsilon$ and $\sigma_c^- = \sigma - \varepsilon$, where $\varepsilon > 0$ is small enough so that σ_c is the only critical value in the interval $]\sigma_c^-, \sigma_c^+[$.

The *flip loci* are defined as:

$$S_{\sigma_c^+} := \left\{T \in \mathcal{N}_{\sigma_c^+} : T \text{is } \sigma_c^- - \text{unstable}\right\} \subset \mathcal{N}_{\sigma_c^+},$$

$$S_{\sigma_c^-} := \left\{T \in \mathcal{N}_{\sigma_c^-} : T \text{is } \sigma_c^+ - \text{unstable}\right\} \subset \mathcal{N}_{\sigma_c^-},$$

and $S^s_{\sigma_c^\pm} := S_{\sigma_c^\pm} \cap \mathcal{N}^s_{\sigma_c^\pm}$ for the stable part of the flip loci.

Note that for $\sigma_c = \sigma_m$, $\mathcal{N}_{\sigma_m^-} = \emptyset$, hence $\mathcal{N}_{\sigma_m^+} = S_{\sigma_m^+}$. Also $\mathcal{N}^s_{\sigma_m} = \emptyset$, by the last part of the last proposition. Anologously, when $r_1 \neq r_2$, $\mathcal{N}_{\sigma_M^+} = \emptyset$, $\mathcal{N}_{\sigma_M^-} = S_{\sigma_M^-}$ and $\mathcal{N}^s_{\sigma_M} = \emptyset$.

For the rank three case, using the isomorphisms between the $(1,2)$-VHS and the moduli spaces of triples $F^k_{d_1} \cong \mathcal{N}_{\sigma_H(k)}(2,1,\tilde{d}_1,\tilde{d}_2)$, together with the restrictions $F^k_{d_1} \hookrightarrow F^{k+1}_{d_1}$ of the inclusions, we find very nice and interesting results in terms of triples:

$$\mathcal{N}_{\sigma_H(k)}(2,1,\tilde{d}_1,\tilde{d}_2) \hookrightarrow \mathcal{N}_{\sigma_H(k+1)}(2,1,\tilde{d}_1+2,\tilde{d}_2) :$$

Lemma (Lemma 3.1.1). *A triple T is σ-stable $\Leftrightarrow$ $i_k(T)$ is $(\sigma+1)$-stable.*

Using this result we do even more: we extend the embedding to

$$i_k : \mathcal{N}_{\sigma_c(k)}(2,1,\tilde{d}_1,\tilde{d}_2) \to \mathcal{N}_{\sigma_c(k+1)}(2,1,\tilde{d}_1+2,\tilde{d}_2)$$

and hence to

$$i_k : \mathcal{N}_{\sigma_c^-(k)}(2,1,\tilde{d}_1,\tilde{d}_2) \to \mathcal{N}_{\sigma_c^-(k+1)}(2,1,\tilde{d}_1+2,\tilde{d}_2)$$

and to

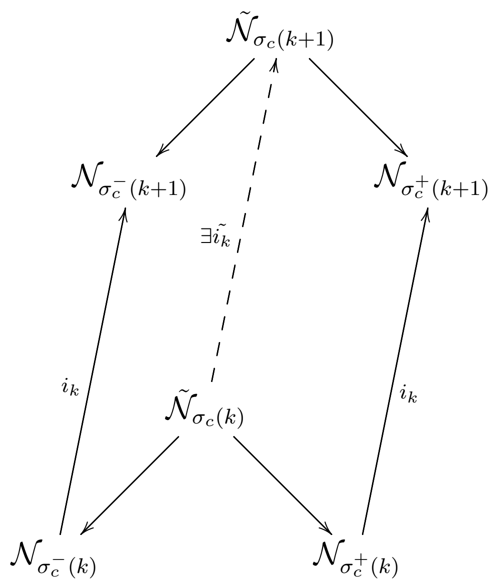
$$i_k : \mathcal{N}_{\sigma_c^+(k)}(2,1,\tilde{d}_1,\tilde{d}_2) \to \mathcal{N}_{\sigma_c^+(k+1)}(2,1,\tilde{d}_1+2,\tilde{d}_2)$$

for any critical value $\sigma_m < \sigma_c(k) < \sigma_M$, and so we extend the embedding to the space $\tilde{\mathcal{N}}_{\sigma_c(k)}$ the blow-up of $\mathcal{N}_{\sigma_c^-(k)}$ along the flip locus $S_{\sigma_c^-(k)}$ and, at the same time, represents the blow-up of $\mathcal{N}_{\sigma_c^+(k)}$ along the flip locus $S_{\sigma_c^+(k)}$:

Proposition (Proposition 3.2.1)**.** *There exists an embedding at the blow-up level*

$$\tilde{i_k} : \tilde{\mathcal{N}}_{\sigma_c(k)} \hookrightarrow \tilde{\mathcal{N}}_{\sigma_c(k+1)}$$

such that the following diagram commutes:



where $\tilde{\mathcal{N}}_{\sigma_c(k)}$ *is the blow-up of* $\mathcal{N}_{\sigma_c^-(k)} = \mathcal{N}_{\sigma_c^-(k)}(2,1,\tilde{d}_1,\tilde{d}_2)$ *along the flip locus* $S_{\sigma_c^-(k)}$ *and, at the same time, represents the blow-up of* $\mathcal{N}_{\sigma_c^+(k)} = \mathcal{N}_{\sigma_c^+(k)}(2,1,\tilde{d}_1,\tilde{d}_2)$ *along the flip locus* $S_{\sigma_c^+(k)}$*.*

We study then, the stabilization of the cohomology groups of the moduli space $\mathcal{N}_{\sigma_c(k)}$, leaving for future work the approach to the $(1,2)$-VHS:

Theorem (Corollary 3.3.7)**.**

$$i_k^* : H^j(\mathcal{N}_{\sigma_c(k+1)}, \mathbb{Z}) \xrightarrow{\cong} H^j(\mathcal{N}_{\sigma_c(k)}, \mathbb{Z}) \quad \forall j \leqslant \tilde{n}(k).$$

Similar results can be obtained using the isomorphisms $F_{d_2}^k \cong \mathcal{N}_{\sigma_H(k)}(1,2,\tilde{d}_1,\tilde{d}_2)$ between the $(2,1)$-VHS and the moduli spaces of triples, and the dual isomorphisms

$$\mathcal{N}_{\sigma_H(k)}(2,1,\tilde{d}_1,\tilde{d}_2) \cong \mathcal{N}_{\sigma_H(k)}(1,2,-\tilde{d}_2,-\tilde{d}_1)$$

between moduli spaces of triples. We leave the application of these results to the study of the topology of the moduli space $\mathcal{M}^k(r,d)$ for future work.

In Chapter 4, we study the relationship between the Shatz stratification and the Bialynicki-Birula stratification on $\mathcal{M}(r,d)$ for rank $r = 2$ and rank $r = 3$. Our results should produce a more refined stratification for rank three, which we expect to be useful in generalizing Hausel'results for rank two to rank three.

In this chapter, we present there a different proof for the rank two case stratifications equivalence, obtained by Hausel [19]. Furthermore, we give a description, for the rank three case, of how the Shatz stratification relates to the Morse stratification and also the other way around.

Let $\big[(E,\Phi)\big] \in \mathcal{M}(3,d)$ and denote $(E^0,\Phi^0) := \lim_{z\to 0}(E, z\cdot\Phi)$. The stratum of the Morse stratification where (E,Φ) belongs is determined by (E^0,Φ^0), and depends on the Harder-Narasimhan Type of E, and on certain properties of Φ. Our Principal Theorem describes in detail that dependence.

To state the Theorem, is convenient to use the following notation: for a vector bundle morphism $\phi : E \to F$, we write $\mathrm{im}(\phi) \subset F$ for that subbundle obtained by the saturation of the respective subsheaf.

Theorem (Theorem 4.2.1)**.** *(1.) Suppose that E is an unstable vector bundle of* $\mathrm{rk}(E) = 3$ *with a Harder-Narasimhan Filtration of length* 1*:*

$$HNF(E):\ 0 = E_0 \subset E_1 \subset E_2 = E$$

where E_1 is the maximal destabilizing line subbundle of E, and $\mu(V_1) > \mu(V_2)$ where $V_1 = E_1,\ V_2 = E/E_1$ are semi-stables. In other words, suppose that $E \to X$ is a holomorphic bundle that has $\mathrm{HNT}(E) = (\mu_1,\mu_2,\mu_2)$ *where $\mu_j = \mu(V_j)$. Consider $\phi_{21} : V_1 \to V_2 \otimes K$ induced by*

$$E_1 \xrightarrow{\ \imath\ } E \xrightarrow{\ \Phi\ } E\otimes K \xrightarrow{\jmath\otimes id_K} \big(E/E_1\big)\otimes K.$$

Define $\mathcal{I} := \phi_{21}(E_1)\otimes K^{-1} \subset V_2$ which is a subbundle of V_2, where $\mathrm{rk}(\mathcal{I}) = 1$*, and define also $F := V_1 \oplus \mathcal{I} \subset V_1 \oplus V_2 = E$ where* $\mathrm{rk}(F) = 2$*. Then, we have two possibilities:*

(1.1.) Suppose that $\mu(F) < \mu(E)$. *Then,* (E^0, Φ^0) *is a* $(1,2)$-*VHS of the form:*

$$(E^0, \Phi^0) = \Big(V_1 \oplus V_2, \begin{pmatrix} 0 & 0 \\ \phi_{21} & 0 \end{pmatrix}\Big).$$

(1.2.) On the other hand, if $\mu(F) \geqslant \mu(E)$, *then,* (E^0, Φ^0) *is a* $(1,1,1)$-*VHS of the form:*

$$(E^0, \Phi^0) = \Big(L_1 \oplus L_2 \oplus L_3, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix}\Big)$$

where $L_1, L_2,$ *and* L_3 *are line bundles.*

(2.) Similarly, suppose that E *is an unstable vector bundle of* $\mathrm{rk}(E) = 3$ *with a Harder-Narasimhan Filtration of length* 1:

$$HNF(E): \; 0 = E_0 \subset E_1 \subset E_2 = E$$

but this time E_1 *is the maximal destabilizing subbundle of* E *with* $\mathrm{rk}(E_1) = 2$, *and* $\mu(V_1) > \mu(V_2)$ *where* $V_1 = E_1,\ V_2 = E/E_1$ *are semi-stables. In other words, suppose that* $E \to X$ *is a holomorphic bundle that has* $\mathrm{HNT}(E) = (\mu_1, \mu_1, \mu_2)$ *where* $\mu_j = \mu(V_j)$. *Consider* $\phi_{21} : V_1 \to V_2 \otimes K$ *induced by*

$$E_1 \xrightarrow{\imath} E \xrightarrow{\Phi} E \otimes K \xrightarrow{\jmath \otimes id_K} \big(E/E_1\big) \otimes K.$$

Define $N := ker(\phi_{21}) \subset V_1$ *which is a subbundle. Then, we have two possibilities:*

(2.1.) Suppose that $\mu(N) < \mu(E)$. *Then,* (E^0, Φ^0) *is a* $(2,1)$-*VHS of the form:*

$$(E^0, \Phi^0) = \Big(V_1 \oplus V_2, \begin{pmatrix} 0 & 0 \\ \phi_{21} & 0 \end{pmatrix}\Big).$$

(2.2.) On the other hand, if $\mu(N) \geqslant \mu(E)$, *then:* (E^0, Φ^0) *is a* $(1,1,1)$-*VHS of the*

form:

$$(E^0, \Phi^0) = \left(L_1 \oplus L_2 \oplus L_3, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix} \right)$$

where $L_1, L_2,$ and L_3 are line bundles.

(3.) Finally, suppose that (E, Φ) is a Higgs Bundle where E is an unstable vector bundle of $\mathrm{rk}(E) = 3$ with a Harder-Narasimhan Filtration of length 2:

$$HNF(E): \; 0 = E_0 \subset E_1 \subset E_2 \subset E_3 = E$$

where $\mu(V_1) > \mu(V_2) > \mu(V_3)$ and $V_1 = E_1, \; V_2 = E_2/E_1,$ and $V_3 = E/E_2$ are semi-stables.

(3.1.) Suppose that $\mu(E_2/E_1) < \mu(E)$. Then we can define F as we did in (1.), and then, we have two possibilities:

(3.1.1.) Suppose that $\mu(F) < \mu(E)$. Then: (E^0, Φ^0) is a $(1, 2)$-VHS.

(3.1.2.) On the other hand, if $\mu(F) \geqslant \mu(E)$, then: (E^0, Φ^0) is a $(1, 1, 1)$-VHS.

(3.2.) On the other hand, if $\mu(E_2/E_1) > \mu(E)$, then define N as we did in (2.), and then, we have two possibilities:

(3.2.1.) If $\mu(N) < \mu(E)$. Then: (E^0, Φ^0) is a $(2, 1)$-VHS.

(3.2.2.) If $\mu(N) \geqslant \mu(E)$, then: (E^0, Φ^0) is a $(1, 1, 1)$-VHS.

Finally, in Chapter 5 we study the stratification of the Nilpotent Cone given by the Downward Morse Flow. The results presented there complement those of Chapter 4. There, in Chapter 5 we find a filtration that describes the Nilpotent Cone in terms of the Downward Morse Flow, for rank two and rank three cases. Hence, for rank two, we have:

Theorem (Theorem 5.2.1)**.** *Let $[(E, \Phi)] \in \chi^{-1}(0)$ be a Hitchin pair with $\mathrm{rk}(E) = 2$. Then, there is a filtration*

$$E = E_1 \supset E_2 \supset E_3 = 0$$

such that

$$\Phi(E_j) \subset E_{j+1} \otimes L$$

and

$$(E^\infty, \Phi^\infty) := \lim_{z\to\infty}(E, z\cdot\Phi) = \left(L_1 \oplus L_2, \begin{pmatrix} 0 & 0 \\ \varphi & 0 \end{pmatrix}\right) \tag{8}$$

is a $(1,1)$*-VHS where*

$$L_j = E_j/E_{j+1} \quad \textit{and} \quad \varphi : L_1 \to L_2 \otimes L.$$

Similarly, for rank three, we have:

Theorem (Theorem 5.3.1)**.** *Let* $[(E,\Phi)] \in \chi^{-1}(0)$ *be a Hitchin pair with* $\mathrm{rk}(E) = 3$*. Then:*

(a) either there is a filtration

$$E = E_1 \supset E_2 \supset E_3 \supset E_4 = 0$$

such that

$$\Phi(E_j) \subset E_{j+1} \otimes L$$

and

$$(E^\infty, \Phi^\infty) := \lim_{z\to\infty}(E, z\cdot\Phi) = \left(L_1 \oplus L_2 \oplus L_3, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix}\right) \tag{9}$$

is a $(1,1,1)$*-VHS where*

$$L_j = E_j/E_{j+1} \quad \textit{and} \quad \varphi_j : L_{j-1} \to L_j \otimes L$$

(b) or, there is a filtration

$$E = E_1 \supset E_2 \supset E_3 = 0$$

such that

(b.1.) either

$$(E^\infty, \Phi^\infty) := \lim_{z\to\infty}(E, z\cdot\Phi) = \left(V_1 \oplus V_2, \begin{pmatrix} 0 & 0 \\ \varphi_{21} & 0 \end{pmatrix}\right) \tag{10}$$

is a $(1,2)$*-VHS where*

$$V_j = E_j/E_{j+1} \quad \textit{and} \quad \varphi : V_1 \to V_2 \otimes L,$$

and where $\Phi(E_j) \subset E_{j+1} \otimes L$*,*

(b.2.) or

$$(E^\infty, \Phi^\infty) := \lim_{z\to\infty}(E, z\cdot\Phi) = \Big(W_1 \oplus W_2, \begin{pmatrix} 0 & 0 \\ \varphi_{21} & 0 \end{pmatrix}\Big) \tag{11}$$

is a $(2,1)$*-VHS, depending on the rank of* E_2*, and depending also on some properties of* Φ*.*

Chapter 1

General Facts

Let X be a closed and connected Riemann surface of genus $g \geqslant 2$. Let $K = K_X \cong (TX)^*$ be the canonical line bundle over X. Note that, algebraically, X is also a non-singular complex projective algebraic curve.

1.1 Basic Definitions

Definition 1.1.1. For a smooth vector bundle $E \to X$, we denote the *rank* of E by $\mathrm{rk}(E) = r$ and the *degree* of E by $\deg(E) = d$. Then, the *slope* of E is defined to be:

$$\mu(E) := \frac{\deg(E)}{\mathrm{rk}(E)} = \frac{d}{r}. \tag{1.1}$$

Definition 1.1.2. A *connection* d_A on a smooth vector bundle $E \to X$ is a differential operator

$$d_A : \Omega^0(X, E) \longrightarrow \Omega^1(X, E)$$

such that

$$d_A(fs) = df \otimes s + f d_A s$$

for any function $f \in C^\infty(X)$ and any section $s \in \Omega^0(X, E)$ where $\Omega^n(X, E)$ is the set of smooth n-forms of X with values in E. Locally:

$$d_A = d + A = d + Cdz + Bd\bar{z}$$

where A is a matrix of 1-forms: $A_{ij} \in \Omega^1(X, E)$, and B, C are matrix valued functions depending on the hermitian metric on E.

Some authors call the matrix A as a connection and call $d_A = d + A$ as its correspoding covariant derivative. We abuse notation and will not distinguish between them.

Suppose, from now on, that there is a hermitian metric on E. When a connection d_A is compatible with the hermitian metric, *i.e.* when

$$d < s, t >=< d_A s, t > + < s, d_A t >$$

for the hermitian inner product $< \cdot, \cdot >$ and for s, t any couple of sections of E, d_A is a *unitary* connection. Denote $\mathcal{A}(E)$, or sometimes just $\mathcal{A}$, the space of unitary connections on E, for a smooth bundle $E \to X$.

Definition 1.1.3. The fundamental invariant of a connection is its *curvature*:

$$F_A := d_A^2 = d_A \circ d_A : \Omega^0(X, E) \longrightarrow \Omega^2(X, E)$$

where we are extending d_A to n-forms in $\Omega^n(X, E)$ in the obvious way. Locally:

$$F_A = dA + A^2.$$

F_A is $C^\infty(X, E)$-linear and can be considered as a 2-form on X with values in $\mathrm{End}(E)$: $F_A \in \Omega^2(X, \mathrm{End}(E))$, or locally as a matrix-valued 2-form.

Definition 1.1.4. If the curvature vanishes, *i.e* $F_A = 0$, we say that the connection d_A is *flat*. A flat connection gives a family of constant transition functions for E, which in turn defines a representation of the fundamental group of X, $\pi_1(X)$ into $GL_r(\mathbb{C})$:

$$\pi_1(X) \to GL_r(\mathbb{C})$$

$$[\alpha] \longmapsto M_\alpha.$$

Note that the image is in $U(n)$ if A is unitary. Besides, from Chern-Weil theory, if $F_A = 0$, then $\deg(E) = 0$.

Definition 1.1.5. A *gauge transformation* is an automorphism of E. Locally, a gauge transformation $g \in Aut(E)$ is a C^{∞}-function with values in $GL_r(\mathbb{C})$. A gauge transformation g is called *unitary* if g preserves the hermitian inner product.

We will denote by $\mathcal{G}$ the group of unitary gauge transformations. This gauge group $\mathcal{G}$ acts on $\mathcal{A}$ by conjugation:

$$g \cdot d_A = g^{-1} d_A g \quad \forall g \in \mathcal{G} \quad \text{and for } d_A \in \mathcal{A}.$$

Note that conjugation by a unitary gauge transformation takes a unitary connection to a unitary connection.

Definition 1.1.6. A *holomorphic structure* on E is a differential operator:

$$\bar{\partial}_A : \Omega^0(X, E) \longrightarrow \Omega^{0,1}(X, E)$$

such that

$$\bar{\partial}_A(fs) = \bar{\partial} f \otimes s + f \bar{\partial}_A s$$

where $\bar{\partial} f = \frac{\partial f}{\partial \bar{z}} d\bar{z}$, and $\Omega^{p,q}(X, E)$ is the space of smooth (p, q)-forms with values in E. Locally:

$$\bar{\partial}_A = \bar{\partial} + A^{0,1} d\bar{z}$$

where $A^{0,1}$ is a matrix valued function.

Definition 1.1.7. A holomorphic vector bundle $E \to X$, is called *semistable* if $\mu(F) \leqslant \mu(E)$ for any F such that $0 \subsetneq F \subseteq E$. Similarly, a vector bundle $E \to X$ is called *stable* if $\mu(F) < \mu(E)$ for any non-zero proper subbundle $0 \subsetneq F \subsetneq E$. Finally, E is called *polystable* if it is the direct sum of stable subbundles, all of the same slope.

Denote $\mathcal{A}^{0,1}(E)$, or sometimes simply $\mathcal{A}^{0,1}$, as the space of holomorphic structures on smooth bundles $E \to X$ with rank $\mathrm{rk}(E) = r$ and degree $\deg(E) = d$, and denote $\mathcal{A}^{0,1}_s(E)$, $\mathcal{A}^{0,1}_{ss}(E)$ and $\mathcal{A}^{0,1}_{ps}(E)$ as the subspaces of holomorphic strucutres on stable, semistable and polystable smooth bundles respectively.

Remark 1.1.8. Since E has a hermitian metric on it, from a holomorphic structure $\bar{\partial}_A = \bar{\partial} + A^{0,1} d\bar{z}$ on E we can define a unique unitary connection d_A such that $d_A = d + A = d - A^{1,0} dz + A^{0,1} d\bar{z}$ is compatible with the hermitian inner product. This is known as the Chern-correspondence between $\mathcal{A}$ and $\mathcal{A}^{0,1}$ given by:

$$d_A = d + A = d + A^{0,1}d\bar{z} - A^{1,0}dz \longmapsto \bar{\partial} + A^{0,1}d\bar{z} = \bar{\partial}_A.$$

Recall that smooth vector bundles over X are classified by their ranks and degrees. Let $\mathcal{A}^{0,1}(\mathcal{E})$ be the complex affine space of holomorphic structures on $\mathcal{E} \to X$ a fixed smooth complex vector bundle over X of rank $\mathrm{rk}(\mathcal{E}) = r$ and degree $\deg(\mathcal{E}) = d$. Consider the *complexified gauge group* $\mathcal{G}^{\mathbb{C}} = \mathrm{Aut}(\mathcal{E})$ of complex automorphisms of $\mathcal{E}$, which acts naturally on $\mathcal{A}^{0,1}(\mathcal{E})$ by conjugation:

$$g \cdot \bar{\partial}_A = g^{-1}\bar{\partial}_A g \quad \forall g \in \mathcal{G}^{\mathbb{C}}, \quad \forall \bar{\partial}_A \in \mathcal{A}^{0,1}$$

and this action induces an equivalence relation between holomorphic structures:

$$\bar{\partial}_{A_1} \simeq \bar{\partial}_{A_2} \Leftrightarrow \exists g \in \mathcal{G}^{\mathbb{C}} \quad \text{such that} \quad g^{-1}\bar{\partial}_{A_1} g = \bar{\partial}_{A_2}.$$

An orbit of this action is the set of vector bundles isomorphic to a given one, then the problem of classifying all the vector bundles over X, reduces to understand these orbits. Nevertheless, because of the so-called *jumping phenomenon*, the quotient space $\mathcal{A}^{0,1}/\mathcal{G}^{\mathbb{C}}$ is not Hausdorff. However, we can get a Hausdorff space using the polystable subspace $\mathcal{A}^{0,1}_{ps}(\mathcal{E}) \subset \mathcal{A}^{0,1}(\mathcal{E})$, and defining the moduli space of stable bundles as

$$\mathcal{N}(r, d) := \mathcal{A}^{0,1}_{ps}(\mathcal{E})/\mathcal{G}^{\mathbb{C}},$$

which is a projective variety.

Remark 1.1.9. Since $g \in \mathcal{G}^{\mathbb{C}}$ takes solutions s of $\bar{\partial}_{A_2} s = 0$ into solutions gs of $\bar{\partial}_{A_1} s = 0$, g is a holomorphic isomorphism. Besides, note that the gauge group action on holomorphic structures looks in local terms like:

$$g^{-1}\bar{\partial}_A g = \bar{\partial} + g^{-1}(\bar{\partial} g) + (g^{-1}A^{0,1}g)d\bar{z}.$$

Considering the open $\mathcal{G}^{\mathbb{C}}$-invariant subset $\mathcal{A}^{0,1}_s \subset \mathcal{A}^{0,1}$, is possible to construct a smooth quasi-projective algebraic variety for the parameter space of stable vector bundles

$$\mathcal{N}_s(r,d) := \mathcal{A}_s^{0,1}(\mathcal{E})/\mathcal{G}^{\mathbb{C}} \subset \mathcal{N}(r,d)$$

and get the following result:

Theorem 1.1.10. *If $GCD(r,d) = 1$ then $\mathcal{A}_s^{0,1} = \mathcal{A}_{ss}^{0,1}$ and the moduli space $\mathcal{N}_s(r,d)$ is a smooth projective algebraic variety of dimension* $\dim_{\mathbb{C}}(\mathcal{N}_s) = r^2(g-1)+1$.

Actually, Narasimhan and Seshadri [33] explain that the moduli space $\mathcal{N}_s(r,d)$ is compact when $GCD(r,d) = 1$, and its topology is independent of the complex structure of X.

1.2 Harder-Narasimhan Filtrations

We shall introduce two concepts that are really relevant to our purposes and quite close related to stability: the *Harder-Narasimhan Filtration* and the *Harder-Narasimhan Type*. Furthermore, we also present the main result about the Harder-Narasimhan Filtration: the Shatz Theorem.

Definition 1.2.1. Let $E \to X$ be a holomorphic vector bundle. A *Harder-Narasimhan Filtration* of E, is a filtration of the form

$$HNF(E):\ E = E_s \supset E_{s-1} \supset \ldots \supset E_1 \supset E_0 = 0$$

which satisfies the following two properties:

i. $\mu(E_{j+1}/E_j) < \mu(E_j/E_{j-1})$ for $1 \leqslant j \leqslant s-1$.

ii. E_j/E_{j-1} is semistable for $1 \leqslant j \leqslant s$.

Remark 1.2.2. i. For simplicity, we shall denote $V_j := E_j/E_{j-1}$ for $1 \leqslant j \leqslant s$.

ii. From the last definition, property i. $\mu(V_{j+1}) < \mu(V_j)$ for $1 \leqslant j \leqslant s-1$ is equivalent to the condition $\mu(E_{j+1}) < \mu(E_j)$ for $1 \leqslant j \leqslant s-1$, which could be intuitively clear if we take a view to the Harder-Narasimhan Polygon (for more details, see Shatz [35, Proposition 5]):

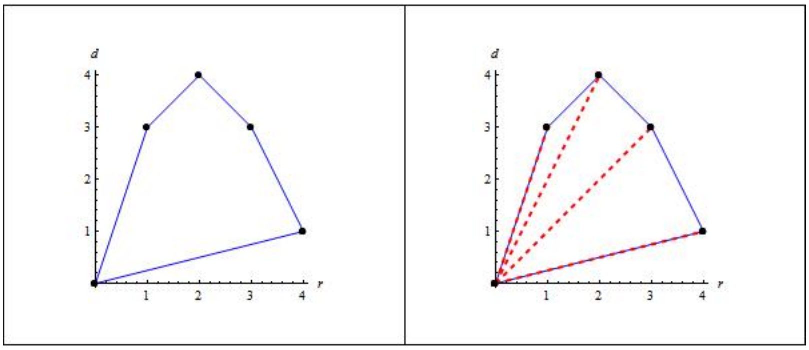


Figure 1: On the left, the Harder-Narasimhan Polygon, where the black points represent the pairs (r_i, d_i) for E_i, and the blue line segments represent segments with slope $\mu(V_i) = \mu(E_i/E_{i-1})$. On the right, the red-dashed line segments represent segments with slope $\mu(E_i)$.

Once we have defined the Harder-Narasimhan Filtration, we will present one of the most important results related to this concept: Shatz [35] has an analogue result of the Jordan-Hölder Theorem for vector bundles, if we take semistable vector bundles as the analogues of simple finite groups:

Theorem 1.2.3 (Shatz [35, Theorem 1])**.** *Every vector bundle $E \to X$ has a unique Harder-Narasimhan Filtration.*

This has been proved in the case when X is a projective non-singular algebraic curve, by Harder and Narasimhan [16]. In the following, we outline Shatz'proof of Theorem 1.2.3. To prove that, Shatz [35] uses the following proposition:

Proposition 1.2.4 (Shatz [35, Proposition 6])**.** *Let E be an unstable vector bundle. Then, there is a unique $V \subset E$ semi-stable subbundle of E such that V is the maximal destabilizing subbundle of E,*

$$i.e.\ \mu(V) > \mu(E) \text{ with maximal rank } \mathrm{rk}(V).$$

The existence of the destabilizing subbundle has been proved first by Narasimhan and Seshadri [33, Proposition 4.5.], and Shatz [35] adds uniqueness and maximality.

Proof. (Theorem 1.2.3)
For the existence, we will use induction on the rank of E. If $\mathrm{rk}(E) = 1$ or E is semi-stable, the existence of the HNF is trivial. Suppose then, that $\mathrm{rk}(E) > 1$ and that it is

unstable. Let $E_1 \subset E$ be the maximal destabilizing subbundle of E mentioned in 1.2.4. The quotient E/E_1 has rank $\mathrm{rk}(E/E_1) = n-1$ so, by hypothesis of induction, it has a HNF of the form:

$$HNF(E/E_1):\ E/E_1 = V_t \supset V_{t-1} \supset \dots \supset V_1 \supset V_0 = 0$$

where the subbundles of E/E_1 satisfy the following two properties:

i. $\mu(V_{j+1}) < \mu(V_j)$ for $1 \leqslant j \leqslant t-1$.

ii. V_j/V_{j-1} is semi-stable for $1 \leqslant j \leqslant t$.

This $HNF(E/E_1)$ lifts to

$$E = E_{t+1} \supset E_t \supset \dots \supset E_1 \supset E_0 = 0$$

where $V_j = E_{j+1}/E_1$ for $1 \leqslant j \leqslant t$. Follows that $E_{j+1}/E_j \cong V_j/V_{j-1}$ are semi-stables for $1 \leqslant j \leqslant t$, and that

$$\mu(E_{j+1}/E_j) < \mu(E_j/E_{j-1}) \text{ for } 2 \leqslant j \leqslant t.$$

Since E_1 is semi-stable, all we need to prove is that $\mu(E_2/E_1) < \mu(E_1)$. Both, E_1 and E_2 are subbundles of E. So, by maximality, 1.2.4 shows that $\mu(E_2) \leqslant \mu(E_1)$. E_2 cannot be semi-stable: if it were, we would have $\mu(E_1) \leqslant \mu(E_2)$, contradicting 1.2.4, since $\mathrm{rk}(E_2) > \mathrm{rk}(E_1)$. As E_2 is unstable, there is a unique maximal subbundle $V \subset E_2$ such that $\mu(V) > \mu(E_2)$. Once again, by 1.2.4: $\mu(E_1) \geqslant \mu(V) > \mu(E_2)$, as required to prove existence.

For the uniqueness, we also proceed by induction on $\mathrm{rk}(E)$. If $\mathrm{rk}(E) = 1$, uniqueness is trivial. More generally, if E is semi-stable, then ii. in 1.2.4 yields uniqueness. Then, assume E is unstable with $\mathrm{rk}(E) = n > 1$. Suppose

$$E = E_t \supset E_{t-1} \supset \dots \supset E_1 \supset E_0 = 0$$

$$E = E'_s \supset E'_{s-1} \supset \dots \supset E'_1 \supset E_0 = 0$$

are two HNF's for E. Let V be the maximal destabilizing subbundle of E given by 1.2.4, and let j be the smallest integer such that the inclusion $V \hookrightarrow E$ factors trough E_j. Then, there is a non-zero homomorphism $V \to E_j \to E_j/E_{j-1}$ where E_j/E_{j-1} is semi-stable. Then, $\mu(V) \leqslant \mu(E_j/E_{j-1})$. However, by 1.2.4, we have:

$$\mu(V) \geqslant \mu(E_1) > \mu(E_2) > ... > \mu(E_j) > \mu(E_j/E_{j-1})$$

which is a contradiction unless $j = 1$. In such a case, $V \subset E_1$, where E_1 is semi-stable. Hence $\mu(V) = \mu(E_1)$. By 1.2.4, $\mathrm{rk}(V) = \mathrm{rk}(E_1)$, and therefore: $V = E_1$. In a very similar way $V = E_1'$. By considering $E/E_1 = E/E_1'$, we reduce the rank of the bundle under consideration, and the induction hypothesis completes the proof. ♠

Definition 1.2.5. For a vector bundle $E \to X$ of rank $\mathrm{rk}(E) = r$, with a Harder-Narasimhan Filtration of the form

$$HNF(E): \ E = E_s \supset E_{s-1} \supset ... \supset E_1 \supset E_0 = 0$$

the Harder-Narasimhan Type, abreviated as HNT, is defined as the vector

$$HNT(E): \vec{\mu} = (\mu_1, ..., \mu_1, \mu_2, ..., \mu_2, ... \ ..., \mu_s, ..., \mu_s) \in \mathbb{Q}^r$$

where $\mu_j = \mu(V_j) = \mu(E_j/E_{j-1})$ appears r_j-times, where $r_j = \mathrm{rk}(V_j)$.

All the bundles of a given HNT $\vec{\mu}$ define a subspace $\mathcal{A}^{0,1}(\vec{\mu})$ of $\mathcal{A}^{0,1}(E)$. Since the HNF is canonical, the subspaces $\mathcal{A}^{0,1}(\vec{\mu})$ are preserved by the gauge group action, so each subspace $\mathcal{A}^{0,1}(\vec{\mu})$ is a union of orbits. For further details, see Atiyah and Bott [2].

1.3 The Moduli Space of Stable Higgs Bundles $\mathcal{M}(r, d)$

Definition 1.3.1. A *Higgs bundle* over X is a pair (E, Φ) where $E \to X$ is a holomorphic vector bundle and $\Phi : E \to E \otimes K$ is an endomorphism of E twisted by K, which is called a *Higgs field*. Note that $\Phi \in H^0\big(X; \mathrm{End}(E) \otimes K\big)$.

Definition 1.3.2. A subbundle $F \subset E$ is said to be Φ-*invariant* if $\Phi(F) \subset F \otimes K$. A Higgs bundle is said to be *semistable* (respectively *stable*) if $\mu(F) \leqslant \mu(E)$ (respectively

$\mu(F) < \mu(E)$) for any non-zero, Φ-invariant subbundle $F \subseteq E$ (respectively$F \subsetneq E$). Finally, (E, Φ) is called *polystable* if it is the direct sum of stable Φ-invariant subbundles, all of the same slope.

Remark 1.3.3. Note that a Higgs bundle (E, Φ) could be stable, while E is unstable: if no destabilizing subbundle of E is Φ-invariant, then (E, Φ) is stable. On the other hand, if (E, Φ) is unstable, it is because E is also unstable.

There is a construction of the moduli space of polystable Higgs bundles due to Simpson [37]:

Proposition 1.3.4 (Simpson [37, Proposition 1.4.]). *There is a quasi-projective variety $\mathcal{M}_{Dol}$ whose points parametrize polystable Higgs bundles (E, Φ) on X with vanishing Chern classes. There is a map from $\mathcal{M}_{Dol}$ to the space of polynomials with coefficients in the symmetric powers of the cotangent bundle of X that takes a Higgs bundle (E, Φ) to the characteristic polynomial of Φ. This map is proper.*

The last proposition has been proved in the case when X is an algebraic curve, by Nitsure [34], without conditions on Chern classes (see Theorem 1.4.2 below).

There is a similar moduli space $\mathcal{M}_B$ for representations of the fundamental group: let $\mathcal{R}_B$ be the affine variety of homomorphisms from $\pi_1(X)$ into $GL_r(\mathbb{C})$ obtained by looking at generators and relators. Then, $\mathcal{M}_B$ is the affine categorical quotient of $\mathcal{R}_B$ by the action of $GL_r(\mathbb{C})$, by conjugation. The points of $\mathcal{M}_B$ parametrize semisimple representations. The correspondence in Simpson [37, Theorem 1.] yields an isomorphism of sets between $\mathcal{M}_B$ and $\mathcal{M}_{Dol}$:

Proposition 1.3.5 (Simpson [37, Proposition 1.5.]). *There is a homemorphism of topological spaces*

$$\mathcal{M}_B \cong \mathcal{M}_{Dol}.$$

On the other hand, recall that Hitchin [24] constructs the moduli space $\mathcal{M}(r, d)$ using gauge theory:

Definition 1.3.6. Consider the space $\mathcal{A}^{0,1}(E) \times \Omega^{1,0}\big(X; \mathrm{End}(E)\big)$. Define

$$\mathcal{B}(r, d) := \big\{(\bar\partial_B, \Phi) : \ \bar\partial_B(\Phi) = 0\big\} \subset \mathcal{A}^{0,1}(E) \times \Omega^{1,0}\big(X; \mathrm{End}(E)\big)$$

and consider the respective subspaces $\mathcal{B}^{ss}(r,d), \mathcal{B}^{s}(r,d)$ and $\mathcal{B}^{ps}(r,d)$ of $\mathcal{B}(r,d)$ for semi-stable, stable and polystable bundles respectively. Then, define $\mathcal{M}(r,d)$, *the moduli space of stable Higgs Bundles* as the quotient of this latter subspace by the complex gauge group action:

$$\mathcal{M}(r,d) := \mathcal{B}^{ps}(r,d)/\mathcal{G}^{\mathbb{C}}.$$

Fixing a Hermitian metric on X, compatible with its Riemann surface structure, since $\dim_{\mathbb{C}} X = 1$, this metric will be Kähler, and so, there is a Kähler form ω that we can choose such that:

$$\int_X \omega = 2\pi, \tag{1.2}$$

and so, has been proved by Hitchin [24], that a stable Higgs bundle $(\bar{\partial}_A, \Phi)$ defined as above, comes from a pair (d_A, Φ) where d_A is a unitary connection on a smooth complex vector bundle $E \to X$ and $\Phi \in \Omega^{1,0}\big(X, End(E)\big)$, satisfying Hitchin's equations:

$$\begin{cases} F_A + [\Phi, \Phi^*] & = & -i \cdot \mu \cdot I_E \cdot \omega \\ \\ \bar{\partial}_A \Phi & = & 0 \end{cases} \tag{1.3}$$

a set of non-linear differential equations for d_A and Φ, related through the curvature F_A, where Φ^* is the adjoint of Φ with respect to a hermitian metric on E (see Theorem 1.3.7), where $I_E \in \mathrm{End}(E)$ is the identity and $\mu = \mu(E)$ is the slope of E, and one consequence is that Φ is holomorphic with respect to the holomorphic structure of E induced by d_A:

$$\text{i.e.} \quad \bar{\partial}_E \Phi = 0$$

where $\bar{\partial}_E = \bar{\partial}_A$ comes from the Chern-correspondence:

$$d_A = d + A = d + A^{0,1} d\bar{z} - A^{1,0} dz \longmapsto \bar{\partial} + A^{0,1} d\bar{z} = \bar{\partial}_A,$$

and where Φ^* is the adjoint of Φ with respect to a hermitian metric on E (given by Theorem 1.3.7 below).

Furthermore, one can see that any solution to (1.3) produces a polystable Higgs bundle. Nevertheless, the converse is quite hard to prove, but also true (see for instance

Wentworth [39, Theorem 2.17]):

Theorem 1.3.7. *If (E, Φ) is polystable, then it admits a hermitian metric satisfying the equations (1.3).*

This result comes indeed from the work of Hitchin [24] and more general from Simpson [36]. This last result, together with the results from the works of Donaldson [9] and also Corlette [8], generalizes the theorem presented by Narasimhan and Seshadri [33], known as the Non-Abelian Hodge Theorem and says that the character variety is homeomorphic to the moduli space of Higgs bundles (see Proposition 1.3.5 above).

There is an alternative construction of $\mathcal{M}(r, d)$ presented by Nitsure [34] using Geometric Invariant Theory. To do that, Nitsure first defines *Higgs bundles* like we do at the begining of this section: as pairs (E, Φ) where $E \to X$ is a holomorphic vector bundle, and $\Phi : E \to E \otimes K$ is an endomorphism twisted by the canonical line bundle $K \to X$, where $\Phi \in H^0(X; End(E))$. We elaborate on this in the next section.

1.4 The Moduli Space of Stable k-Higgs Bundles $\mathcal{M}^k(r, d)$

Definition 1.4.1. Hitchin Pairs and k-Higgs Bundles

i. A *Hitchin pair* is a generalization of a Higgs bundle. Instead of consider K, the canonical line bundle of X, if we consider a general line bundle $L \to X$, we get a *Hitchin pair* where now $\Phi \in H^0(X, \mathrm{End}(E) \otimes L)$.

ii. For $k \geqslant 0$, a *k-Higgs bundle* or *Higgs bundle with poles of order* k is the particular case of a Hitchin pair where $L = K \otimes L_p^{\otimes k}$. More clearly, if we consider a fixed point $p \in X$ as a divisor $p \in \mathrm{Sym}^1(X) = X$, and $L_p = \mathcal{O}_X(p)$ the line bundle that corresponds to that divisor p, we get a complex of the form

$$E \xrightarrow{\Phi^k} E \otimes K \otimes L_p^{\otimes k}$$

where $\Phi^k : E \to E \otimes K \otimes L_p^{\otimes k}$ is a *Higgs field with poles of order* k. So, we call such a complex as a k-Higgs bundle and Φ^k as its k-Higgs field.

iii. A k-Higgs bundle (E, Φ^k) is stable (respectively semistable) if the slope of any Φ^k-invariant subbundle of E is strictly less (respectively less or equal) than the slope of E : $\mu(E)$. Finally, (E, Φ^k) is called *polystable* if it is the direct sum of stable Φ^k-invariant subbundles, all of the same slope.

The moduli space of stable k-Higgs bundles $\mathcal{M}^k(r, d)$, and more generally, the moduli space of Hitchin pairs $\mathcal{M}_L(r, d)$, can be constructed either using gauge theory:

$$\mathcal{M}^k(r, d) = \mathcal{M}^k := \mathcal{B}_k^{ps}(r, d)/\mathcal{G}^{\mathbb{C}}$$

where

$$\mathcal{B}_k^{ps}(r, d) = \left\{(\bar{\partial}_B, \Phi^k) : \bar{\partial}_B(\Phi^k) = 0\right\} \subset \Big(\mathcal{A}_{ps}^{0,1}(r, d) \times \Omega_k^{1,0}(X; \mathrm{End}(E)\Big),$$

or using Geometric Invariant Theory, considering Φ^k as a 0-section:

$$\Phi^k \in H^0(X; End(E) \otimes L_p^{\otimes k}).$$

The moduli space of k-Higgs bundles is constructed by Nitsure [34]:

Theorem 1.4.2 (Nitsure [34, Proposition 7.4.]). *The space $\mathcal{M}^k(r, d)$ is a quasi-projective variety of complex dimension*

$$\dim_{\mathbb{C}}\big(\mathcal{M}^k(r, d)\big) = (r^2 - 1)\deg(K \otimes L_p^{\otimes k}) = (r^2 - 1)(2g - 2 + k).$$

From now on, we will suppose that $\mathrm{GCD}(r, d) = 1$. This co-prime condition implies that $\mathcal{M}^k(r, d)$ is smooth.

1.5 The Hitchin Map

From Proposition 1.3.4, the characteristic polynomial of Φ, the so-called Hitchin map is defined by:

$$\chi : \mathcal{M}^k(r, d) \longrightarrow H^0(X, L) \oplus \cdots \oplus H^0(X, L^r) \tag{1.4}$$

where $L = K \otimes L_p^{\otimes k}$ (see Nitsure [34, Theorem 6.1]). The Hitchin map is proper, and it is also an algebraically completely integrable system.

Definition 1.5.1. The set

$$\chi^{-1}(0) := \big\{[(E, \Phi)] \in \mathcal{M}_L(r, d) : \quad \chi(\Phi) = 0\big\}$$

is called *the Nilpotent Cone*.

1.6 The Moduli Space of Stable Triples $\mathcal{N}_\sigma(r_1, r_2, d_1, d_2)$

The reader may see the work of Bradlow and García-Prada [5], the work of Bradlow, García-Prada and Gothen [6] and the work of Muñoz, Ortega, Vázquez-Gallo [32] for the details on the results sumarized here.

Definition 1.6.1. Holomorphic Triples

i. A holomorphic triple on X is a triple $T = (E_1, E_2, \phi)$ consisting of two holomorphic vector bundles $E_1 \to X$ and $E_2 \to X$ and a homomorphism $\phi : E_2 \to E_1$, i.e. an element $\phi \in H^0(Hom(E_2, E_1))$.

ii. A homomorphism from a triple $T' = (E_1', E_2', \phi')$ to another triple $T = (E_1, E_2, \phi)$ is a commutative diagram of the form:

$$\begin{array}{ccc} E_1' & \xrightarrow{\phi'} & E_2' \\ \downarrow & & \downarrow \\ E_1 & \xrightarrow{\phi} & E_2 \end{array}$$

where the vertical arrows represent holomorphic maps.

iii. $T' \subset T$ is a subtriple if the sheaf homomorphisms $E_1' \to E_1$ and $E_2' \to E_2$ are injective. As usual, a subtriple is called proper if $0 \neq T' \subsetneq T$.

Definition 1.6.2. σ-Stability, σ-Semistability and σ-Polystability

i. For any $\sigma \in \mathbb{R}$ the σ-degree and the σ-slope of $T = (E_1, E_2, \phi)$ are defined as:

$$deg_\sigma(T) := \deg(E_1) + \deg(E_2) + \sigma \cdot \mathrm{rk}(E_2)$$

and

$$\mu_\sigma(T) := \frac{deg_\sigma(T)}{\mathrm{rk}(E_1) + \mathrm{rk}(E_2)} =$$

$$\frac{\deg(E_1) + \deg(E_2) + \sigma \cdot \mathrm{rk}(E_2)}{\mathrm{rk}(E_1) + \mathrm{rk}(E_2)} = \mu(E_1 \oplus E_2) + \sigma \frac{\mathrm{rk}(E_2)}{\mathrm{rk}(E_1) + \mathrm{rk}(E_2)}.$$

ii. T is then called σ-stable (respectively σ-semistable) if $\mu_\sigma(T') < \mu_\sigma(T)$ (respectively $\mu_\sigma(T') \leqslant \mu_\sigma(T)$) for any proper subtriple $0 \neq T' \subsetneq T$.

iii. A triple is called σ-polystable if it is the direct sum of σ-stable triples of the same σ-slope.

Now, we may use the following notation for Moduli Spaces of Triples:

i. Denote $\mathbf{r} = (r_1, r_2)$ and $\mathbf{d} = (d_1, d_2)$, and then consider

$$\mathcal{N}_\sigma = \mathcal{N}_\sigma(\mathbf{r}, \mathbf{d}) = \mathcal{N}_\sigma(r_1, r_2, d_1, d_2)$$

as the moduli space of σ-polystable triples $T = (E_1, E_2, \phi)$ such that $\mathrm{rk}(E_j) = r_j$ and $\deg(E_j) = d_j$.

ii. Denote $\mathcal{N}_\sigma^s = \mathcal{N}_\sigma^s(\mathbf{r}, \mathbf{d})$ as the subspace of σ-stable triples.

iii. Refer $(\mathbf{r}, \mathbf{d}) = (r_1, r_2, d_1, d_2)$ as the type of the triple $T = (E_1, E_2, \phi)$.

As mentioned by Bradlow, García-Prada and Gothen [6], there are certain necessary conditions in order for σ-polystable triples to exist. Denote $\mu_j = \mu(E_j) = \frac{d_j}{r_j}$ and define then:

$$\sigma_m := \mu_1 - \mu_2 \tag{1.5}$$

and

$$\sigma_M := \left(1 + \frac{r_1 + r_2}{|r_1 - r_2|}\right)(\mu_1 - \mu_2), \text{ when } r_1 \neq r_2. \tag{1.6}$$

Then, there is a couple of nice results from Bradlow, García-Prada and Gothen [6], the first one in terms of these necessary conditions for the existence, and the second one in terms of a duality isomorphism between moduli spaces:

Proposition 1.6.3 (Bradlow, García-Prada and Gothen [6, Proposition 2.2.]). *The moduli space $\mathcal{N}_\sigma(r_1, r_2, d_1, d_2)$ is a complex analytic variety, which is projective when $\sigma \in \mathbb{Q}$. A necessary condition for $\mathcal{N}_\sigma(r_1, r_2, d_1, d_2) \neq \emptyset$ is:*

$$0 \leqslant \sigma_m < \sigma < \sigma_M \text{ when } r_1 \neq r_2,$$

or

$$0 \leqslant \sigma_m < \sigma \text{ when } r_1 = r_2.$$

Remark 1.6.4 (Bradlow, García-Prada and Gothen [6, Remark 2.3.]). If $\sigma_m = 0$ and $r_1 \neq r_2$ then $\sigma_m = \sigma_M = 0$ and $\mathcal{N}^s_\sigma(r_1, r_2, d_1, d_2) = \emptyset$ unless $\sigma = 0$.

We denote by $I \subset \mathbb{R}$ the following interval:

$$I = \begin{cases} [\sigma_m, \sigma_M] & \text{if } r_1 \neq r_2, r_1 \neq 0, r_2 \neq 0, \\ [\sigma_m, \infty[& \text{if } r_1 = r_2 \neq 0, \\ \mathbb{R} & \text{if } r_1 = 0 \text{ or } r_2 = 0. \end{cases} \tag{1.7}$$

Given a triple $T = (E_1, E_2, \phi)$, its dual triple is $T^* = (E_2^*, E_1^*, \phi^*)$, where E_j^* is the dual of E_j and $\phi^* : E_1^* \to E_2^*$ is the transpose of $\phi : E_2 \to E_1$. So:

Theorem 1.6.5 (Bradlow and García-Prada [5, Proposition 3.16]). *T is σ-stable (respectively σ-semistable) if and only if T^* is σ-stable (respectively σ-semistable). The map $T \mapsto T^*$ induces an isomorphism:*

$$\mathcal{N}_\sigma(r_1, r_2, d_1, d_2) \cong \mathcal{N}_\sigma(r_2, r_1, -d_2, -d_1).$$

Last theorem is an important tool since it can be used to restrict the study of triples to the case $r_1 \geqslant r_2$ and appeal to duality when dealing with $r_1 < r_2$. For more triples details, the reader may see [5] or [6].

Here we present the main results of Bradlow, García-Prada and Gothen [6], where they describe the general moduli spaces of triples:

Theorem 1.6.6 (Bradlow, García-Prada and Gothen [6, Theorem A]). *(1) A triple $T = (E_1, E_2, \varphi)$ is σ_m-polystable if and only if $\varphi = 0$, and E_1 and E_2 are polystable. We thus have*

$$\mathcal{N}_{\sigma_m}(r_1, r_2, d_1, d_2) \cong M(r_1, d_1) \times M(r_2, d_2)$$

where $M(r, d)$ represents the moduli space of polystable bundles of rank r and degree d. In particular, $\mathcal{N}_{\sigma_m}(r_1, r_2, d_1, d_2)$ is non-empty and irreducible.

(2) If $\sigma > \sigma_m$ is any value such that $\sigma > 2g - 2$ (and $\sigma < \sigma_M$ if $r_1 \neq r_2$) then $\mathcal{N}^s_\sigma(r_1, r_2, d_1, d_2)$ is smooth, non-empty and irreducible of dimension

$$\dim_{\mathbb{C}}(\mathcal{N}^s_\sigma(r_1, r_2, d_1, d_2)) = (g-1)(r_1^2 + r_2^2 - r_1 r_2) - r_1 d_2 + r_2 d_1 + 1.$$

Moreover:

- *If $r_1 = r_2 = r$ then the moduli space $\mathcal{N}^s_\sigma(r, r, d_1, d_2)$ is birrationally equivalent to a $\mathbb{P}^N$-fibration over $\mathcal{N}(r, d_2) \times \mathrm{Sym}^{d_1 - d_2}(X)$ where the fiber dimension is $N = r(d_1 - d_2) - 1$.*
- *If $r_1 > r_2$ then the moduli space $\mathcal{N}^s_\sigma(r_1, r_2, d_1, d_2)$ is birrationally equivalent to a $\mathbb{P}^N$-fibration over $\mathcal{N}(r_1 - r_2, d_1 - d_2) \times \mathcal{N}(r_2, d_2)$ where the fiber dimension is $N = r_2 d_1 - r_1 d_2 + r_2(r_1 - r_2)(g-1) - 1$.*
- *If $r_1 < r_2$ then the moduli space $\mathcal{N}^s_\sigma(r_1, r_2, d_1, d_2)$ is birrationally equivalent to a $\mathbb{P}^N$-fibration over $\mathcal{N}(r_2 - r_1, d_2 - d_1) \times \mathcal{N}(r_1, d_1)$ where the fiber dimension is $N = r_2 d_1 - r_1 d_2 + r_1(r_2 - r_1)(g-1) - 1$.*

(3) If $r_1 \neq r_2$ then $\mathcal{N}_{\sigma_M}(r_1, r_2, d_1, d_2)$ is non-empty and irreducible. Moreover:

$$\mathcal{N}_{\sigma_M}(r_1, r_2, d_1, d_2) \cong \mathcal{N}(r_2, d_2) \times \mathcal{N}(r_1 - r_2, d_1 - d_2) \text{ if } r_1 > r_2$$

and

$$\mathcal{N}_{\sigma_M}(r_1, r_2, d_1, d_2) \cong \mathcal{N}(r_1, d_1) \times \mathcal{N}(r_2 - r_1, d_2 - d_1) \text{ if } r_1 < r_2.$$

Using the results above, Muñoz, Ortega and Vásquez-Gallo [32] conclude some useful results that we will use later:

Lemma 1.6.7 (Muñoz, Ortega and Vásquez-Gallo [32, Lemma 3.5]). *There are isomorphisms*

$$\mathcal{N}_\sigma(r,0,d,0) \cong M(r,d) \quad \textit{and} \quad \mathcal{N}^s_\sigma(r,0,d,0) \cong \mathcal{N}(r,d) \quad \forall \sigma \in \mathbb{R}.$$

In particular

$$\mathcal{N}_\sigma(1,0,d,0) = \mathcal{N}^s_\sigma(1,0,d,0) \cong \mathcal{J}^d(X) \quad \forall \sigma \in \mathbb{R}.$$

Proposition 1.6.8 (Muñoz, Ortega and Vásquez-Gallo [32, Proposition 3.7]). *Let $\sigma_0 \in I$ and let $T = (E_1, E_2, \phi) \in \mathcal{N}_{\sigma_0}(r_1, r_2, d_1, d_2)$ be a strictly σ_0-semistable triple. Then one of the following conditions holds:*

(1) For all σ_0-destabilizing subtriples $T' = (E_1', E_2', \phi')$, we have

$$\frac{r_2'}{r_1' + r_2'} = \frac{r_2}{r_1 + r_2}.$$

Then T is strictly σ-semistable for $\sigma \in \,]\sigma_0 - \varepsilon, \sigma_0 + \varepsilon[$, for some $\varepsilon > 0$ small enough.

(2) There exists a σ_0-destabilizing subtriple $T' = (E_1', E_2', \phi')$ with

$$\frac{r_2'}{r_1' + r_2'} \neq \frac{r_2}{r_1 + r_2}.$$

Then:

- *either*
$$\frac{r_2'}{r_1' + r_2'} > \frac{r_2}{r_1 + r_2},$$
and so T is σ-unstable for any $\sigma > \sigma_0$,
- *or*
$$\frac{r_2'}{r_1' + r_2'} < \frac{r_2}{r_1 + r_2},$$
and so T is σ-unstable for any $\sigma < \sigma_0$.

Definition 1.6.9. Those values of σ for which Case (2) in Proposition 1.6.8 occurs are called *critical values*.

Lemma 1.6.10 (Muñoz, Ortega and Vásquez-Gallo [32, Lemma 3.16]). *(1) If $d_1 < d_2$ then $\mathcal{N}_\sigma(1,1,d_1,d_2) = \emptyset$.*

(2) If $d_1 > d_2$ then:

- $\mathcal{N}_{\sigma_m}(1,1,d_1,d_2) \cong \mathcal{J}^{d_1} \times \mathcal{J}^{d_2}$ *and* $\mathcal{N}^s_{\sigma_m}(1,1,d_1,d_2) = \emptyset$.
- $\mathcal{N}_\sigma(1,1,d_1,d_2) = \mathcal{N}^s_\sigma(1,1,d_1,d_2) \cong \mathcal{J}^{d_1} \times \mathrm{Sym}^{d_1-d_2}(X)\ \forall \sigma > \sigma_m$.
- $\mathcal{N}_\sigma(1,1,d_1,d_2) = \mathcal{N}^s_\sigma(1,1,d_1,d_2) = \emptyset$ *for* $\sigma < \sigma_m$.

Fixing the type $(\mathbf{r},\mathbf{d}) = (r_1,r_2,d_1,d_2)$ for the moduli spaces of holomorphic triples, Muñoz, Ortega and Vásquez-Gallo [32] describe the differences between two spaces $\mathcal{N}_{\sigma_1}$ and $\mathcal{N}_{\sigma_2}$ when σ_1 and σ_2 are separated by a critical value. For a critical value $\sigma_c \in I$ set $\sigma_c^+ = \sigma + \varepsilon$ and $\sigma_c^- = \sigma - \varepsilon$, where $\varepsilon > 0$ is small enough so that σ_c is the only critical value in the interval $]\sigma_c^-, \sigma_c^+[$.

Definition 1.6.11. The *flip loci* are defined as:

$$S_{\sigma_c^+} := \left\{T \in \mathcal{N}_{\sigma_c^+} : T \text{is } \sigma_c^- - \text{unstable}\right\} \subset \mathcal{N}_{\sigma_c^+},$$

$$S_{\sigma_c^-} := \left\{T \in \mathcal{N}_{\sigma_c^-} : T \text{is } \sigma_c^+ - \text{unstable}\right\} \subset \mathcal{N}_{\sigma_c^-},$$

and $S^s_{\sigma_c^\pm} := S_{\sigma_c^\pm} \cap \mathcal{N}^s_{\sigma_c^\pm}$ for the stable part of the flip loci.

Remark 1.6.12. Note that for $\sigma_c = \sigma_m$, $\mathcal{N}_{\sigma_m^-} = \emptyset$, hence $\mathcal{N}_{\sigma_m^+} = S_{\sigma_m^+}$. Also $\mathcal{N}^s_{\sigma_m} = \emptyset$, by the last part of Proposition 1.6.8. Anologously, when $r_1 \neq r_2$, $\mathcal{N}_{\sigma_M^+} = \emptyset$, $\mathcal{N}_{\sigma_M^-} = S_{\sigma_M^-}$ and $\mathcal{N}^s_{\sigma_M} = \emptyset$.

1.7 Stratifications on the Moduli Space of Higgs Bundles

Definition 1.7.1. As a consequence of Shatz [35, Proposition 10 and Proposition 11], there is a finite stratification of $\mathcal{M}(r,d)$ by the Harder-Narasimhan type of the underlying vector bundle E of a Higgs bundle (E,Φ):

$$\mathcal{M}(r,d) = \bigcup_t U'_t$$

where $U'_t \subset \mathcal{M}(r,d)$ is the subspace of Higgs bundles (E,Φ) which associated vector bundle E has $\mathrm{HNT}(E) = t$, and where we are taking this union over the existing types in $\mathcal{M}(r,d)$. This stratification is known as the *Shatz stratification*.

On the other hand, according to Hitchin [24], $(\mathcal{M}, I, \Omega)$ is a Kähler manifold, where I is its complex structure and Ω its corresponding Kähler form. Furthermore, $\mathbb{C}^*$ acts on $\mathcal{M}$ biholomorphically with respect to the complex structure I by the action $z \cdot (E,\Phi) = (E, z\cdot\Phi)$, where the Kähler form Ω is invariant under the induced action $e^{i\theta}\cdot(E,\Phi) = (E, e^{i\theta}\cdot\Phi)$ of the circle $\mathbb{S}^1 \subset \mathbb{C}^*$. Besides, this circle action is Hamiltonian with proper momentum map

$$f : \mathcal{M} \longrightarrow \mathbb{R}$$

defined by:

$$f(E,\Phi) = \frac{1}{2\pi}\|\Phi\|^2_{L^2} = \frac{i}{2\pi}\int_X tr(\Phi\Phi^*). \tag{1.8}$$

where Φ^* is the adjoint of Φ with respect to the hermitian metric on E given by Theorem 1.3.7, and f has finitely many critical values.

There is another important fact mentioned by Hitchin [24](see the original version in Frankel [10], and its application to Higgs bundles in Hitchin [24]): the critical points of f are exactly the fixed points of the circle action on $\mathcal{M}$.

If $(E,\Phi) = (E, e^{i\theta}\Phi)$ then $\Phi = 0$ with critical value $c_0 = 0$. The corresponding critical submanifold is $F_0 = f^{-1}(c_0) = f^{-1}(0) = \mathcal{N}$, the moduli space of stable bundles.

On the other hand, when $\Phi \neq 0$, there is a type of algebraic structure for Higgs bundles introduced by Simpson [36]: a *Variation of Hodge Structure*, or simply a *VHS*, for a Higgs bundle (E,Φ) is a decomposition:

$$E = \bigoplus_{j=1}^{n} E_j \text{ such that } \Phi : E_j \to E_{j+1}\otimes K \text{ for } 1 \leqslant j \leqslant n-1. \tag{1.9}$$

Has been proved by Simpson [37] that the fixed points of the circle action on $\mathcal{M}(r,d)$, and so, the critical points of f, are these Variations of the Hodge Structure, VHS, where the critical values $c_\lambda = f(E,\Phi)$ will depend on the degrees d_j of the components

$E_j \subset E$. By Morse theory, we can stratify $\mathcal{M}$ in such a way that there is a non-zero critical submanifold $F_\lambda := f^{-1}(c_\lambda)$ for each non-zero critical value $0 \neq c_\lambda = f(E, \Phi)$ where (E, Φ) represents a fixed point of the circle action, or equivalently, a VHS. We said then that (E, Φ) is a $(\mathrm{rk}(E_1),\ ...,\ \mathrm{rk}(E_n))$-VHS.

For rank $\mathrm{rk}(E) = 2$ and degree $\deg(E) = d$ Higgs bundles, Hitchin [24, Proposition (7.1)] establishes that:

1. The momentum map f is proper.
2. f has a finite number of critical values: $c_0 = 0$ and $c_{d_1} = d_1 - \frac{d}{2}$ for $d_1 \in \{1,\ ...,\ g-1\}$.
3. $F_0 = f^{-1}(0)$ is a non-degenerate critical manifold of index 0, and is isomorphic to the moduli space $\mathcal{N}$ of stable bundles.

Hence, a point $(E, \Phi) \in \mathcal{N} = F_0$ is a pair where $E \to X$ is an indecomposable holomorphic bundle of $\mathrm{rk}(E) = r$ and $\Phi \equiv 0$. This statement holds in general, as well as the first one: f is also proper in higher rank. The second statement is proved just for rank two by Hitchin [24] and for rank three by Gothen [14]; even so, it holds in general: it follows from the results in García-Prada and Heinloth [12], where they describe the possibles VHS that can exist as fixed loci in the moduli of Higgs bundles.

We will use some results of Kirwan [27] in terms of stratifications. Since $\mathcal{M}$ is not a compact manifold, we shall need the following:

Theorem 1.7.2 (Kirwan [27, (9.1.)])**.** *Let Σ be any symplectic manifold. Let K be any compact group that acts on Σ. Suppose there is a moment map $f : \Sigma \to \mathbb{R}$. Then one can obtain the same results of Kirwan [27] as for compact manifolds (except for Kirwan [27, Theorem (5.8.)]) subject only to one condition: for some metric on Σ, every path of steepest descent under the function $h := \|f\|$ is contained in some compact subset of Σ.*

In our particular case, $\Sigma = \mathcal{M},\ K = \mathbb{S}^1 \subset \mathbb{C}^*,\ f : \mathcal{M} \to \mathbb{R}$ defined as before, and everything holds. Recall that there is a holomorphic action of the multiplicative group

$\mathbb{C}^*$ on $\mathcal{M}(r,d)$ defined by the multiplication: $z \cdot (E,\Phi) \mapsto (E, z\cdot\Phi)$. Recall also that Hausel [19] proves that the limit $\lim_{z\to 0} z\cdot(E,\Phi) = \lim_{z\to 0}(E, z\cdot\Phi)$ exists and is well defined for all $(E,\Phi) \in \mathcal{M}(r,d)$. Moreover, this limit is fixed by the $\mathbb{C}^*$-action. Let $\{F_\lambda\}$ be the irreducible components of the fixed points loci of $\mathbb{C}^*$ on $\mathcal{M}(r,d)$.

In general, when we have a Kähler manifold (Σ, I, ω) with complex structure I and Kähler form ω, where a compact group K acts biholomorphically with respect to I and such that ω is invariant under this action, where besides, the action is Hamiltonian with proper momentum map $f : \Sigma \to \mathbb{R}$, with finitely many critical values, being $(0, c_0)$ the absolute minimum, we may then consider the set of components of the fixed points of the K-action: $\{F_\lambda\}_{\lambda\in\Lambda}$ and then, we may consider two stratifications on Σ: the *Bialynicki-Birula stratification* and the *Morse stratification*. We shall define both.

Definition 1.7.3. Consider the set

$$U_\lambda^{BB} := \{(E,\Phi) \in \mathcal{M} | \lim_{z\to 0} z\cdot(E,\Phi) \in F_\lambda\}.$$

This set U_λ^{BB} is the *upward stratum* of the *Bialynicki-Birula stratification*:

$$\mathcal{M} = \bigcup_\lambda U_\lambda^{BB}.$$

Definition 1.7.4. Similarly, consider the set

$$D_\lambda^{BB} := \{(E,\Phi) \in \mathcal{M} | \lim_{z\to\infty} z\cdot(E,\Phi) \in F_\lambda\},$$

is known as the *downward stratum* of the *Bialynicki-Birula stratification*.

Remark 1.7.5. This time, we must be careful, $\bigcup_\lambda D_\lambda^{BB}$ is not the whole space $\mathcal{M}$, but a deformation retraction of it.

Definition 1.7.6. Let U_λ^M be the set of points $(E,\Phi) \in \mathcal{M}$ such that its path of steepest descent for the Morse function f and the Kähler metric have limit points in F_λ. This set is called the *upward Morse flow of* F_λ, and it gives another stratification of $\mathcal{M}$:

$$\mathcal{M} = \bigcup_\lambda U_\lambda^M$$

Definition 1.7.7. As well as we did above, we may define D_λ^M as the set of points $(E, \Phi) \in \mathcal{M}$ such that its path of steepest descent for the Morse function $-f$ and the Kähler metric have limit points in F_λ. This set is called the *downward Morse flow of* F_λ.

Remark 1.7.8. Once again, we must be careful, because $\bigcup_\lambda D_\lambda^M$ is not the whole $\mathcal{M}$, it is just a deformation retraction of it.

1.7.2 is a very strong result that allows us to use all the work of Kirwan [27], except Theorem Kirwan [27, Theorem (5.8.)]. In particular, we have:

Theorem 1.7.9 (Kirwan [27, Theorem (6.16.)])**.** *The Bialynicki-Birula stratification and the Morse stratification coincide. In other words, using the above notation, we get:*

$$U_\lambda^{BB} = U_\lambda^M \text{ and } \quad D_\lambda^{BB} = D_\lambda^M \quad \forall\lambda$$

From now on, we will denote simply $U_\lambda^+ := U_\lambda^{BB} = U_\lambda^M$.

Remark 1.7.10. Everything in this section can be generalized to Hitchin pairs.

1.8 Morse Theory

In this section we shall use the abbreviated notations $\mathcal{M} = \mathcal{M} = \mathcal{M}^k(r, d)$, whenever no confusion is likely to arise. We assume here, as everywhere else, that r and d are co-prime.

The Hitchin functional $f\colon \mathcal{M} \to \mathbb{R}$ is a perfect Bott–Morse function. This was observed by Hitchin [24] and follows from a Theorem of Frankel [10], using the fact that f is a non-negative proper moment map for a circle action on a Kähler manifold.

We denote the (connected) critical submanifolds of f by $\{F_\lambda\}$. Write $N_\lambda = T_{\mathcal{M}/F_\lambda}$ for the normal bundle to F_λ in $\mathcal{M}$. The fact that f is Bott–Morse means that the restriction of the tangent bundle of $\mathcal{M}$ to F_λ decomposes as

$$T\mathcal{M}_{|F_\lambda} = TF_\lambda \oplus N_\lambda^+ \oplus N_\lambda^-,$$

where $N_\lambda^\pm$ denote the subbundles of N_λ on which the Hessian of f is positive and negative definite, respectively. Thus the normal bundle to F_λ in $\mathcal{M}$ is

$$N_\lambda = N_\lambda^+ \oplus N_\lambda^-$$

The *Bott–Morse index* of the critical submanifold F_λ is by definition the rank of the negative part of the normal bundle:

$$I_\lambda^- := \mathrm{rk}(N_\lambda^-).$$

It is a standard procedure in Morse theory to perturb the Bott–Morse function f, so that it takes different values in each of the critical submanifolds F_λ (see, for example, Hirsch [23]). In what follows we shall assume that this has been done, so that we may write $f(F_\lambda) = \lambda \in \mathbb{R}$, with the absolute minimum of f being the moduli space of stable bundles $N_0 = f^{-1}(0) = \mathcal{N}(r, d)$.

We shall use the standard Morse theory notation

$$\mathcal{M}_\lambda = f^{-1}([0, \lambda]).$$

Denote by $S(N_\lambda^-)$ and $D(N_\lambda^-)$ the sphere and disk bundles in N_λ^-, respectively. It is a basic fact of Bott-Morse theory that, for each λ, there is a homotopy equivalence

$$\mathcal{M}_\lambda \sim \mathcal{M}_{\lambda-\epsilon} \cup_{S(N_\lambda^-)} D(N_\lambda^-), \tag{1.10}$$

for $\epsilon > 0$ small enough that there are no critical values of f in $[\lambda - \epsilon, \lambda[$. Moreover, the fact that f is perfect means that, even with integer coefficients,

$$H^*(\mathcal{M}_\lambda) = H^*(\mathcal{M}_{\lambda-\epsilon}) \oplus H^*(D(N_\lambda^-), S(N_\lambda^-)). \tag{1.11}$$

Moreover, the Thom isomorphism gives

$$H^*(D(N_\lambda^-), S(N_\lambda^-)) \cong H^{*+I_\lambda}(N_\lambda), \tag{1.12}$$

so that, with $\mathbb{Z}$-coefficients,

$$H^*(\mathcal{M}_\lambda) = H^*(\mathcal{M}_{\lambda-\epsilon}) \oplus H^{*+I_\lambda}(N_\lambda). \tag{1.13}$$

It follows that

$$H^*(\mathcal{M}, \mathbb{Z}) = \bigoplus_\lambda H^{*+I_\lambda}(F_\lambda, \mathbb{Z}). \tag{1.14}$$

When the rank is $\mathrm{rk}(E) = 2$, Bento [3, Theorem 2.1.7.] shows that the Bott-Morse index $I_\lambda = I_{d_1} = 2(2d_1 - d + g - 1)$, for

$$F_\lambda = F_{d_1} = \left\{ (E, \Phi) = (E_1 \oplus E_2, \begin{pmatrix} 0 & 0 \\ \varphi_{21} & 0 \end{pmatrix}) \middle| \begin{array}{c} \deg(E_1) = d_1, \quad \deg(E_2) = d_2, \\ \mathrm{rk}(E_1) = 1, \qquad \mathrm{rk}(E_2) = 1, \\ \varphi_{21} : E_1 \to E_2 \otimes L \end{array} \right\}.$$

So, there is a explicit description of the additive cohomology of $\mathcal{M}(2, d)$:

$$H^*(\mathcal{M}(2, d), \mathbb{Z}) = H^*(\mathcal{N}, \mathbb{Z}) \oplus \bigoplus_{d_1 > \frac{d}{2}}^{\frac{d+d_L}{2}} H^{*+2(2d_1-d+g-1)}(F_{d_1}, \mathbb{Z}), \tag{1.15}$$

where $d_L = \deg(L)$ and $\frac{d}{2} < d_1 < \frac{d+d_L}{2}$.

When the rank is $\mathrm{rk}(E) = 3$, there are three kinds of non-trivial critical submanifolds, or equivalently, three different VHS:

1. $(1, 2)$-VHS of the form

$$F_{d_1} = \left\{ (E, \Phi) = (E_1 \oplus E_2, \begin{pmatrix} 0 & 0 \\ \varphi_{21} & 0 \end{pmatrix}) \middle| \begin{array}{c} \deg(E_1) = d_1, \quad \deg(E_2) = d_2, \\ \mathrm{rk}(E_1) = 1, \qquad \mathrm{rk}(E_2) = 2, \\ \varphi_{21} : E_1 \to E_2 \otimes L \end{array} \right\},$$

with Bott-Morse index $I_\lambda = I_{d_1} = 2(3d_1 - d + 2g - 2)$, and $\frac{d}{3} < d_1 < \frac{d}{3} + \frac{d_L}{2}$.

2. $(2,1)$-VHS of the form

$$F_{d_2} = \left\{ (E,\Phi) = (E_2 \oplus E_1, \begin{pmatrix} 0 & 0 \\ \varphi_{21} & 0 \end{pmatrix}) \middle| \begin{array}{c} \deg(E_2) = d_2, \quad \deg(E_1) = d_1, \\ \mathrm{rk}(E_2) = 2, \qquad \mathrm{rk}(E_1) = 1, \\ \varphi_{21} : E_2 \to E_1 \otimes L \end{array} \right\},$$

with Bott-Morse index $I_\lambda = I_{d_2} = 2(3d_2 - 2d + 2g - 2)$, and $\frac{2d}{3} < d_2 < \frac{2d}{3} + \frac{d_L}{2}$.

3. $(1,1,1)$-VHS of the form

$$F_{m_1 m_2} = \left\{ (E,\Phi) = (E_1 \oplus E_2 \oplus E_3, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix}) \middle| \begin{array}{c} \deg(E_j) = d_j, \\ rk(E_j) = 1, \\ \varphi_{ij} : E_j \to E_i \otimes L \end{array} \right\},$$

with Bott-Morse index $I_\lambda = I_{m_1 m_2} = 2(3d_L - (m_1 + m_2) + 2g - 2)$, where $(m_1, m_2) \in \Omega$ where $M_j := E_j^* E_{j+1} L,\ m_j := \deg(M_j) = d_{j+1} - d_j + d_L$, and

$$\Omega = \left\{ (m_1, m_2) \in \mathbb{N} \times \mathbb{N} \middle| \begin{array}{c} 2m_1 + m_2 < 3d_L \\ m_1 + 2m_2 < 3d_L \\ m_1 + 2m_2 \equiv 0 (mod 3) \end{array} \right\}.$$

For more details of the description of Ω, the reader can see Gothen [14], or Bento [3]. Therefore, there is a explicit description of the additive cohomology of $\mathcal{M}(3,d)$:

$$\begin{gathered} H^*(\mathcal{M}(3,d),\mathbb{Z}) = H^*(\mathcal{N},\mathbb{Z}) \oplus \bigoplus_{d_1 > \frac{d}{3}}^{\frac{d}{3} + \frac{d_L}{2}} H^{*+I_{d_1}}(F_{d_1},\mathbb{Z}) \oplus \\ \bigoplus_{d_2 > \frac{2d}{3}}^{\frac{2d}{3} + \frac{d_L}{2}} H^{*+I_{d_2}}(F_{d_2},\mathbb{Z}) \oplus \bigoplus_{(m_1,m_2) \in \Omega} H^{*+I_{m_1 m_2}}(F_{m_1 m_2},\mathbb{Z}). \end{gathered} \tag{1.16}$$

Chapter 2

Stabilization of the Homotopy Groups of the Moduli Space of k-Higgs Bundles

Fix a point $p \in X$, and let $L_p = \mathcal{O}_X(p)$ be the associated line bundle to the divisor $p \in \mathrm{Sym}^1(X) = X$. Recall that a *$k$-Higgs bundle* (or *Higgs bundle with poles of order k*) is a pair (E, Φ^k) where:

$$E \xrightarrow{\quad \Phi^k \quad} E \otimes K \otimes L_p^{\otimes k}$$

and where the morphism $\Phi^k \in H^0(X, \mathrm{End}(E) \otimes K \otimes L_p^{\otimes k})$ is what we call as a *Higgs field with poles of order k*. The moduli space of k-Higgs bundles of rank r and degree d is denoted by $\mathcal{M}^k(r,d)$. Recall that $\mathrm{GCD}(r,d) = 1$, so $\mathcal{M}^k(r,d)$ is smooth.

Furthermore, there is an embedding

$$i_k : \mathcal{M}^k(r,d) \to \mathcal{M}^{k+1}(r,d)$$

$$\big[(E, \Phi^k)\big] \longmapsto \big[(E, \Phi^k \otimes s_p)\big]$$

where $0 \neq s_p \in H^0(X, L_p)$ is a non-zero fixed section of L_p.

2.1 Generators for the Cohomology Ring

According to Hausel and Thaddeus [21, (4.4)], there is a universal family $(\mathbb{E}^k, \mathbf{\Phi}^k)$ over $X \times \mathcal{M}^k$ where

$$\begin{cases} \mathbb{E}^k & \to & X \times \mathcal{M}^k(r,d) \\ \mathbf{\Phi}^k & \in & H^0\big(\mathrm{End}(\mathbb{E}^k) \otimes K \otimes L_p^{\otimes k}\big) \end{cases}$$

and from now on, we will refer $(\mathbb{E}^k, \mathbf{\Phi}^k)$ as a *universal k-Higgs bundle*. Note that $(\mathbb{E}^k, \mathbf{\Phi}^k)$ satisfies the *Universal Property*, it means that, whenever exists $(\mathbb{F}^k, \mathbf{\Psi}^k)$ such that

$$(\mathbb{E}^k, \mathbf{\Phi}^k)_p \cong (\mathbb{F}^k, \mathbf{\Psi}^k)_p \quad \forall p = (E, \Phi^k) \in \mathcal{M}^k(r,d),$$

then, there exists a unique bundle morphism $\xi : \mathbb{F}^k \to \mathbb{E}^k$ such that

$$\begin{array}{ccc} \mathbb{F}^k & \overset{\exists ! \xi}{\dashrightarrow} & \mathbb{E}^k \\ & {\scriptstyle p_2} \searrow \quad \swarrow {\scriptstyle p_1} & \\ & X \times \mathcal{M}^k(r,d) & \end{array} \tag{2.1}$$

commutes: $p_2 = p_1 \circ \xi$. Equivalently, if $(\mathbb{E}^k, \mathbf{\Phi}^k)$ and $(\mathbb{F}^k, \mathbf{\Psi}^k)$ are families of stable k-Higgs bundles parametrized by $\mathcal{M}^k(r,d)$, such that $(\mathbb{E}^k, \mathbf{\Phi}^k)_p \cong (\mathbb{F}^k, \mathbf{\Psi}^k)_p$ for all $p = (E, \Phi^k) \in \mathcal{M}^k(r,d)$, then, there is a line bundle $\mathcal{L} \to \mathcal{M}^k(r,d)$ such that $(\mathbb{E}^k, \mathbf{\Phi}^k) \cong (\mathbb{F}^k \otimes \pi_2^*(\mathcal{L}), \mathbf{\Phi}^k)$, where $\pi_2 : X \times \mathcal{M}^k(r,d) \to \mathcal{M}^k(r,d)$ is the natural projection. For more details, see Hausel and Thaddeus [21, (4.2)].

If we consider the embedding $i_k : \mathcal{M}^k(r,d) \to \mathcal{M}^{k+1}(r,d)$ for general rank, we get that:

Proposition 2.1.1. *Let $(\mathbb{E}^k, \mathbf{\Phi}^k)$ be a universal Higgs bundle. Then:*

$$(\mathrm{Id}_X \times i_k)^*(\mathbb{E}^{k+1}) \cong \mathbb{E}^k.$$

Proof. Note that

$$\big(\mathbb{E}^k, \mathbf{\Phi}^k \otimes \pi_1^*(s_p)\big) \to X \times \mathcal{M}^k$$

is a family of $(k+1)$-Higgs bundles on X, where $\pi_1 : X \times \mathcal{M}^k \to X$ is the natural projection. So, by the univeral property:

$$\big(\mathbb{E}^k, \mathbf{\Phi}^k \otimes \pi_1^*(s_p)\big) = j^*\big(\mathbb{E}^{k+1}, \varPhi^{k+1}\big)$$

where

$$j : X \times \mathcal{M}^k \to X \times \mathcal{M}^{k+1}$$
$$\big(x, (E, \Phi^k)\big) \mapsto \big(x, (E, \Phi^k \otimes s_p)\big).$$

♠

Now, define $\mathrm{Vect}^r(X)$ as

$$\mathrm{Vect}^r(X) := \big\{V \to X : \ V \text{ is a topological vector bundle of rank } \quad \mathrm{rk}(V) = r\big\}/ \cong,$$

and take the operation

$$[V] \oplus [W] := [V \oplus W]$$

where the equivalence classes are taken by isomorphism between vector bundles. Then, $\big(\mathrm{Vect}^r(X), \oplus\big)$ is an abelian semi-group. Let $K(X)$ be the K-theory group of X where

$$K(X) = K\big(\mathrm{Vect}^r(X)\big) := \big\{[V] - [W]\big\}/ \sim$$

and where

$$[V] - [W] \sim [V \oplus U] - [W \oplus U] \ \forall U \to X \text{ topological vector bundle},$$

and recall that it is an abelian group (see Atiyah [1] or Hatcher [18]). Recall also that the Chern classes factor through K-theory:

$$\begin{array}{ccc} \mathrm{Vect}^r(X) & \xrightarrow{\ c\ } & H^*(X, \mathbb{Z}) \\ \downarrow & \nearrow_{c} & \\ K(X) & & \end{array} \tag{2.2}$$

where, for a complex vector bundle $V \to X$ of rank $\mathrm{rk}(V) = r$, c is defined as

$$c\big([V]\big) := \sum_{i=0}^{r} c_i(V) \quad \forall [V] \in \mathrm{Vect}^r(X)$$

and where $c_i(V) \in H^{2i}(X, \mathbb{Z})$. This map is a homomorphism since

$$c\big([W] \oplus [W]\big) = c\big([V]\big) \cdot c\big([W]\big) \in H^*(X, \mathbb{Z}).$$

We now describe a result of Markman [30]. Choose a basis:

$$\{x_1,\ ...,\ x_{2g},\ x_{2g+1},\ x_{2g+2}\} \subset K(X) = K^0(X) \oplus K^1(X),$$

where $\{x_1,\ ...,\ x_{2g}\} \subset K^1(X)$, and $\{x_{2g+1},\ x_{2g+2}\} \subset K^0(X)$ and so, since there is a universal bundle $\mathbb{E}^k \to X \times \mathcal{M}^k$, we can get the Künneth decomposition (see Atiyah [1, Corollary 2.7.15]):

$$[\mathbb{E}^k] = \sum_{j=0}^{2g} x_j \otimes e_j^k$$

for $e_j^k \in K(\mathcal{M}^k)$, since $K(X \times \mathcal{M}^k) \cong K(X) \otimes K(\mathcal{M})$.

Then, Markman [30] considers the Chern classes $c_j(e_i^k) \in H^{2j}(\mathcal{M}^k, \mathbb{Z})$ for $e_i^k \in K(\mathcal{M}^k)$ and proves that:

Theorem 2.1.2 (Markman [30, Theorem 3])**.** *The cohomology ring $H^*\big(\mathcal{M}^k(r,d), \mathbb{Z}\big)$ is generated by the Chern classes of the Künneth factors of the universal vector bundle.*

2.2 Main Result

Recall that we want to prove that the map

$$\pi_j(i_k) : \pi_j\big(\mathcal{M}^k(r,d)\big) \to \pi_j\big(\mathcal{M}^{k+1}(r,d)\big)$$

stabilizes as $k \to \infty$. But first, we need to present some previous results to conclude that.

Proposition 2.2.1. *Consider the classes $e_i^k \in K(\mathcal{M}^k)$. Then $i_k^*\big(c_j(e_i^{k+1})\big) = c_j(e_i^k)$.*

Proof. By 2.1.1, and by the naturality of the Chern classes:

$$\sum_{j=0}^{2g} x_j \otimes e_j^k = [\mathbb{E}^k] = [(\mathrm{Id}_X \times i_k)^*(\mathbb{E}^{k+1})] = \sum_{j=0}^{2g} x_j \otimes i_k^*(e_j^{k+1})$$

we have that $i_k^*\big(e_i^{k+1}\big) = e_i^k$ and hence $i_k^*\big(c_j(e_i^{k+1})\big) = c_j(e_i^k)$. ♠

An immediate consequence will be

Corollary 2.2.2. $i_k^* : H^*(\mathcal{M}^{k+1}, \mathbb{Z}) \twoheadrightarrow H^*(\mathcal{M}^k, \mathbb{Z})$ *is surjective.*

Recall that a gauge transformation g is called *unitary* if g preserves the hermitian inner product. We will denote $\mathcal{G}$ as the group of unitary gauge transformations. Atiyah and Bott [2] denote $\bar{\mathcal{G}}$ as the quotient of $\mathcal{G}$ by its constant central $U(1)$-subgroup. We will follow this notation too. Moreover, denote $B\mathcal{G}$ and $B\bar{\mathcal{G}}$ as the classifying spaces of $\mathcal{G}$ and $\bar{\mathcal{G}}$, respectively.

There are a couple of results of Atiyah and Bott [2] that will be very useful for us:

Theorem 2.2.3 (Atiyah and Bott [2, (2.7)])**.** $H^*(B\mathcal{G}, \mathbb{Z})$ *is torsion free and has Poincaré polynomial:*

$$P_t(B\mathcal{G}) = \frac{\big((1+t)(1+t^3)\big)^{2g}}{(1-t^2)^2(1-t^4)}.$$

Corollary 2.2.4 (Atiyah and Bott [2, (9.7)])**.** $H^*(B\bar{\mathcal{G}}, \mathbb{Z})$ *is also torsion free with Poincaré polynomial:*

$$P_t(B\bar{\mathcal{G}}) = (1-t^2)P_t(B\mathcal{G}) = \frac{\big((1+t)(1+t^3)\big)^{2g}}{(1-t^2)(1-t^4)}.$$

Let $\mathcal{M}^\infty := \lim_{k\to\infty} \mathcal{M}^k = \bigcup_{k=0}^\infty \mathcal{M}^k$ be the direct limit of the spaces $\big\{\mathcal{M}^k(r,d)\big\}_{k=0}^\infty$. Hausel and Thaddeus [21, (9.7)] prove that:

Theorem 2.2.5. $B\bar{\mathcal{G}} \cong \mathcal{M}^\infty = \lim\limits_{k\to\infty} \mathcal{M}^k$.

Proof. By the last corollary, $H^j(B\bar{\mathcal{G}}, \mathbb{Z}) \cong H^j(\mathcal{M}^\infty, \mathbb{Z}) \twoheadrightarrow H^j(\mathcal{M}^k, \mathbb{Z})$ must be surjective, since all the groups $H^j(B\bar{\mathcal{G}}, \mathbb{Z})$ are finitely generated free abelian groups. The result follows then from Corollary 2.2.2. ♠

We will make the following conjecture, which has not been proved before, since the general form of critical manifolds has not been described before.

Conjecture 2.2.6. *$H^n\big(\mathcal{M}^k(r,d)\big)$ is torsion free $\forall k \in \mathbb{N}$ and $\forall n \in \mathbb{N}$.*

A possible sketch of a proof would be the use of the result of Frankel [10, Corollary 1]:

$$F_\lambda^k \quad \text{is torsion free} \quad \forall \lambda \Leftrightarrow \mathcal{M}^k \quad \text{is torsion free},$$

but unfortunately, we have not proved that F_λ is torsion free for all λ for general rank $\mathrm{rk}(E) = r$. Nevertheless, the last result is certainly true for rank two and rank three k-Higgs bundles:

Theorem 2.2.7. *$H^n\big(\mathcal{M}^k(2,d)\big)$ and $H^n\big(\mathcal{M}^k(3,d)\big)$ are torsion free for all $k \in \mathbb{N}$ and for all $n \in \mathbb{N}$.*

Proof. 1. When $\mathrm{rk}(E) = 2$, the non-trivial critical submanifolds, or $(1,1)$-VHS, are of the form

$$F_{d_1}^k = \left\{ (E,\Phi^k) = (E_1 \oplus E_2, \begin{pmatrix} 0 & 0 \\ \varphi_{21}^k & 0 \end{pmatrix}) \middle| \begin{array}{cc} \deg(E_1) = d_1, & \deg(E_2) = d_2, \\ \mathrm{rk}(E_1) = 1, & \mathrm{rk}(E_2) = 1, \\ \multicolumn{2}{c}{\varphi_{21}^k : E_1 \to E_2 \otimes K \otimes L_p^{\otimes k}} \end{array} \right\}$$

and $F_{d_1}^k$ is isomorphic to the moduli space of σ_H-stable triples $\mathcal{N}_{\sigma_H}(1,1,\bar{d},d_1)$, where $\sigma_H = \deg(K \otimes L_p^{\otimes k}) = 2g-2+k$ and $\bar{d} = d_2 + 2g - 2 + k - d_1$, by the map:

$$(E_1 \otimes E_2, \Phi^k) \mapsto (E_2 \otimes K \otimes L_p^{\otimes k}, E_1, \varphi_{21}^k).$$

Furthermore, by Muñoz, Ortega, Vázquez-Gallo [32, Lemma 3.16.], $\mathcal{N}_{\sigma_H}(1,1,\bar{d},d_1)$, is isomorphic to the cartesian product $\mathcal{J}^{d_1}(X) \times \mathrm{Sym}^{\bar{d}-d_1}(X)$. Hence:

$$F_{d_1}^k \cong \mathcal{J}^{d_1}(X) \times \mathrm{Sym}^{\bar{d}-d_1}(X)$$

which, by Macdonald [28, (12.3)], is indeed torsion free.

2. When $\mathrm{rk}(E) = 3$, there are three kinds of non-trivial critical submanifolds:

2.1. $(1,2)$-VHS of the form

$$F_{d_1}^k = \left\{ (E,\Phi^k) = (E_1 \oplus E_2, \begin{pmatrix} 0 & 0 \\ \varphi_{21}^k & 0 \end{pmatrix}) \middle| \begin{array}{c} \deg(E_1) = d_1, \quad \deg(E_2) = d_2, \\ \mathrm{rk}(E_1) = 1, \qquad \mathrm{rk}(E_2) = 2, \\ \varphi_{21}^k : E_1 \to E_2 \otimes K \otimes L_p^{\otimes k} \end{array} \right\}.$$

In this case, there are isomorphisms between the $(1,2)$-VHS and the moduli spaces of triples $F_{d_1}^k \cong \mathcal{N}_{\sigma_H(k)}(2,1,\tilde{d}_1,\tilde{d}_2)$, where $\tilde{d}_1 = d_2 + 2(2g-2+k)$ and $\tilde{d}_2 = d_1$, and where the isomorphism is giving by a map similar to the mentioned above. By Muñoz, Ortega, Vázquez-Gallo [32], the flip loci $S_{\sigma_c}^+$ and $S_{\sigma_c}^-$ are free of torsion for all $\sigma_c \in I$, and so is $\mathcal{N}_{\sigma_H(k)}(2,1,\tilde{d}_1,\tilde{d}_2)$. Hence, $F_{d_1}^k$ is torsion free. The fact that $\mathcal{N}_{\sigma_H(k)}(2,1,\tilde{d}_1,\tilde{d}_2)$ is torsion free since the flip loci are, follows from the next lemma:

Lemma 2.2.8. *Let M be a complex manifold, and let $\Sigma \subset M$ be a complex submanifold. Let $\tilde{M}$ be the blow-up of M along Σ. Let $E = \mathbb{P}(N_{\Sigma/M})$ be the projectivized normal bundle of Σ in M, sometimes called exceptional divisor. Then*

$$H^*(\tilde{M},\mathbb{Z}) \cong H^*(M,\mathbb{Z}) \oplus H^{*+2}(\Sigma,\mathbb{Z}) \oplus \dots H^{*+2n-2}(\Sigma,\mathbb{Z})$$

where n is the rank of $N_{\Sigma/M}$.

Proof. (Lemma 2.2.8)
It follows from the fact that the additive cohomology of the blow-up $H^*(\tilde{M},\mathbb{Z})$, can be expressed as:

$$H^*(\tilde{M}) \cong \pi^* H^*(M) \oplus H^*(E)/\pi^* H^*(\Sigma)$$

(see for instance Griffiths and Harris [15, Chapter 4.,Section 6.]), and the fact that $H^*(E)$ is a free module over $H^*(\Sigma)$ via the injective map $\pi^*\colon H^*(\Sigma) \to H^*(E)$ with basis

$$1, c, \dots, c^{n-1},$$

where $c \in H^2(E)$ is the first Chern class of the tautological line bundle along the fibres of the projective bundle $E \to \Sigma$ (see the general version at Husemoller [25, Chapter 17.,Theorem 2.5.]). ♠

2.2. $(2,1)$-VHS of the form

$$F_{d_2}^k = \left\{ (E, \Phi^k) = (E_2 \oplus E_1, \begin{pmatrix} 0 & 0 \\ \varphi_{21}^k & 0 \end{pmatrix}) \middle| \begin{array}{cc} \deg(E_2) = d_2, & \deg(E_1) = d_1, \\ \mathrm{rk}(E_2) = 2, & \mathrm{rk}(E_1) = 1, \\ \multicolumn{2}{c}{\varphi_{21}^k : E_2 \to E_1 \otimes K \otimes L_p^{\otimes k}} \end{array} \right\}.$$

By symmetry, similar results can be obtained using the isomorphisms between the $(2,1)$-VHS and the moduli spaces of triples: $F_{d_2}^k \cong \mathcal{N}_{\sigma_H(k)}(1, 2, \tilde{d}_1, \tilde{d}_2)$, and the dual isomorphisms

$$\mathcal{N}_{\sigma_H(k)}(2, 1, \tilde{d}_1, \tilde{d}_2) \cong \mathcal{N}_{\sigma_H(k)}(1, 2, -\tilde{d}_2, -\tilde{d}_1)$$

between moduli spaces of triples.

2.3. $(1,1,1)$-VHS of the form

$$F_{d_1 d_2 d_3}^k = \left\{ (E, \Phi^k) = (E_1 \oplus E_2 \oplus E_3, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21}^k & 0 & 0 \\ 0 & \varphi_{32}^k & 0 \end{pmatrix}) \middle| \begin{array}{c} \deg(E_j) = d_j, \\ rk(E_j) = 1, \\ \varphi_{ij} : E_j \to E_i \otimes K \end{array} \right\}.$$

Finally, we know that

$$F_{d_1 d_2 d_3}^k \xrightarrow{\cong} \mathrm{Sym}^{m_1}(X) \times \mathrm{Sym}^{m_2}(X) \times \mathcal{J}^{d_3}(X)$$

$$(E, \Phi^k) \mapsto (\mathrm{div}(\varphi_{21}^k), \mathrm{div}(\varphi_{32}^k), E_3),$$

where $m_i = d_{i+1} - d_i + \sigma_H$, and so, by Macdonald [28, (12.3)] there is nothing to worry about torsion.

♠

Using all the facts above, if the Conjecture 2.2.6 is true, Hausel and Thaddeus [21, (10.1)] conclude that:

Corollary 2.2.9. *If $H^*(\mathcal{M}^k, \mathbb{Z})$ is torsion free, then $H^*(B\bar{\mathcal{G}}, \mathbb{Z}) = \varprojlim H^*(\mathcal{M}^k, \mathbb{Z})$.*

And so, we may conclude also that:

Theorem 2.2.10. *If $H^n\big(\mathcal{M}^k(r,d)\big)$ is torsion free $\forall k \in \mathbb{N}$ and $\forall n \in \mathbb{N}$, then $\forall j \quad \exists k_0 = k_0(j)$ such that*

$$i_k^* : H^j(\mathcal{M}^{k+1}, \mathbb{Z}) \xrightarrow{\cong} H^j(\mathcal{M}^k, \mathbb{Z}) \quad \forall k \geqslant k_0.$$

By the Universal Coefficient Theorem for Cohomology (see for instance Hatcher [17, Theorem 3.2. and Corollary 3.3.]), we would get

Lemma 2.2.11. *If $H^n\big(\mathcal{M}^k(r,d)\big)$ is torsion free $\forall k \in \mathbb{N}$ and $\forall n \in \mathbb{N}$, then for all n there exists $k_0 = k_0(n)$ such that $H_j(\mathcal{M}^\infty, \mathcal{M}^k; \mathbb{Z}) = 0$ for all $k \geqslant k_0$ and for all $j \leqslant n$. In particular, this statement holds true for rank 2 and rank 3.*

Proof. The embedding $i_k : \mathcal{M}^k(r,d) \to \mathcal{M}^{k+1}(r,d)$ is injective, and by Theorem 2.2.10, we know that $i_k^* : H^j(\mathcal{M}^k, \mathbb{Z}) \leftarrow H^j(\mathcal{M}^{k+1}, \mathbb{Z})$ is surjective $\forall k$. Hence, by the Universal Coefficient Theorem, we get that the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \operatorname{Ext}\big(H_{j-1}(\mathcal{M}^k), \mathbb{Z}\big) & \longrightarrow & H^j(\mathcal{M}^k, \mathbb{Z}) & \longrightarrow & \operatorname{Hom}\big(H_j(\mathcal{M}^k), \mathbb{Z}\big) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle (i_{k*})^*} & & \uparrow{\scriptstyle i_k^*} & & \uparrow{\scriptstyle (i_{k*})^*} & & \\
0 & \longrightarrow & \operatorname{Ext}\big(H_{j-1}(\mathcal{M}^{k+1}), \mathbb{Z}\big) & \longrightarrow & H^j(\mathcal{M}^{k+1}, \mathbb{Z}) & \longrightarrow & \operatorname{Hom}\big(H_j(\mathcal{M}^{k+1}), \mathbb{Z}\big) & \longrightarrow & 0
\end{array} \tag{2.3}$$

commutes. So, if the Conjecture 2.2.6 is true, then $\forall n \quad \exists k_0 = k_0(n)$ such that

$$H_j\big(\mathcal{M}^k(r,d), \mathbb{Z}\big) \xrightarrow{\cong} H_j\big(\mathcal{M}^{k+1}(r,d), \mathbb{Z}\big) \xrightarrow{\cong} H_j\big(\mathcal{M}^\infty(r,d), \mathbb{Z}\big)$$

$\forall k \geqslant k_0$ and $\forall j \leqslant n \Rightarrow H_j(\mathcal{M}^\infty, \mathcal{M}^k; \mathbb{Z}) = 0 \quad \forall k \geqslant k_0 \quad$ and $\quad \forall j \leqslant n.$ ♠

Proposition 2.2.12.

$$\begin{array}{ccc} \pi_1(\mathcal{M}^k) & \xrightarrow{\cong} & \pi_1(\mathcal{M}^{k+1}) \\ \uparrow \cong & & \uparrow \cong \\ \pi_1(\mathcal{N}) & \xrightarrow[=]{} & \pi_1(\mathcal{N}) \end{array} \tag{2.4}$$

Proof. It is an immediate consequence of the result proved by Bradlow, García-Prada and Gothen [7, Proposition 3.2.] using Morse theory. ♠

Proposition 2.2.13.

$$\pi_1(\mathcal{M}^k) \xrightarrow{\cong} \pi_1(\mathcal{M}^{\infty}).$$

Proof. Using the generalization of Van Kampen's Theorem presented by Fulton [11], and using the fact that $\mathcal{M}^k \hookrightarrow \mathcal{M}^{k+1}$ are embeddings of *Deformation Neighborhood Retracts* (DNR), *i.e.* every $\mathcal{M}^k(r,d)$ is the image of a map defined on some open neighborhood of itself and homotopic to the identity (see Hausel and Thaddeus [21, (9.1)]), we can conlcude that $\pi_1\big(\lim\limits_{k\to\infty} \mathcal{M}^k\big) = \lim\limits_{k\to\infty} \pi_1\big(\mathcal{M}^k\big)$. ♠

We will need the following version of Hurewicz Theorem, presented by Hatcher [17, Theorem 4.37.] (see also James [26]). Hatcher first mentions that, in the relative case when (X, A) is an $(n-1)$-connected pair of path-connected spaces, the kernel of the Hurewicz map

$$h : \pi_n(X, A) \to H_n(X, A; \mathbb{Z})$$

contains the elements of the form $[\gamma][f] - [f]$ for $[\gamma] \in \pi_1(A)$. Hatcher defines $\pi_1'(X, A)$ to be the quotient group of $\pi_n(X, A)$ obtained by factoring out the subgroup generated by the elements of the form $[\gamma][f] - [f]$, or the normal subgroup generated by such elements in the case $n = 2$ when $\pi_2(X, A)$ may not be abelian, then h induces a homomorphism $h' : \pi_n'(X, A) \to H_n(X, A; \mathbb{Z})$. The general form of Hurewicz Theorem presented by Hatcher deals with this homomorphism:

Theorem 2.2.14. *If (X, A) is an $(n-1)$-connected pair of path-connected spaces, with $n \geqslant 2$ and $A \neq \emptyset$, then $h' : \pi_n'(X, A) \to H_n(X, A; \mathbb{Z})$ is an isomorphism and $H_j(X, A; \mathbb{Z}) = 0$ for $j \leqslant n - 1$.*

One would expect the action of $\pi_1(\mathcal{M}^k)$ on the higher homotopy groups to be trivial but we did not manage to prove it. Therefore, the following group of results is conditional on the following conjecture being true:

Conjecture 2.2.15. *$\pi_1(\mathcal{M}^k)$ acts trivially on $\pi_n(\mathcal{M}^\infty, \mathcal{M}^k) \quad \forall n \in \mathbb{N}, \quad \forall k \in \mathbb{N}$.*

Remark 2.2.16. If we restrict our attention to $\mathcal{M}^k(r, \Lambda)$, the moduli space of Higgs bundles with fixed determinant Λ, then, the fundamental group will be trivial, since $\mathcal{M}^k(r, \Lambda)$ is simply connected. So, the conjecture will be trivially true for $\mathcal{M}^k(r, \Lambda)$.

Now, supposing that Conjecture 2.2.6 and Conjecture 2.2.15 mentioned above are true, we could get that:

Proposition 2.2.17. *If $H^n\big(\mathcal{M}^k(r,d), \mathbb{Z}\big)$ is torsion free $\forall k \in \mathbb{N}$ and $\forall n \in \mathbb{N}$, and if $\pi_1(\mathcal{M}^k)$ acts trivially on $\pi_n(\mathcal{M}^\infty, \mathcal{M}^k)$ for all $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$, then for all n exists $k_0 = k_0(n)$ such that $\pi_j(\mathcal{M}^\infty, \mathcal{M}^k) = 0$ for all $k \geqslant k_0$ and for all $j \leqslant n$.*

Proof. Assume that Conjecture 2.2.6 and Conjecture 2.2.15 are true. We proceed by induction on $m \in \mathbb{N}$ for $2 \leqslant m \leqslant n$. The first induction step is trivial because

$$\pi_1(\mathcal{N}) = \pi_1(\mathcal{M}) = \pi_1(\mathcal{M}^k) = \pi_1(\mathcal{M}^\infty)$$

by Proposition 2.2.12. For $m = 2$ we need $\pi_2(\mathcal{M}^\infty, \mathcal{M}^k)$ to be abelian. Consider the sequence

$$\pi_2(\mathcal{M}^\infty) \to \pi_2(\mathcal{M}^\infty, \mathcal{M}^k) \to \pi_1(\mathcal{M}^k) \to \pi_1(\mathcal{M}^\infty) \to \pi_1(\mathcal{M}^\infty, \mathcal{M}^k) \to 0$$

where $\pi_2(\mathcal{M}^\infty) \twoheadrightarrow \pi_2(\mathcal{M}^\infty, \mathcal{M}^k)$ is surjective, $\pi_1(\mathcal{M}^k) \xrightarrow{\cong} \pi_1(\mathcal{M}^\infty)$, and hence $\pi_1(\mathcal{M}^\infty, \mathcal{M}^k) = 0$. So, $\pi_2(\mathcal{M}^\infty, \mathcal{M}^k)$ is a quotient of the abelian group $\pi_2(\mathcal{M}^\infty)$, and so it is also abelian.
Finally, suppose that the statement is true for all $j \leqslant m-1$ for $2 \leqslant m \leqslant n$. So, $(\mathcal{M}^\infty, \mathcal{M}^k)$ is $(m-1)$-connected, *i.e.*

$$\pi_j(\mathcal{M}^\infty, \mathcal{M}^k) = 0 \quad \forall j \leqslant m-1.$$

For $m \geqslant 2$, by Hurewicz Theorem 2.2.14,

$$h' : \pi'_m(\mathcal{M}^\infty, \mathcal{M}^k) \xrightarrow{\cong} H_m(\mathcal{M}^\infty, \mathcal{M}^k; \mathbb{Z})$$

is an isomorphism. If $H_n\big(\mathcal{M}^k(r,d),\mathbb{Z}\big)$ is torsion free $\forall k \in \mathbb{N}$ and $\forall n \in \mathbb{N}$, by Lemma 2.2.11, $H_m(\mathcal{M}^\infty, \mathcal{M}^k; \mathbb{Z}) = 0$. Hence, if $\pi_1(\mathcal{M}^k)$ acts trivially on $\pi_n(\mathcal{M}^\infty, \mathcal{M}^k)$ for all $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$, then

$$\pi_m(\mathcal{M}^\infty, \mathcal{M}^k) = \pi'_m(\mathcal{M}^\infty, \mathcal{M}^k) = 0$$

finishing the induction process. ♠

Corollary 2.2.18. *If $H^n\big(\mathcal{M}^k(r,d),\mathbb{Z}\big)$ is torsion free $\forall k \in \mathbb{N}$ and $\forall n \in \mathbb{N}$, and if $\pi_1(\mathcal{M}^k)$ acts trivially on $\pi_n(\mathcal{M}^\infty, \mathcal{M}^k)$ for all $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$, then for all n exists $k_0 = k_0(n)$ such that*

$$\pi_j(\mathcal{M}^k) \xrightarrow{\cong} \pi_j(\mathcal{M}^\infty)$$

for all $k \geqslant k_0$ and for all $j \leqslant n-1$.

Chapter 3

Moduli Space of Triples

Motivated by the result of Hausel for rank two, the derive result for general rank $\mathrm{rk}(E) = r$, finding the value of the bounds for j and k, we investigate when the embedding

$$i_k : \mathcal{M}^k(r,d) \to \mathcal{M}^{k+1}(r,d)$$

$$\left[(E,\Phi^k)\right] \longmapsto \left[(E,\Phi^k \otimes s_p)\right],$$

is well defined, where $s_p \in H^0(X, L_p),\ s_p \neq 0$ is a non-zero fixed section of L_p. We show that i_k induces embeddings of the form

$$F_\lambda^k \xrightarrow{\ i_k\ } F_\lambda^{k+1}\ \forall\lambda,$$

and that those embeddings induce isomorphisms in cohomology:

$$H^j(F_\lambda^{k+1},\mathbb{Z}) \xrightarrow{\ \cong\ } H^j(F_\lambda^k,\mathbb{Z})$$

for certain values of j and k. It turns out that is difficult to find the range for j for which the isomorphism holds. Hence, it is not obvious how to apply this approach to $\mathcal{M}^k(3,d)$.

In the particular case of the moduli space of rank three k-Higgs bundles, if we restrict the embedding to the critical manifolds of type $(1,2)$:

$$\begin{array}{ccc} F^k_{d_1} & \xrightarrow{i_k} & F^{k+1}_{d_1} \\ (E_1\oplus E_2, \begin{pmatrix} 0 & 0 \\ \varphi^k_{21} & 0 \end{pmatrix}) & \longmapsto & (E_1\oplus E_2, \begin{pmatrix} 0 & 0 \\ \varphi^k_{21}\otimes s_p & 0 \end{pmatrix}) \end{array} \tag{3.1}$$

(see for instance Gothen [14] or Bento [3]) then, the isomorphisms

$$\begin{array}{ccc} F^k_{d_1} & \xrightarrow{\cong} & \mathcal{N}_{\sigma^k_H}(2,1,\tilde{d}_1,\tilde{d}_2) \\ (E_1\oplus E_2, \begin{pmatrix} 0 & 0 \\ \varphi^k_{21} & 0 \end{pmatrix}) & \longmapsto & (V_1,V_2,\varphi) \end{array}$$

between $(1,2)$-VHS and the moduli space of triples, where we denote by $V_1 = E_2\otimes K\otimes L_p^{\otimes k}$, by $V_2 = E_1$, by $\varphi = \varphi^k_{21}$ and $\sigma^k_H = \deg(K\otimes L_p^{\otimes k}) = 2g-2+k$, induces another embedding:

$$i_k : \mathcal{N}_{\sigma^k_H}(2,1,\tilde{d}_1,\tilde{d}_2) \to \mathcal{N}_{\sigma^{k+1}_H}(2,1,\tilde{d}_1+2,\tilde{d}_2)$$

$$(V_1,V_2,\varphi)\mapsto (V_1\otimes L_p, V_2, \varphi\otimes s_p)$$

where $\tilde{d}_1 = \deg(V_1) = d_2+2\sigma^k_H$ and $\tilde{d}_2 = \deg(V_2) = d_1$, and so, it induces embeddings of the kind:

$$i_k : \mathcal{N}_{\sigma_c^-(k)}(2,1,\tilde{d}_1,\tilde{d}_2) \to \mathcal{N}_{\sigma_c^-(k+1)}(2,1,\tilde{d}_1+2,\tilde{d}_2)$$

and

$$i_k : \mathcal{N}_{\sigma_c^+(k)}(2,1,\tilde{d}_1,\tilde{d}_2) \to \mathcal{N}_{\sigma_c^+(k+1)}(2,1,\tilde{d}_1+2,\tilde{d}_2)$$

for $\sigma_m < \sigma_c(k) < \sigma_M$.

3.1 σ-Stability

Our first result has to do with σ-stability:

Lemma 3.1.1. *A triple T is σ-stable $\Leftrightarrow i_k(T)$ is $(\sigma+1)$-stable.*

Proof. Recall that $T = (V_1,V_2,\varphi)$ is σ-stable if and only if $\mu_\sigma(T') < \mu_\sigma(T)$ for any T'

proper subtriple of T.

Denote by $S = i_k(T) = (V_1 \otimes L_p, V_2, \varphi \otimes s_p)$. Is easy to check that $\mu_{\sigma+1}(S) = \mu_\sigma(T) + 1$.

Without lost of generality, we may suppose that any S' proper subtriple of S is of the form $S' = i_k(T')$ for some T' subtriple of T, or equivalently:

$$S' = (V_1' \otimes L_p, V_2', \varphi \otimes s_p)$$

and that there are injective sheaf homomorphisms $V_1' \to V_1$ and $V_2' \to V_2$. This statement is justified since the following diagram commutes:

$$\begin{array}{cccc}
S: & V_2 & \xrightarrow{\varphi\otimes s_p} & V_1 \otimes L_p \\
\downarrow & \cup & & \cup \\
S': & B & \xrightarrow{(\varphi\otimes s_p)|_B} & A \\
\downarrow & \downarrow & & \downarrow \\
T': & B & \xrightarrow{(\varphi\otimes s_p)|_B \otimes s_p^{-1}} & A \otimes L_{(-p)} \\
\downarrow & \cap & & \cap \\
T: & V_2 & \xrightarrow{\varphi} & V_1 \otimes L_p
\end{array}$$

i.e. there is a $(1-1)$-correspodence between the proper subtriples $S' \subset S$ and the proper subtriples $T' \subset T$. Taking $A = V_1' \otimes L_p,\ B = V_2'$ and $T' = (V_1', V_2', \varphi)$, we can easily see that $\mu_{\sigma+1}(S') = \mu_\sigma(T') + 1$ and hence:

$$\mu_{\sigma+1}(S') < \mu_{\sigma+1}(S) \Leftrightarrow \mu_\sigma(T') + 1 < \mu_\sigma(T) + 1 \Leftrightarrow \mu_\sigma(T') < \mu_\sigma(T).$$

Therefore, T is σ-stable $\Leftrightarrow S = i_k(T)$ is $(\sigma + 1)$-stable. ♠

Corollary 3.1.2. *The embedding*

$$i_k : \mathcal{N}_{\sigma(k)}(2, 1, \tilde{d}_1, \tilde{d}_2) \to \mathcal{N}_{\sigma(k+1)}(2, 1, \tilde{d}_1 + 2, \tilde{d}_2)$$

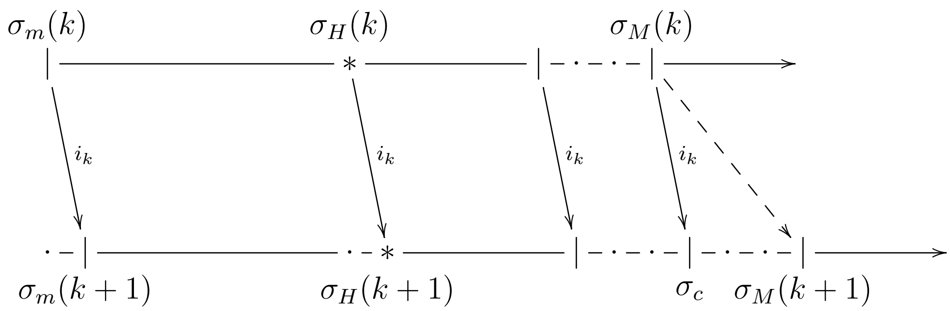
is well defined for any $\sigma(k)$ such that $\sigma_m < \sigma(k) < \sigma_M$. In particular, the embedding i_k restricted to $F^k_{d_1}$ (see (3.1)) is well defined and we have a commutative diagram of the

form:

$$
\begin{array}{ccc}
(E_1 \oplus E_2, \Phi^k) & \longmapsto & (E_1 \oplus E_2, \Phi^k \otimes s_p) \\
F_{d_1}^k & \xrightarrow{i_k} & F_{d_1}^{k+1} \\
\cong \updownarrow & & \updownarrow \cong \\
\mathcal{N}_{\sigma_H(k)} & \xrightarrow[i_k]{} & \mathcal{N}_{\sigma_H(k+1)} \\
(\tilde{E}_1, \tilde{E}_2, \varphi_{21}^k) & \longmapsto & (\tilde{E}_1, \tilde{E}_2, \varphi_{21}^k \otimes s_p),
\end{array}
$$

where $\tilde{E}_1 = E_2 \otimes K \otimes L_p^{\otimes k}$, $\tilde{E}_2 = E_1$, *and* $\varphi_{21}^k : E_1 \to E_2 \otimes K \otimes L_p^{\otimes k}$.

These results give us an interesting and important correspondence between the σ-stability values of moduli spaces of triples:



where $\sigma_m(k) = \tilde{\mu}_1 - \tilde{\mu}_2$, $\sigma_M(k) = 4(\tilde{\mu}_1 - \tilde{\mu}_2)$, $\sigma_H(k) = \deg(K \otimes L_p^{\otimes k}) = 2g - 2 + k$, and the correspodence gives us $\sigma_m(k+1) = \sigma_m(k) + 1$, $\sigma_c = \sigma_M(k) + 1$, $\sigma_M(k+1) = \sigma_M(k) + 3$, and $\sigma_H(k+1) = \sigma_H(k) + 1$.

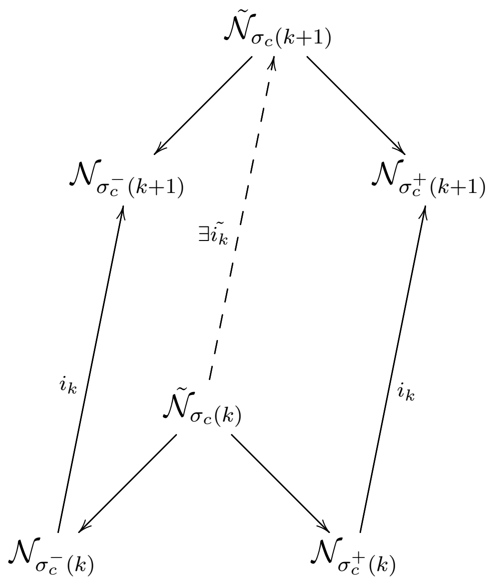
3.2 Blow-UP and The Roof Theorem

Recall that the blow-up of $\mathcal{N}_{\sigma_c^-(k)} = \mathcal{N}_{\sigma_c^-(k)}(2, 1, \tilde{d}_1, \tilde{d}_2)$ along the flip locus $S_{\sigma_c^-(k)}$, $\tilde{\mathcal{N}}_{\sigma_c^-(k)}$ is isomorphic to $\tilde{\mathcal{N}}_{\sigma_c^+(k)}$, the blow-up of $\mathcal{N}_{\sigma_c^+(k)} = \mathcal{N}_{\sigma_c^+(k)}(2, 1, \tilde{d}_1, \tilde{d}_2)$ along the flip locus $S_{\sigma_c^+(k)}$. From now on, we will denote just $\tilde{\mathcal{N}}_{\sigma_c(k)}$ whenever no confusion is likely to arise.

Proposition 3.2.1. *There exists an embedding at the blow-up level*

$$\tilde{i_k} : \tilde{\mathcal{N}}_{\sigma_c(k)} \hookrightarrow \tilde{\mathcal{N}}_{\sigma_c(k+1)}$$

such that the following diagram commutes:



where $\tilde{\mathcal{N}}_{\sigma_c(k)}$ is the blow-up of $\mathcal{N}_{\sigma_c^-(k)} = \mathcal{N}_{\sigma_c^-(k)}(2,1,\tilde{d}_1,\tilde{d}_2)$ along the flip locus $S_{\sigma_c^-(k)}$ and, at the same time, represents the blow-up of $\mathcal{N}_{\sigma_c^+(k)} = \mathcal{N}_{\sigma_c^+(k)}(2,1,\tilde{d}_1,\tilde{d}_2)$ along the flip locus $S_{\sigma_c^+(k)}$.

Remark 3.2.2. The construction of the blow-up may be found in the book of Griffiths and Harris [15].

Proof. Recall that T is σ-stable if and only if $i_k(T)$ is $(\sigma+1)$-stable. Furthermore, by Muñoz, Ortega and Vásquez-Gallo [32], note that any triple $T \in S_{\sigma_c^+(k)} \subset \mathcal{N}_{\sigma_c^+(k)}(2,1,\tilde{d}_1,\tilde{d}_2)$ is a non-trivial extension of a subtriple $T' \subset T$ of the form $T' = (V_1', V_2', \varphi') = (M, 0, \varphi')$ by a quotient triple of the form $T'' = (V_1'', V_2'', \varphi'') = (L, V_2, \varphi'')$, where M is a line bundle of degree $\deg(M) = d_M$ and L is a line bundle of degree $\deg(L) = d_L = \tilde{d}_1 - d_M$. Besides, also by Muñoz, Ortega and Vásquez-Gallo [32], the non-trivial critical values $\sigma_c \neq \sigma_m$ for $\sigma_m < \sigma < \sigma_M$ are of the form $\sigma_c = 3d_M - \tilde{d}_1 - \tilde{d}_2$. Then, we can visualize the embedding $i_k : T \to i_k(T)$ as follows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & T' & \longrightarrow & T & \longrightarrow & T'' & \longrightarrow & 0 \\
 & & & & & & & & \\
0 & \longrightarrow & 0 & \longrightarrow & V_2 & \xrightarrow{=} & V_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi''} & & \\
0 & \longrightarrow & M & \longrightarrow & V_1 & \longrightarrow & L & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle i_k} & & & & \\
0 & \longrightarrow & 0 & \longrightarrow & V_2 & \xrightarrow{=} & V_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi'\otimes s_p} & & \downarrow{\scriptstyle \varphi\otimes s_p} & & \downarrow{\scriptstyle \varphi''\otimes s_p} & & \\
0 & \longrightarrow & M \otimes L_p & \longrightarrow & V_1 \otimes L_p & \longrightarrow & L \otimes L_p & \longrightarrow & 0
\end{array}
$$

where $\deg(V_1 \otimes L_p) = \tilde{d}_1 + 2$ and $\deg(M \otimes L_p) = d_M + 1$, and so $L \otimes L_p$ verifies that $\deg(L \otimes L_p) = \deg(V_1 \otimes L_p) - \deg(M \otimes L_p)$:

$$\deg(L\otimes L_p) = d_L+1 = \tilde{d}_1 - d_M + 1 = (\tilde{d}_1+2)-(d_M+1) = \deg(V_1\otimes L_p)-\deg(M\otimes L_p).$$

Hence, $\sigma_c(k+1)$ verifies that $\sigma_c(k+1) = \sigma_c(k) + 1$:

$$\sigma_c(k+1) = 3\deg(M \otimes L_p) - \deg(V_1 \otimes L_p) - \deg(V_2) =$$

$$3d_M + 3 - \tilde{d}_1 - 2 - \tilde{d}_2 = (3d_M - \tilde{d}_1 - \tilde{d}_2) + 1 = \sigma_c(k) + 1$$

and where $i_k(T') = (M \otimes L_p, 0, \varphi' \otimes s_p)$ is the maximal $\sigma_c^+(k+1)$-destabilizing subtriple of $i_k(T)$.

Similarly, also by Muñoz, Ortega and Vásquez-Gallo [32], any triple $T \in S_{\sigma_c^-(k)} \subset \mathcal{N}_{\sigma_c^-(k)}(2, 1, \tilde{d}_1, \tilde{d}_2)$ is a non-trivial extension of a subtriple $T' \subset T$ of the form $T' = (V_1', V_2', \varphi') = (L, V_2, \varphi')$ by a quotient triple of the form $T'' = (V_1'', V_2'', \varphi'') = (M, 0, \varphi'')$, where M is a line bundle of degree $\deg(M) = d_M$ and L is a line bundle of degree

$\deg(L) = d_L = \tilde{d}_1 - d_M$. Then, the embedding $i_k : T \to i_k(T)$ looks like:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T' & \longrightarrow & T & \longrightarrow & T'' & \longrightarrow & 0 \\
\\
0 & \longrightarrow & V_2 & \xrightarrow{=} & V_2 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi''} & & \\
0 & \longrightarrow & L & \longrightarrow & V_1 & \longrightarrow & M & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle i_k} & & & & \\
0 & \longrightarrow & V_2 & \xrightarrow{=} & V_2 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi'\otimes s_p} & & \downarrow{\scriptstyle \varphi\otimes s_p} & & \downarrow{\scriptstyle \varphi''\otimes s_p} & & \\
0 & \longrightarrow & L \otimes L_p & \longrightarrow & V_1 \otimes L_p & \longrightarrow & M \otimes L_p & \longrightarrow & 0
\end{array}$$

where $i_k(T') = (L, V_2, \varphi')$ is the maximal $\sigma_c^+(k+1)$-destabilizing subtriple of $i_k(T)$.

Hence, i_k restricts to the flip loci $S_{\sigma_c^+(k)}$ and $S_{\sigma_c^-(k)}$. Recall that, by definition, the blow-up of $\mathcal{N}_{\sigma_c^+(k)}$ along the flip locus $S_{\sigma_c^+(k)}$, is the space $\tilde{\mathcal{N}}_{\sigma_c(k)}$ together with the projection

$$\pi : \tilde{\mathcal{N}}_{\sigma_c(k)} \to \mathcal{N}_{\sigma_c^+(k)}$$

where π restricted to $\mathcal{N}_{\sigma_c^+(k)} - S_{\sigma_c^+(k)}$ is an isomorphism and the *exceptional divisor* $\mathcal{E}^+ = \pi^{-1}(S_{\sigma_c^+(k)}) \subset \tilde{\mathcal{N}}_{\sigma_c(k)}$ is a fiber bundle over $S_{\sigma_c^+(k)}$ with fiber $\mathbb{P}^{n-k-1}$, where $n = \dim(\mathcal{N}_{\sigma_c^+(k)})$ and $k = \dim(S_{\sigma_c^+(k)})$. So, the embedding can be extended to $\mathcal{E}^+$ in a natural way. Same argument remains valid when we consider $\tilde{\mathcal{N}}_{\sigma_c(k)}$ as the blow-up of $\mathcal{N}_{\sigma_c^-(k)}$ along the flip locus $S_{\sigma_c^-(k)}$ with exceptional divisor $\mathcal{E}^- = \pi^{-1}(S_{\sigma_c^-(k)}) \subset \tilde{\mathcal{N}}_{\sigma_c(k)}$. Therefore, the embedding can be extended to the whole $\tilde{\mathcal{N}}_{\sigma_c(k)}$. ♠

3.3 Cohomology of the $(1,2)$-VHS

We need to prove that the embedding $i_k : F^k_{d_1} \hookrightarrow F^{k+1}_{d_1}$ induces an isomorphism in cohomology:

$$H^j(F^{k+1}_{d_1}, \mathbb{Z}) \xrightarrow{\cong} H^j(F^k_{d_1}, \mathbb{Z})$$

for certain j, or equivalently:

$$H^j(\mathcal{N}^{k+1}_{\sigma_H}, \mathbb{Z}) \xrightarrow{\cong} H^j(\mathcal{N}^k_{\sigma_H}, \mathbb{Z}),$$

where we denote $\mathcal{N}^k_{\sigma_H} = \mathcal{N}_{\sigma_H(k)}(2,1,\tilde{d}_1,\tilde{d}_2)$. Nevertheless, what we get so far, is that

$$H^j(\mathcal{N}^{k+1}_{\sigma_c}, \mathbb{Z}) \xrightarrow{\cong} H^j(\mathcal{N}^k_{\sigma_c}, \mathbb{Z})$$

for all $\sigma_c = \sigma_c(k)$ critical such that $\sigma_m(k) < \sigma_c(k) < \sigma_M(k)$, and for all $j \leqslant \tilde{n}(k)$, where the bound $\tilde{n}(k)$ is known. We first analize the embedding restricted to the flip loci: $i_k : S_{\sigma_c^-(k)} \hookrightarrow S_{\sigma_c^-(k+1)}$ and $i_k : S_{\sigma_c^+(k)} \hookrightarrow S_{\sigma_c^+(k+1)}$. For simplicity, we will denote from now on $S^k_- = S_{\sigma_c^-(k)}$ and $S^k_+ = S_{\sigma_c^+(k)}$ whenever no confusion is likely to arise about the critical value.

Theorem 3.3.1.

$$i_k^* : H^j(S^{k+1}_-, \mathbb{Z}) \xrightarrow{\cong} H^j(S^k_-, \mathbb{Z})$$

for all $j \leqslant \tilde{d}_1 - d_M - \tilde{d}_2 - 1 = d_2 - d_1 + 2\sigma_H(k) - d_M$*, where* $d_j = \deg(E_j)$, $\tilde{d}_j = \deg(\tilde{E}_j)$, $d_M = \deg(M)$*, and* $\sigma_H(k) = \deg(K \otimes L_p^{\otimes k}) = 2g - 2 + k$.

Proof. Recall that, according to Muñoz, Ortega, Vázquez-Gallo [32, Theorem 4.8.], $S^k_- = \mathbb{P}(\mathcal{V})$ is the projectivization of a bundle $\mathcal{V} \to \mathcal{N}'_c \times \mathcal{N}''_c$ of rank $\mathrm{rk}(\mathcal{V}) = -\chi(T'', T')$, where $\mathcal{N}'_c = \mathcal{N}_c(1,1,\tilde{d}_1 - d_M, \tilde{d}_2) \cong \mathcal{J}^{\tilde{d}_2} \times \mathrm{Sym}^{\tilde{d}_1 - d_M - \tilde{d}_2}(X)$ and $\mathcal{N}''_c = \mathcal{N}_c(1,0,d_M,0) \cong \mathcal{J}^{d_M}(X)$, and where any triple $T = (V_1, V_2, \varphi) \in S^k_- \subset \mathcal{N}_{\sigma_c^-(k)}(2,1,\tilde{d}_1,\tilde{d}_2)$ is a non-trivial extension of a subtriple $T' \subset T$ of the form $T' = (V_1', V_2', \varphi') = (L, V_2, \varphi')$ by a quotient triple of the form $T'' = (V_1'', V_2'', \varphi'') = (M, 0, \varphi'')$, where M is a line bundle of degree $\deg(M) = d_M$ and L is a line bundle of degree $\deg(L) =$

$d_L = \tilde{d}_1 - d_M$. Then, the embedding $i_k : T \to i_k(T)$ restricts to:

$$\begin{array}{ccc}
(V_1', V_2', \varphi') & \longmapsto & (V_1' \otimes L_p, V_2', \varphi' \otimes s_p) \\
\mathcal{N}'_{\sigma_c} & \xrightarrow{\ i_k\ } & \mathcal{N}'_{\sigma_c+1} \\
\cong \updownarrow & & \updownarrow \cong \\
\mathcal{J}^{\tilde{d}_2} \times \mathrm{Sym}^{\tilde{d}_1 - d_M - \tilde{d}_2}(X) & \xrightarrow[\ i_k\ ]{} & \mathcal{J}^{\tilde{d}_2} \times \mathrm{Sym}^{\tilde{d}_1 - d_M - \tilde{d}_2 + 1}(X) \\
\big([V_2'], \mathrm{div}(\varphi')\big) & \longmapsto & \big([V_2'], \mathrm{div}(\varphi' \otimes s_p)\big)
\end{array}$$

because $\sigma_c(k+1) = \sigma_c(k) + 1$, and $d_M(k+1) = d_M(k) + 1$, and because, by the proof of the Roof Theorem 3.2.1, i_k restricts to the flip locus S_-^k.
Similarly, i_k restricts to:

$$\begin{array}{ccc}
(V_1'', 0, 0) & \longmapsto & (V_1'' \otimes L_p, 0, 0) \\
\mathcal{N}''_{\sigma_c} & \xrightarrow{\ i_k\ } & \mathcal{N}''_{\sigma_c+1} \\
\cong \updownarrow & & \updownarrow \cong \\
\mathcal{J}^{d_M} & \xrightarrow[\ i_k\ ]{} & \mathcal{J}^{d_M} \\
[V_1''] & \longmapsto & [V_1'' \otimes L_p]
\end{array}$$

So, by Macdonald [28, (12.2)], $i_k^* : H^j(\mathcal{N}'_{\sigma_c+1}, \mathbb{Z}) \xrightarrow{\cong} H^j(\mathcal{N}'_{\sigma_c}, \mathbb{Z})$ for all $j \leqslant \tilde{d}_1 - d_M - \tilde{d}_2 - 1$, and hence

$$i_k^* : H^j(S_-^{k+1}, \mathbb{Z}) \xrightarrow{\cong} H^j(S_-^k, \mathbb{Z}) \quad \forall j \leqslant \tilde{d}_1 - d_M - \tilde{d}_2 - 1.$$

♠

Similarly, for the flip locus $S_+^k = S_{\sigma_c^+(k)}$ we have:

Theorem 3.3.2.

$$i_k^* : H^j(S_+^{k+1}, \mathbb{Z}) \xrightarrow{\cong} H^j(S_+^k, \mathbb{Z})$$

for all $j \leqslant 2d_M - \tilde{d}_1 + g - 2 = 2d_M - \big(d_2 + 2\sigma_H(k)\big) + g - 2$*, where* $d_j = \deg(E_j)$*,* $\tilde{d}_j = \deg(\tilde{E}_j)$*,* $d_M = \deg(M)$*, and* $\sigma_H(k) = \deg(K \otimes L_p^{\otimes k}) = 2g - 2 + k$*.*

Proof. Quite similar argument to the one presented above, except for the detail that this time is the other way around: according also to Muñoz, Ortega, Vázquez-Gallo [32, Theorem 4.8.], $S_+^k = \mathbb{P}(\mathcal{V})$ is the projectivization of a bundle $\mathcal{V} \to \mathcal{N}_c' \times \mathcal{N}_c''$ of rank $\mathrm{rk}(\mathcal{V}) = -\chi(T'', T')$, but this time $\mathcal{N}_c' = \mathcal{N}_c(1, 0, d_M, 0) \cong \mathcal{J}^{d_M}(X)$, and $\mathcal{N}_c'' = \mathcal{N}_c(1, 1, \tilde{d}_1 - d_M, \tilde{d}_2) \cong \mathcal{J}^{\tilde{d}_2} \times \mathrm{Sym}^{\tilde{d}_1 - d_M - \tilde{d}_2}(X)$ and where any triple $T = (V_1, V_2, \varphi) \in S_+^k \subset \mathcal{N}_{\sigma_c^+(k)}(2, 1, \tilde{d}_1, \tilde{d}_2)$ is a non-trivial extension of a subtriple $T' \subset T$ of the form $T' = (V_1', V_2', \varphi') = (M, 0, \varphi')$ by a quotient triple of the form $T'' = (V_1'', V_2'', \varphi'') = (L, V_2, \varphi'')$, where M is a line bundle of degree $\deg(M) = d_M$ and L is a line bundle of degree $\deg(L) = d_L = \tilde{d}_1 - d_M$. ♠

Theorem 3.3.3.

$$i_k^* : H^j(\mathcal{N}_{\sigma_c^-(k+1)}, \mathbb{Z}) \xrightarrow{\cong} H^j(\mathcal{N}_{\sigma_c^-(k)}, \mathbb{Z}) \quad \forall j \leqslant 2\big(\tilde{d}_1 - 2\tilde{d}_2 - (2g-2)\big) + 1.$$

Since the behavior of $\mathcal{N}_{\sigma_c^-}$, where $\sigma_c^- = \sigma_c - \varepsilon$, is the same that the one of $\mathcal{N}_{\sigma_m^+}$, where $\sigma_m^+ = \sigma_m + \varepsilon$, is enough to prove the following lemma:

Lemma 3.3.4.

$$H^j(\mathcal{N}_{\sigma_m^+(k+1)}, \mathcal{N}_{\sigma_m^+(k)}; \mathbb{Z}) = 0 \quad \forall j \leqslant 2\big(\tilde{d}_1 - 2\tilde{d}_2 - (2g-2)\big).$$

Proof. Note that $\mathcal{N}_{\sigma_m^-(k)} = \emptyset$, hence $\mathcal{N}_{\sigma_m^+(k)} = S_+^k$, and according to Muñoz, Ortega, Vázquez-Gallo [32, Theorem 4.10.], any triple $T = (V_1, V_2, \varphi) \in S_+^k = \mathcal{N}_{\sigma_m^+(k)}(2, 1, \tilde{d}_1, \tilde{d}_2)$ is a non-trivial extension of a subtriple $T' \subset T$ of the form $T' = (V_1', V_2', \varphi') = (V_1, 0, 0)$ by a quotient triple of the form $T'' = (V_1'', V_2'', \varphi'') = (0, V_2, 0)$. Hence, there is a map

$$\pi : \mathcal{N}_{\sigma_m^+} \to \mathcal{N}(2, \tilde{d}_1) \times \mathcal{J}^{\tilde{d}_2}(X)$$

$$(V_1, V_2, \varphi) \mapsto ([V_1], [V_2])$$

where the inverse image $\pi^{-1}\big(\mathcal{N}(2,\tilde{d}_1)\times\mathcal{J}^{\tilde{d}_2}(X)\big)=\mathbb{P}^N$ has rank $N=-\chi(T'',T')=\tilde{d}_1-2\tilde{d}_2-(2g-2)$, and the proof follows. ♠

Theorem 3.3.5.

$$\tilde{i}_k^*: H^j(\tilde{\mathcal{N}}_{\sigma_c(k+1)},\mathbb{Z})\xrightarrow{\cong}H^j(\tilde{\mathcal{N}}_{\sigma_c(k)},\mathbb{Z})\quad\forall j\leqslant n(k)$$

at the blow-up level, where $n(k):=\min(\tilde{d}_1-d_M-\tilde{d}_2-1,\quad 2\big(\tilde{d}_1-2\tilde{d}_2-(2g-2)\big)+1)$.

Proof. By the Roof Theorem 3.2.1, i_k lifts to the blow-up level. We will denote $\mathcal{N}_-^k=\mathcal{N}_{\sigma_c^-(k)}(2,1,\tilde{d}_1,\tilde{d}_2)$ and $\tilde{\mathcal{N}}^k=\tilde{\mathcal{N}}_{\sigma_c(k)}$ its blow-up along the flip locus $S_-^k=S_{\sigma_c^-(k)}$. Recall that, from the construction of the blow-up, there is a map $\pi_-:\tilde{\mathcal{N}}^k\to\mathcal{N}_-^k$ such that

$$0\to\pi_-^*\big(H^j(\mathcal{N}_-^k)\big)\to H^j(\tilde{\mathcal{N}}^k)\to H^j(\mathcal{E}^k)/\pi_-^*\big(H^j(\mathcal{S}_-^k)\big)\to 0$$

splits where $\mathcal{E}^k=\pi_-^{-1}(S_-^k)$ is the so-called exceptional divisor. Hence, the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_-^*\big(H^j(\mathcal{N}_-^k)\big) & \longrightarrow & H^j(\tilde{\mathcal{N}}^k) & \longrightarrow & H^j(\mathcal{E}^k)/\pi_-^*\big(H^j(\mathcal{S}_-^k)\big) & \longrightarrow & 0\\
 & & \cong\uparrow & & \uparrow\tilde{i}_k^* & & \uparrow\cong & & \\
0 & \longrightarrow & \pi_-^*\big(H^j(\mathcal{N}_-^{k+1})\big) & \longrightarrow & H^j(\tilde{\mathcal{N}}^{k+1}) & \longrightarrow & H^j(\mathcal{E}^{k+1})/\pi_-^*\big(H^j(\mathcal{S}_-^{k+1})\big) & \longrightarrow & 0
\end{array}\tag{3.2}$$

commutes for all $j\leqslant n(k)$, and the theorem follows. ♠

Corollary 3.3.6.

$$i_k^*: H^j(\mathcal{N}_{\sigma_c^+(k+1)},\mathbb{Z})\xrightarrow{\cong}H^j(\mathcal{N}_{\sigma_c^+(k)},\mathbb{Z})\quad\forall j\leqslant\tilde{n}(k)$$

where $\tilde{n}(k):=\min\big(n(k),\quad 2d_M-\tilde{d}_1+g-2\big)$.

Proof. Recall that $\tilde{\mathcal{N}}^k=\tilde{\mathcal{N}}_{\sigma_c(k)}$ is also the blow-up of $\mathcal{N}_+^k=\mathcal{N}_{\sigma_c^+(k)}(2,1,\tilde{d}_1,\tilde{d}_2)$ along the flip locus $S_+^k=S_{\sigma_c^+(k)}$, so there is a map $\pi_+:\tilde{\mathcal{N}}^k\to\mathcal{N}_+^k$ such that

$$0\to\pi_+^*\big(H^j(\mathcal{N}_+^k)\big)\to H^j(\tilde{\mathcal{N}}^k)\to H^j(\mathcal{E}^k)/\pi_+^*\big(H^j(\mathcal{S}_+^k)\big)\to 0$$

splits:

$$H^j(\tilde{\mathcal{N}}^k) = \pi_+^*\big(H^j(\mathcal{N}_+^k)\big) \oplus H^j(\mathcal{E}^k)/\pi_+^*\big(H^j(\mathcal{S}_+^k)\big),$$

and by Theorem 3.3.2 and Theorem 3.3.5, the result follows. ♠

Corollary 3.3.7.

$$i_k^* : H^j(\mathcal{N}_{\sigma_c(k+1)}, \mathbb{Z}) \xrightarrow{\cong} H^j(\mathcal{N}_{\sigma_c(k)}, \mathbb{Z}) \quad \forall j \leqslant \tilde{n}(k).$$

Chapter 4

Stratifications on the Moduli Space of Higgs Bundles

Recall that we are supposing that $\mathrm{GCD}(r,d)=1$. In this chapter, we study the relationship between the Shatz stratification and the Bialynicki-Birula stratification on $\mathcal{M}(r,d)$ for rank $r=2$ and rank $r=3$. Our results should produce a more refined stratification for rank three, which we expect to be useful in generalizing Hausel'results for rank two to rank three.

4.1 Equivalent Stratifications on the Moduli Space of Rank Two Higgs Bundles

Recall that a point $(E,\Phi)\in\mathcal{N}=F_0$ is a pair where $E\to X$ is a stable holomorphic bundle of $\mathrm{rk}(E)=2$ and $\Phi\equiv 0$.

On the other hand, for $d_1>0$ and $\Phi\neq 0$, define then F_{d_1} as follow:

$$F_{d_1}=\left\{(E,\Phi)=(E_1\oplus E_2,\begin{pmatrix}0&0\\ \varphi&0\end{pmatrix})\middle|\begin{array}{cc}\deg(E_1)=d_1, & \deg(E_2)=d_2,\\ \mathrm{rk}(E_1)=1, & \mathrm{rk}(E_2)=1,\\ \varphi:E_1\to E_2\otimes K & \end{array}\right\}.$$

The description of these critical submanifolds has been done by Bento [3]:

Proposition 4.1.1 (Bento [3, Proposição 2.1.1.]). *There is a critical submanifold F_{d_1} for each $d_1 \in \ \left]\frac{d}{2}, \frac{d+d_K}{2}\right[\ \cap \ \mathbb{Z}$.*

Proof. Let F_{d_1} be a critical submanifold as described above, with $d_1 > 0$ and $\Phi \neq 0$, where $\Phi = \begin{pmatrix} 0 & 0 \\ \varphi & 0 \end{pmatrix}$ and $\varphi : E_1 \to E_2 \otimes K$, so we can consider $0 \neq \varphi \in H^0(Hom(E_1, E_2) \otimes K)$. Then, E_2 is Φ-invariant and so, the stability of (E, Φ) implies:

$$\mu(E_2) < \mu(E) \Leftrightarrow d - d_1 = d_2 < \frac{d}{2} \Leftrightarrow \frac{d}{2} < d_1.$$

But, if $d_1 > \frac{d}{2}$ then E_1 can not be Φ-invariant, and so, since $\varphi \neq 0$, we get that:

$$\deg(E_1^* E_2 K) > 0 \Leftrightarrow d_2 + d_K - d_1 > 0 \Leftrightarrow d - d_1 + d_K - d_1 > 0 \Leftrightarrow d_1 < \frac{d + d_K}{2}.$$

Since $d_1 \in \mathbb{Z}$, the Proposition follows. ♠

The holomorphic splitting $(E, \Phi) = (E_1 \oplus E_2, \begin{pmatrix} 0 & 0 \\ \varphi & 0 \end{pmatrix})$ is the so-called *Variation of Hodge Structure* of type $(1, 1)$, and denoted $(1, 1)$-VHS.

In such a case:

$$[\Phi, \Phi^*] = \Phi\Phi^* + \Phi^*\Phi =$$

$$\begin{pmatrix} 0 & 0 \\ \varphi & 0 \end{pmatrix} \begin{pmatrix} 0 & \varphi^* \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \varphi^* \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \varphi & 0 \end{pmatrix} = \begin{pmatrix} \varphi^*\varphi & 0 \\ 0 & \varphi\varphi^* \end{pmatrix}$$

Hence, the first Hitchin-Equation becomes:

$$\begin{pmatrix} F_A(E_1) - \varphi\varphi^* & 0 \\ 0 & F_A(E_2) + \varphi\varphi^* \end{pmatrix} = -i \cdot \mu \cdot \begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix} \cdot \omega$$

where $\mu = \mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)} = \frac{d}{2}$, so it is constant. In the first entry we have:

$$F_A(E_1) - \varphi\varphi^* = -i \cdot \mu \cdot I_1 \cdot \omega.$$

Taking the trace $tr(\cdot)$ and integrating on X, we get:

$$\int_X tr(F_A(E_1)) - \int_X tr(\varphi\varphi^*) = -i \cdot \mu \int_X tr(I_1)\omega$$

which is equivalent to:

$$-2\pi \cdot i \cdot c_1(E_1) - \int_X tr(\Phi\Phi^*) = -i \cdot \frac{d}{2} \cdot \mathrm{rk}(E_1) \cdot 2\pi$$

which implies:

$$-2\pi i(d_1 - \frac{d}{2}) = \int_X tr(\Phi\Phi^*)$$

Therefore:

$$f(E, \Phi) = d_1 - \frac{d}{2}$$

for each $(E, \Phi) \in F_{d_1}$, for every $1 \leqslant d_1 \leqslant g - 1$. The non-zero critical values for the $\mathrm{rk}(E) = 2$ case, were computed by Hitchin [24] (and also by Hausel [20]) with the assumption of $\deg(E) \equiv 1$, and the stability of $(E, \Phi) \in F_{d_1}$ gives the bound $d_1 < g$. See the work of Hitchin [24] for more details.

Recall that the sets

$$U_{d_1}^{BB} := \{(E, \Phi) \in \mathcal{M}(2, d) | \lim_{z \to 0} z \cdot (E, \Phi) \in F_{d_1}\}$$

are the *upward stratum sets* of the *Bialynicki-Birula stratification*:

$$\mathcal{M}(2, d) = \bigcup_{d_1=0}^{g-1} U_{d_1}^{BB}.$$

On the other hand, recall also that, as a consequence of Shatz [35, Proposition 10 and Proposition 11], there is a finite stratification of $\mathcal{M}(r, d)$ by the Harder-Narasimhan type of the underlying vector bundle E of a Higgs bundle (E, Φ):

$$\mathcal{M}(r, d) = \bigcup_t U'_t$$

where $U'_t \subset \mathcal{M}(r, d)$ is the subspace of Higgs bundles (E, Φ) which associated vector bundle E has $\mathrm{HNT}(E) = t$, and where we are taking this union over the existing types in $\mathcal{M}(r, d)$. This is the *Shatz stratification*. Nevertheless, for rank two Higgs bundles, the HNT is a vector of the form $t = (d_1 d - d_1)$, where $d = \deg(E)$ is a known parameter. So, Hausel labels the Shatz stratum as follows: Let $U'_0 \subset \mathcal{M}$ be the locus of points $(E, \Phi) \in \mathcal{M}(2, d)$ such that E is stable, and let $U'_{d_1} \subset \mathcal{M}$ be the locus of points $(E, \Phi) \in \mathcal{M}(2, d)$ such that E is unstable and its destabilizing line bundle E_1 is of degree $d_1 > 0$. This family $\{U'_{d_1}\}_{d_1=0}^{g-1}$ gives us the Shatz stratification of $\mathcal{M}$:

$$\mathcal{M} = \bigcup_{d_1=0}^{g-1} U'_{d_1}.$$

We shall give an alternative proof for the statement of Hausel [19]: that Shatz stratification and Hitchin stratification are essentially the same thing when $\mathrm{rk}(E) = 2$:

Theorem 4.1.2 (Hausel [19, Proposition 4.3.2])**.** *The Shatz stratification coincides with the Hitchin stratification,*

$$\textit{i.e. } U'_{d_1} = U_{d_1} \textit{ for } 0 \leqslant d_1 \leqslant g - 1$$

using the above notation.

Proof. The inclusion $U_{d_1} \subseteq U'_{d_1}$ is trivial:
Just take a point $(E, \Phi) \in U_{d_1}$ and consider its limit:

$$(E^0, \Phi^0) := \lim_{z \to 0} z \cdot (E, \Phi) = \lim_{z \to 0} (E, z \cdot \Phi) \in F_{d_1}.$$

Since $(E^0, \Phi^0) \in F_{d_1}$, it has the form:

$$(E^0, \Phi^0) = (L_1 \oplus L_2, \begin{pmatrix} 0 & 0 \\ \phi_{21} & 0 \end{pmatrix})$$

where $d_1 = \deg(L_1),\ \mathrm{rk}(L_1) = 1$, and $\phi_{21} : L_1 \to L_2 \otimes K$.

The Harder-Narasimhan Type of the limit bundle (E^0, Φ^0) is then the vector

$$HNT(E^0, \Phi^0) : \vec{\mu} = (\mu_1, \mu_2) = (d_1, d_2)$$

where $\deg(L_2) = d_2 = d - d_1$. Then, it is enough to consider $E_1 := L_1$ as maximal destabilizing line bundle of E with degree d_1. It is destabilizing since

$$c_{d_1} = d_1 - \frac{d}{2} > 0, \text{ so } \mu(E_1) = d_1 > \frac{d}{2} = \mu(E);$$

and it is trivially maximal. Besides, E_1 and E/E_1 are semi-stables.
So, we get the Harder-Narasimhan Filtration:

$$0 \subset E_1 \subset E$$

and hence:

$$U_{d_1} \subseteq U'_{d_1}.$$

Remark 4.1.3. Note that $E = E^0$ as smooth vector bundles, but not as holomorphic vector bundles, since we are varying its holomorphic structure when we take the limit when $z \to 0$. That is why $E_1 = L_1$ is its maximal destabilizing subundle as smooth vector bundle, but not as Higgs bundle, since E_1 is not even Φ^0-invariant.

The other inclusion, $U'_{d_1} \subseteq U_{d_1}$, is not so trivial. Suppose E is an unstable bundle with maximal destabilizing line bundle E_1 with $\deg(E_1) = d_1$.

$$i.e.\ HNF(E) : 0 \subset E_1 \subset E$$

where $\mu(E_1) > \mu(E)$ and E_1 is the already mentioned maximal destabilizing subbundle of E. Then, there is a smooth decomposition $E = L_1 \oplus L_2$ comming from the short exact sequence:

$$0 \longrightarrow L_1 \longrightarrow E \longrightarrow L_2 \longrightarrow 0$$

where $L_1 = E_1$ and $L_2 \cong E/E_1$.

So far, we have been abusing of the notation before, since the points of $\mathcal{M}$, and so the elements of the subsets F_{d_1}, U_{d_1}, and U'_{d_1}, are not the pairs (E, Φ), but their equivalence classes $[(E, \Phi)]$ under the gauge group action. So, for an element $[(E, \Phi)] \in U'_{d_1}$ it will be enough to find a gauge transformation $g \in \mathcal{G}$ such that

$$(E^0, \Phi^0) = \lim_{z \to 0} g(z)^{-1}(E, z \cdot \Phi) g(z) \in F_{d_1}.$$

We may suppose that $g(z) \in GL_2(\mathbb{C})$ is diagonal, so, $g_{12}(z) \equiv 0$ and $g_{21}(z) \equiv 0$. In such a case, we have:

$$g(z) = \begin{pmatrix} g_{11}(z) & g_{12}(z) \\ g_{21}(z) & g_{22}(z) \end{pmatrix} = \begin{pmatrix} g_{11}(z) & 0 \\ 0 & g_{22}(z) \end{pmatrix} \text{ for } z \in \mathbb{C}^*$$

and then:

$$g(z)^{-1} = \frac{1}{det(g)} \begin{pmatrix} g_{22}(z) & 0 \\ 0 & g_{11}(z) \end{pmatrix} = \frac{1}{g_{11}(z) g_{22}(z)} \begin{pmatrix} g_{22}(z) & 0 \\ 0 & g_{11}(z) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{g_{11}(z)} & 0 \\ 0 & \frac{1}{g_{22}(z)} \end{pmatrix} \text{ for } z \in \mathbb{C}^*.$$

Recall also that a representative pair (E, Φ) of the equivalence class $[(E, \Phi)] \in U'_{d_1}$ has a representative holomorphic structure $\bar{\partial}_E = \bar{\partial}_A = \bar{\partial} + B d\bar{z}$ of the form:

$$\bar{\partial}_A = \begin{pmatrix} \bar{\partial}_1 & \beta \\ 0 & \bar{\partial}_2 \end{pmatrix} = \begin{pmatrix} \bar{\partial} + b_{11} d\bar{z} & b_{12} d\bar{z} \\ 0 & \bar{\partial} + b_{22} d\bar{z} \end{pmatrix}$$

and its Higgs field takes the form:

$$\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$$

where $\phi_{ij} : L_j \to L_i \otimes K$. Then:

$$g^{-1}(z \cdot \Phi) g = \begin{pmatrix} \frac{1}{g_{11}(z)} & 0 \\ 0 & \frac{1}{g_{22}(z)} \end{pmatrix} \begin{pmatrix} z \cdot \phi_{11} & z \cdot \phi_{12} \\ z \cdot \phi_{21} & z \cdot \phi_{22} \end{pmatrix} \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}$$

$$= \begin{pmatrix} z \cdot \phi_{11} & \frac{g_{22}}{g_{11}} z \cdot \phi_{12} \\ \frac{g_{11}}{g_{22}} z \cdot \phi_{21} & z \cdot \phi_{22} \end{pmatrix}$$

where, once again, $g_{ij} = g_{ij}(z)$ is an abuse of notation. Since an element of F_{d_1} has a Higgs field of the form:

$$\Psi = \begin{pmatrix} 0 & 0 \\ \psi & 0 \end{pmatrix}$$

it will be enough if the g_{ij}'s satisfy:

$$\lim_{z \to 0} \frac{g_{11}(z)}{g_{22}(z)} z = 1$$

and

$$\lim_{z \to 0} \frac{g_{22}(z)}{g_{11}(z)} z = 0$$

We may choose

$$g_{11}(z) \equiv 1,\ g_{22}(z) = z \text{ for } z \in \mathbb{C}^* :$$

$$g^{-1}(z)(z \cdot \Phi)g(z) = \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} z \cdot \phi_{11} & z \cdot \phi_{12} \\ z \cdot \phi_{21} & z \cdot \phi_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} =$$

$$\begin{pmatrix} z \cdot \phi_{11} & z^2 \cdot \phi_{12} \\ \phi_{21} & z \cdot \phi_{22} \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ \phi_{21} & 0 \end{pmatrix} \text{ when } z \to 0.$$

Furthermore:

$$g^{-1}\bar{\partial}_E g = g^{-1}\bar{\partial}_A g = \bar{\partial} + g^{-1}(\bar{\partial} g) + (g^{-1}Bg)d\bar{z}$$

In this case $g(z)$ doesn't depend on $\bar{z}$, so $\bar{\partial} g = \frac{\partial g}{\partial \bar{z}} d\bar{z} \equiv 0$. Then:

$$g^{-1}\bar{\partial}_A g = \bar{\partial} + (g^{-1}Bg)d\bar{z}$$

where

$$g(z)^{-1}Bg(z) = \begin{pmatrix} b_{11} & zb_{12} \\ 0 & b_{22} \end{pmatrix} \to \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix} \text{ when } z \to 0.$$

So:

$$g^{-1}\bar{\partial}_E g \to \begin{pmatrix} \bar{\partial}_1 & 0 \\ 0 & \bar{\partial}_2 \end{pmatrix} \text{ when } z \to 0.$$

We are almost done. It remains to verify two things: first, that Φ^0 is holomorphic since Φ is; and second, that (E^0, Φ^0) is stable.

If we look carefully to how is ϕ_{21} defined, we can see that:

$$\phi_{21} : L_1 \xrightarrow{\imath_1} E \xrightarrow{\Phi} E \otimes K \xrightarrow{\jmath_2 \otimes 1} L_2 \otimes K$$

where $\imath_1 : L_1 \to E$ and $\jmath_2 : E \to L_2$ are the canonical inclusion and projection respectively, and the three components are holomorphic, so ϕ_{21} is. Since ϕ_{21} is the only non-trivial component, Φ^0 is also holomorphic.

Since $L_1 = E_1$ is not Φ^0-invariant, the line subbundles which are Φ^0-invariant are those that are isomorphic to L_2. But we know that $\mu(L_2) < \mu(E^0)$ trivially, since $\mu(E_1) > \mu(E) = \mu(E^0)$.

$$\therefore [(E^0, \Phi^0)] \in F_{d_1}.$$

♠

Remark 4.1.4. Recall that 4.1.2 doesn't remain valid for the general case:

$$\mathrm{rk}(E) = r \geqslant 3.$$

See for instance the works of Hausel and Thaddeus [21] and [22]. It will follow also from our work in the next section.

4.2 Stratifications on the Moduli Space of Rank Three Higgs Bundles

Denote as above $d = \deg(E)$. Recall that we are considering the coprime case $GCD(3, d) = 1$.

If E is stable, then $(E, \Phi) \in \mathcal{N} = F_0 \subset \mathcal{M}(3, d)$ is a pair where $E \to X$ is a stable holomorphic bundle of $\mathrm{rk}(E) = 3$ and $\Phi \equiv 0$.

Suppose then that (E, Φ) is a pair where E is an unstable vector bundle and $\Phi \neq 0$ is not trivial. Hence, we must consider three non-trivial cases for the Harder-Narasimhan

Filtration of E.

Let $\big[(E,\Phi)\big] \in \mathcal{M}(3,d)$ and denote $(E^0,\Phi^0) := \lim_{z\to 0}(E, z\cdot\Phi)$. The stratum of the Morse stratification where (E,Φ) belongs is determined by (E^0,Φ^0), and depends on the Harder-Narasimhan Type of E, and on certain properties of Φ. Our Principal Theorem describes in detail that dependence.

To state the Theorem, is convenient to use the following notation: for a vector bundle morphism $\phi : E \to F$, we write $\ker(\phi) \subset E$ and $\mathrm{im}(\phi) \subset F$ for those subbundles obtained by the saturation of the respective subsheaves.

Theorem 4.2.1. *Let $\big[(E,\Phi)\big] \in \mathcal{M}(3,d)$ and denote $(E^0,\Phi^0) := \lim_{z\to 0}(E, z\cdot\Phi)$.*

(1.) Suppose that E is an unstable vector bundle of $\mathrm{rk}(E) = 3$ with a Harder-Narasimhan Filtration of length 1:

$$HNF(E):\ 0 = E_0 \subset E_1 \subset E_2 = E$$

where E_1 is the maximal destabilizing line subbundle of E, and $\mu(V_1) > \mu(V_2)$ where $V_1 = E_1,\ V_2 = E/E_1$ are semi-stables. In other words, suppose that $E \to X$ is a holomorphic bundle that has $\mathrm{HNT}(E) = (\mu_1,\mu_2,\mu_2)$ where $\mu_j = \mu(V_j)$. Consider $\phi_{21} : V_1 \to V_2 \otimes K$ induced by

$$E_1 \xrightarrow{\ \imath\ } E \xrightarrow{\ \Phi\ } E\otimes K \xrightarrow{\jmath\otimes id_K} \big(E/E_1\big)\otimes K.$$

Define $\mathcal{I} := \phi_{21}(E_1)\otimes K^{-1} \subset V_2$ which is a subbundle of V_2, where $\mathrm{rk}(\mathcal{I}) = 1$, and define also $F := V_1 \oplus \mathcal{I} \subset V_1 \oplus V_2 = E$ where $\mathrm{rk}(F) = 2$. Then, we have two possibilities:

(1.1.) Suppose that $\mu(F) < \mu(E)$. Then, (E^0,Φ^0) is a $(1,2)$-VHS of the form:

$$(E^0,\Phi^0) = \left(V_1 \oplus V_2, \begin{pmatrix} 0 & 0 \\ \phi_{21} & 0 \end{pmatrix}\right).$$

(1.2.) On the other hand, if $\mu(F) \geqslant \mu(E)$, then, (E^0,Φ^0) is a $(1,1,1)$-VHS of the

form:

$$(E^0, \Phi^0) = \Big(L_1 \oplus L_2 \oplus L_3, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix}\Big)$$

where $L_1, L_2,$ and L_3 are line bundles.

(2.) Analogously, suppose that E is an unstable vector bundle of $\mathrm{rk}(E) = 3$ with a Harder-Narasimhan Filtration of length 1:

$$HNF(E):\ 0 = E_0 \subset E_1 \subset E_2 = E$$

but this time E_1 is the maximal destabilizing subbundle of E with $\mathrm{rk}(E_1) = 2$, and $\mu(V_1) > \mu(V_2)$ where $V_1 = E_1,\ V_2 = E/E_1$ are semi-stables. In other words, suppose that $E \to X$ is a holomorphic bundle that has $\mathrm{HNT}(E) = (\mu_1, \mu_1, \mu_2)$ where $\mu_j = \mu(V_j)$. Consider $\phi_{21} : V_1 \to V_2 \otimes K$ induced by

$$E_1 \xrightarrow{\ \imath\ } E \xrightarrow{\ \Phi\ } E \otimes K \xrightarrow{\jmath \otimes id_K} \big(E/E_1\big) \otimes K.$$

Define $N := ker(\phi_{21}) \subset V_1$ which is a subbundle. Then, we have two possibilities:

(2.1.) Suppose that $\mu(N) < \mu(E)$. Then, (E^0, Φ^0) is a $(2,1)$-VHS of the form:

$$(E^0, \Phi^0) = \Big(V_1 \oplus V_2, \begin{pmatrix} 0 & 0 \\ \phi_{21} & 0 \end{pmatrix}\Big).$$

(2.2.) On the other hand, if $\mu(N) \geqslant \mu(E)$, then: (E^0, Φ^0) is a $(1,1,1)$-VHS of the form:

$$(E^0, \Phi^0) = \Big(L_1 \oplus L_2 \oplus L_3, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix}\Big)$$

where $L_1, L_2,$ and L_3 are line bundles.

(3.) Finally, suppose that (E, Φ) is a Higgs Bundle where E is an unstable vector bun-

dle of $\operatorname{rk}(E) = 3$ *with a Harder-Narasimhan Filtration of length* 2*:*

$$HNF(E):\ 0 = E_0 \subset E_1 \subset E_2 \subset E_3 = E$$

where $\mu(V_1) > \mu(V_2) > \mu(V_3)$ *and* $V_1 = E_1,\ V_2 = E_2/E_1,$ *and* $V_3 = E/E_2$ *are semi-stables.*

(3.1.) Suppose that $\mu(E_2/E_1) < \mu(E)$*. Then we can define* F *as we did in (1.), and then, we have two possibilities:*

(3.1.1.) Suppose that $\mu(F) < \mu(E)$*. Then:* (E^0, Φ^0) *is a* $(1,2)$*-VHS.*

(3.1.2.) On the other hand, if $\mu(F) \geqslant \mu(E)$*, then:* (E^0, Φ^0) *is a* $(1,1,1)$*-VHS.*

(3.2.) On the other hand, if $\mu(E_2/E_1) > \mu(E)$*, then define* N *as we did in (2.), and then, we have two possibilities:*

(3.2.1.) If $\mu(N) < \mu(E)$*. Then:* (E^0, Φ^0) *is a* $(2,1)$*-VHS.*

(3.2.2.) If $\mu(N) \geqslant \mu(E)$*, then:* (E^0, Φ^0) *is a* $(1,1,1)$*-VHS.*

This theorem shall be proved, case by case, step by step, considering every single Harder-Narasimhan Type.

4.2.1 Case (1)

Suppose that E is an unstable vector bundle of $\operatorname{rk}(E) = 3$ with a Harder-Narasimhan Filtration of length 1:

$$HNF(E):\ 0 = E_0 \subset E_1 \subset E_2 = E$$

where E_1 is the maximal destabilizing line subbundle of $E:\ \mu(E_1) > \mu(E)$ and $V_1,\ V_2$ are semi-stables. Then, there is a smooth decomposition $E = V_1 \oplus V_2$ from the short exact sequence:

$$0 \longrightarrow V_1 \longrightarrow E \longrightarrow V_2 \longrightarrow 0$$

where $V_1 = E_1,$ and $V_2 \cong E/E_1$. Then, the Higgs field Φ takes the form:

$$\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$$

where $0 \neq \phi_{21} : V_1 \longrightarrow V_2 \otimes K$ is a (1×2)-size block, and every block $\phi_{ij} \in \Omega^{1,0}(X, Hom(V_j, V_i) \otimes K)$. Besides, the representative holomorphic structure of E, $\bar{\partial}_E$ becomes:

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_1 & \beta \\ 0 & \bar{\partial}_2 \end{pmatrix}$$

where $\bar{\partial}_j$ is the corresponding holomorphic structure of V_j, and $\beta \in \Omega^{0,1}(X, Hom(V_2, V_1))$.

Denote by $d_1 = \deg(V_1)$ and $d_2 = \deg(V_2)$. Recall that V_2 satisfies the following:

a. $\mathrm{rk}(V_2) = 2$

b. $d_2 = d - d_1$

c. V_2 is semi-stable

d. $\mu(V_2) < \mu(E) < \mu(V_1)$

These are general properties of the Harder-Narasimhan Filtration. The last one can be easily proved, since $\mu(E_1) > \mu(E)$.

Define $\mathcal{I} := \phi_{21}(E_1) \otimes K^{-1} \subset V_2$ and recall that we understand this as the subbundle that we obtain saturating the respective subsheaf. Besides, $\mathrm{rk}(\mathcal{I}) = 1$, and define also $F := V_1 \oplus \mathcal{I} \subset V_1 \oplus V_2 = E$ where $\mathrm{rk}(F) = 2$. Denote $d_{\mathcal{I}} = \deg(\mathcal{I})$ and $d_F = \deg(F)$, then $d_F = d_1 + d_{\mathcal{I}}$.

Define the pair $(E^0, \Phi^0) := \lim\limits_{z \to 0}(E, z \cdot \Phi)$. We must consider then, two subcases:

(1) $\mu(F) < \mu(E)$

(2) $\mu(F) \geqslant \mu(E)$

Case (1.1)

Proposition 4.2.2. *Suppose that $\mu(F) < \mu(E)$. Then: (E^0, Φ^0) is a $(1, 2)$-VHS.*

Proof. All we need to do, is to consider

$$g(z) := \begin{pmatrix} 1 & 0 \\ 0 & z \cdot I \end{pmatrix} \in GL_3(\mathbb{C})$$

where $I \in GL_2(\mathbb{C})$ is the identity matrix, and $g \in \mathcal{G}$ defines a gauge transformation. Then:

$$g(z) * (z \cdot \Phi) = g(z)^{-1}(z \cdot \Phi)g(z) =$$

$$= \begin{pmatrix} z \cdot \phi_{11} & z^2 \cdot \phi_{12} \\ \phi_{21} & z \cdot \phi_{22} \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ \phi_{21} & 0 \end{pmatrix} \text{ when } z \to 0$$

and also:

$$g(z) * \bar{\partial}_E = g(z)^{-1}\bar{\partial}_E g(z) =$$

$$= \begin{pmatrix} \bar{\partial}_1 & z \cdot \beta \\ 0 & \bar{\partial}_2 \end{pmatrix} \to \begin{pmatrix} \bar{\partial}_1 & 0 \\ 0 & \bar{\partial}_2 \end{pmatrix} \text{ when } z \to 0.$$

We can easily showed that Φ^0 is holomorphic since Φ is:

$$\phi_{21} : V_1 \xrightarrow{\imath_1} E \xrightarrow{\Phi} E \otimes K \xrightarrow{\jmath_2 \otimes id} V_2 \otimes K$$

where $\imath_1 : V_1 \to E$ and $\jmath_2 : E \to V_2$ are the canonical inclusion and projection respectively, and the three components are holomorphic, so ϕ_{21} is. Since ϕ_{21} is the only non-trivial component, Φ^0 is also holomorphic.

There are three kinds of Φ^0-invariant subbundles: Thoseones isomorphic to F, those ones isomorphic to V_2, and any line bundle $L \subset V_2$.

1. F :
   By hypothesis $\mu(F) < \mu(E) = \mu(E^0)$ in this subcase, so there is nothing to worry about.

2. V_2 :
   We already have seen that $\mu(V_2) < \mu(E) = \mu(E^0)$.

3. $L \subset V_2$:
   Since V_2 is semi-stable, $\mu(L) \leqslant \mu(V_2)$, and since $\mu(V_2) < \mu(E^0)$, we get $\mu(L) < \mu(E^0)$ for any line bundle $L \subset V_2$.

Hence:

$$(E^0, \Phi^0) := \lim_{z\to 0}(E, z \cdot \Phi) = (V_1 \oplus V_2, \begin{pmatrix} 0 & 0 \\ \phi_{21} & 0 \end{pmatrix}) \text{ is stable.}$$

♠

Remark 4.2.3. In this case, the Harder-Narasimhan Type of the limit bundle (E^0, Φ^0) is the vector:

$$HNT(E^0, \Phi^0) : \vec{\nu} = (\nu_1, \nu_2, \nu_2)$$

where $\nu_j = \mu(V_j)$ coincides with $\mu_j = \mu(V_j)$. So, in this subcase we get

$$HNT(E^0, \Phi^0) = HNT(E, \Phi).$$

Case (1.2)

On the other hand:

Proposition 4.2.4. *If* $\mu(F) \geqslant \mu(E)$*, then:* (E^0, Φ^0) *is a* $(1, 1, 1)$*-VHS.*

Proof. Suppose $\mu(F) \geqslant \mu(E)$, define $\mathcal{Q} := V_2/\mathcal{I}$ and consider the short exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow V_2 \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Then, there is a smooth splitting $V_2 = \mathcal{I} \oplus \mathcal{Q}$, and then a new smooth splitting

$$E = V_1 \oplus \mathcal{I} \oplus \mathcal{Q} = L_1 \oplus L_2 \oplus L_3$$

where $L_1 := V_1,\ L_2 := \mathcal{I},$ and $L_3 := \mathcal{Q}$. Hence, we may re-write the Higgs field Φ as:

$$\Phi = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{pmatrix} = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ 0 & \varphi_{32} & \varphi_{33} \end{pmatrix}$$

where every block $\varphi_{ij} \in \Omega^{1,0}(X, Hom(L_j, L_i) \otimes K)$ using the new notation, and $\varphi_{31} \equiv 0$ since:

$$\begin{array}{ccc} E & \xrightarrow{\ \Phi\ } & E \otimes K \\ {\scriptstyle \imath_1}\uparrow & & \downarrow{\scriptstyle \jmath_3 \otimes id} \\ L_1 & \xrightarrow[\ \varphi_{31}\ ]{} & L_3 \otimes K \end{array}$$

where, by definition, $L_1 = E_1$, $L_3 = \mathcal{Q} = V_2/\mathcal{I}$ and $\mathcal{I} = \phi_{21}(E_1) \otimes K^{-1} \subset V_2$, then $\varphi_{31} \equiv 0$.

This time, we shall take

$$g(z) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z^2 \end{pmatrix} \in GL_3(\mathbb{C}).$$

Then, $g \in \mathcal{G}$ defines a gauge transformation, and then:

$$g(z) * (z \cdot \Phi) = g(z)^{-1}(z \cdot \Phi)g(z) =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & z^{-1} & 0 \\ 0 & 0 & z^{-2} \end{pmatrix} \begin{pmatrix} z \cdot \varphi_{11} & z \cdot \varphi_{12} & z \cdot \varphi_{13} \\ z \cdot \varphi_{21} & z \cdot \varphi_{22} & z \cdot \varphi_{23} \\ 0 & z \cdot \varphi_{32} & z \cdot \varphi_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z^2 \end{pmatrix} =$$

$$\begin{pmatrix} z\cdot\varphi_{11} & z^2\cdot\varphi_{12} & z^3\cdot\varphi_{13} \\ \varphi_{21} & z\cdot\varphi_{22} & z^2\cdot\varphi_{23} \\ 0 & \varphi_{32} & z\cdot\varphi_{33} \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix} \text{ when } z \to 0.$$

Besides, $\bar{\partial}_E$ the holomorphic structure of E may be expressed as

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_1 & \beta_{12} & \beta_{13} \\ 0 & \bar{\partial}_2 & \beta_{23} \\ 0 & 0 & \bar{\partial}_3 \end{pmatrix}$$

in terms of $\bar{\partial}_j$, which corresponds to the holomorphic structure of L_j, and $\beta_{ij} \in \Omega^{0,1}(X, Hom(L_j, L_i))$.

Then:

$$g(z) * \bar{\partial}_E = g(z)^{-1}\bar{\partial}_E g(z) =$$

$$= \begin{pmatrix} \bar{\partial}_1 & z\cdot\beta_{12} & z^2\cdot\beta_{13} \\ 0 & \bar{\partial}_2 & z\cdot\beta_{23} \\ 0 & 0 & \bar{\partial}_3 \end{pmatrix} \longrightarrow \begin{pmatrix} \bar{\partial}_1 & 0 & 0 \\ 0 & \bar{\partial}_2 & 0 \\ 0 & 0 & \bar{\partial}_3 \end{pmatrix} \text{ when } z \to 0.$$

Remains to answer two important questions. First, is Φ^0 holomorphic since Φ is? And second, is

$$(E^0, \Phi^0) = \lim_{z\to 0}(E, z\cdot\Phi) = (L_1 \oplus L_2 \oplus L_3, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix}) \text{ stable?}$$

We shall start by answering the first question:

$$\phi_{21} : L_1 \xrightarrow{\imath_1} E \xrightarrow{\Phi} E \otimes K \xrightarrow{\jmath_2 \otimes id} L_2 \otimes K$$

and

$$\phi_{32} : L_2 \xrightarrow{\imath_2} E \xrightarrow{\Phi} E \otimes K \xrightarrow{\jmath_3 \otimes id} L_3 \otimes K$$

are both holomorphic since Φ, the inclusions and the projections are. Then Φ^0 is also holomorphic, since ϕ_{21} and ϕ_{32} are the only two non-trivial components of Φ^0.

To answer the second question, is necessary to consider the Φ^0-invariant subbundles of E^0, and there are two kinds: those ones isomorphic to $L_3 := \mathcal{Q}$, and those ones isomorphic to $L_2 \oplus L_3 = \mathcal{I} \oplus \mathcal{Q}$.

$$\mu(L_3) \leqslant \mu(E^0) :$$

Recall that we are supposing that $\mu(F) \geqslant \mu(E^0)$ where $F = L_1 \oplus L_2 = E_1 \oplus \mathcal{I}$

$$i.e. \quad \mu(F) = \frac{1}{2}(\mu(L_1) + \mu(L_2)) \geqslant \frac{1}{3}(\mu(L_1) + \mu(L_2) + \mu(L_3)) = \mu(E^0) \Longleftrightarrow$$

$$3(\mu(L_1) + \mu(L_2)) \geqslant 2(\mu(L_1) + \mu(L_2) + \mu(L_3)) \Longleftrightarrow \mu(L_1) + \mu(L_2) \geqslant 2\mu(L_3) \Longleftrightarrow$$

$$\mu(L_1) + \mu(L_2) + \mu(L_3) \geqslant 3\mu(L_3) \Longleftrightarrow \frac{1}{3}(\mu(L_1) + \mu(L_2) + \mu(L_3)) \geqslant \mu(L_3)$$

$$\therefore \mu(E^0) \geqslant \mu(L_3).$$

$$\mu(L_2 \oplus L_3) < \mu(E^0) :$$

Recall that $\mu(E) < \mu(L_1)$ since $L_1 = E_1$ is the maximal destabilizing line subbundle of E. Then:

$$\frac{1}{3}(\mu(L_1) + \mu(L_2) + \mu(L_3)) < \mu(L_1) \Longleftrightarrow \mu(L_1) + \mu(L_2) + \mu(L_3) < 3\mu(L_1) \Longleftrightarrow$$

$$\mu(L_2) + \mu(L_3) < 2\mu(L_1) \Longleftrightarrow 3(\mu(L_2) + \mu(L_3)) < 2(\mu(L_1) + \mu(L_2) + \mu(L_3)) \Longleftrightarrow$$

$$\frac{1}{2}(\mu(L_2) + \mu(L_3)) < \frac{1}{3}(\mu(L_1) + \mu(L_2) + \mu(L_3)) \Longleftrightarrow \mu(L_2 \oplus L_3) < \mu(E^0).$$

We have shown that (E^0, Φ^0) is semistable, but we are taking $GCD(3, d) = 1$, and it implies stability.

$$\therefore (E^0, \Phi^0) = \lim_{z \to 0}(E, z \cdot \Phi) = (L_1 \oplus L_2 \oplus L_3, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix}) \text{ is stable.}$$

♠

Remark 4.2.5. Since E/E_1 is semi-stable, $\mu(\mathcal{I}) \leqslant \mu(E/E_1)$, and so $\mu(\mathcal{I}) \leqslant \mu(\mathcal{Q})$. Then, in this case, the Harder-Narasimhan Type of the limit bundle (E^0, Φ^0) is the vector:

$$HNT(E^0, \Phi^0) : \vec{\lambda} = (\lambda_1, \lambda_3, \lambda_2)$$

where $\lambda_j = \mu(L_j)$. In this subcase, $HNT(E^0, \Phi^0)$ coincides with $HNT(E, \Phi)$ if and only if $\lambda_3 = \lambda_2 = \mu_2 = \mu(V_2)$.

4.2.2 Case (2)

Similarly, suppose that E is an unstable vector bundle of $\mathrm{rk}(E) = 3$ with a Harder-Narasimhan Filtration of length 1:

$$HNF(E) : \ 0 \subset E_1 \subset E$$

but this time E_1 is the maximal destabilizing subbundle of E : $\mu(E_1) > \mu(E)$ with $\mathrm{rk}(E_1) = 2$ where $V_1 = E_1, \ V_2 = E/E_1$ are semi-stables. Then, there is a smooth decomposition $E = V_1 \oplus V_2$ from the short exact sequence:

$$0 \longrightarrow V_1 \longrightarrow E \longrightarrow V_2 \longrightarrow 0.$$

Hence, once again, the Higgs field Φ takes the form:

$$\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$$

where this time $0 \neq \phi_{21} : V_1 \longrightarrow V_2 \otimes K$ is a block of size (2×1), and every block $\phi_{ij} \in \Omega^{1,0}(X, Hom(V_j, V_i) \otimes K)$. Furthermore, the representative holomorphic structure of E, $\bar{\partial}_E$ takes the upper triangular form:

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_1 & \beta \\ 0 & \bar{\partial}_2 \end{pmatrix}$$

where $\bar{\partial}_j$ is the corresponding holomorphic structure of V_j, and $\beta \in \Omega^{0,1}(X, Hom(V_2, V_1))$.

Denote by $d_1 = \deg(V_1)$ and $d_2 = \deg(V_2)$, where $d_2 = d - d_1$. We also note that V_2 satisfies:

a. $\mathrm{rk}(V_2) = 1$

b. V_2 is semi-stable

c. $\mu(V_2) < \mu(E) < \mu(V_1)$

Once again, these are general properties of the Harder-Narasimhan Filtration.

Define $N := ker(\phi_{21}) \subset V_1$ and recall once again that we understand by this the subbundle that we obtain saturating the respective subsheaf. Besides, $\mathrm{rk}(N) = 1$.

Recall that we have defined $(E^0, \Phi^0) := \lim_{z \to 0}(E, z \cdot \Phi)$. We must consider then, two subcases:

(1) $\mu(N) < \mu(E)$

(2) $\mu(N) \geqslant \mu(E)$

Case (2.1)

Proposition 4.2.6. *Suppose that $\mu(N) < \mu(E)$. Then: (E^0, Φ^0) is a $(2, 1)$-VHS.*

Proof. All we need to do, is to consider

$$g(z) := \begin{pmatrix} I & 0 \\ 0 & z \end{pmatrix} \in GL_3(\mathbb{C})$$

where $I \in GL_2(\mathbb{C})$ is the identity matrix, and $g \in \mathcal{G}$ defines a gauge transformation. Then:

$$g(z) * (z \cdot \Phi) = g(z)^{-1}(z \cdot \Phi)g(z) =$$

$$= \begin{pmatrix} z \cdot \phi_{11} & z^2 \cdot \phi_{12} \\ \phi_{21} & z \cdot \phi_{22} \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ \phi_{21} & 0 \end{pmatrix} \text{ when } z \to 0$$

and also:

$$g(z) * \bar{\partial}_E = g(z)^{-1}\bar{\partial}_E g(z) =$$

$$= \begin{pmatrix} \bar{\partial}_1 & z \cdot \beta \\ 0 & \bar{\partial}_2 \end{pmatrix} \to \begin{pmatrix} \bar{\partial}_1 & 0 \\ 0 & \bar{\partial}_2 \end{pmatrix} \text{ when } z \to 0.$$

To prove that Φ^0 is holomorphic since Φ is, we may proceed as before, as what we have done in 4.2.1. The proof is the same.

There are three kinds of Φ^0-invariant subbundles: those ones isomorphic to N, those ones isomorphic to V_2, and those ones isomorphic to $F = L \oplus V_2$ where $L \subset V_1$ is any line bundle. Everything is fine with N since, by hypothesis, in this subcase $\mu(N) < \mu(E)$. On the other hand, $\mu(V_2) < \mu(E)$ since $\mu(E) < \mu(E_1)$ and $V_2 = E/E_1$. Let's see what happen to $F = L \oplus V_2$:
Since $V_1 = E_1$ is the maximal destabilizing subbundle of E, and since $\mu(V_2) < \mu(E) < \mu(V_1)$ where V_1 and V_2 are semistable, we have:

$$\mu(F) = \frac{1}{2}(\mu(L) + \mu(V_2)) \leqslant \frac{1}{2}(\mu(V_1) + \mu(V_2)) < \frac{2}{3}\mu(V_1) + \frac{1}{3}\mu(V_2) = \mu(E) = \mu(E^0)$$

Hence:

$$(E^0, \Phi^0) := \lim_{z \to 0}(E, z \cdot \Phi) = (V_1 \oplus V_2, \begin{pmatrix} 0 & 0 \\ \phi_{21} & 0 \end{pmatrix}) \text{ is stable.}$$

♠

Remark 4.2.7. In this case, the Harder-Narasimhan Type of the limit bundle (E^0, Φ^0) is the vector:

$$HNT(E^0, \Phi^0) : \vec{\nu} = (\nu_1, \nu_1, \nu_2)$$

where $\nu_j = \mu(V_j)$, and besides $\nu_j = \mu_j$. So, in this subcase we got

$$HNT(E^0, \Phi^0) = HNT(E, \Phi).$$

Case (2.2)

Proposition 4.2.8. *If $\mu(N) \geqslant \mu(E)$, then: (E^0, Φ^0) is a $(1,1,1)$-VHS.*

Proof. Suppose $\mu(N) \geqslant \mu(E)$, consider then the smooth splitting $V_1 = N \oplus Q$, from the short exact sequence

$$0 \longrightarrow N \longrightarrow V_1 \longrightarrow Q \longrightarrow 0$$

where $Q := V_1/N$. Then, there is a new smooth splitting

$$E = N \oplus Q \oplus V_2 = L_1 \oplus L_2 \oplus L_3$$

where $L_1 := N,\ L_2 := Q,$ and $L_3 := V_2$. Hence, we may re-write the Higgs field Φ as:

$$\Phi = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{pmatrix} = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ 0 & \varphi_{32} & \varphi_{33} \end{pmatrix}$$

where every block $\varphi_{ij} \in \Omega^{1,0}(X, Hom(L_j, L_i) \otimes K)$, and $\varphi_{31} \equiv 0$ since:

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & E \otimes K \\ \uparrow{\scriptstyle \imath_1} & & \downarrow{\scriptstyle \jmath_3 \otimes id} \\ L_1 & \xrightarrow[\varphi_{31}]{} & L_3 \otimes K \end{array}$$

where, by definition, $L_1 = N$, $L_3 = V_2$ and N is the saturated sheaf of $ker(\phi_{21})$, hence $\varphi_{31} \equiv 0$.

We shall take

$$g(z) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z^2 \end{pmatrix} \in GL_3(\mathbb{C})$$

where $g \in \mathcal{G}$ defines a gauge transformation, and then:

$$g(z) * (z \cdot \Phi) = g(z)^{-1}(z \cdot \Phi)g(z) =$$

$$\begin{pmatrix} z \cdot \varphi_{11} & z^2 \cdot \varphi_{12} & z^3 \cdot \varphi_{13} \\ \varphi_{21} & z \cdot \varphi_{22} & z^2 \cdot \varphi_{23} \\ 0 & \varphi_{32} & z \cdot \varphi_{33} \end{pmatrix} \xrightarrow[z \to 0]{} \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix}.$$

Besides, $\bar{\partial}_E$ the holomorphic structure of E may be expressed as

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_1 & \beta_{12} & \beta_{13} \\ 0 & \bar{\partial}_2 & \beta_{23} \\ 0 & 0 & \bar{\partial}_3 \end{pmatrix}$$

in terms of $\bar{\partial}_j$, which corresponds to the holomorphic structure of L_j, and $\beta_{ij} \in \Omega^{0,1}(X, Hom(L_j, L_i))$.

Then:

$$g(z) * \bar{\partial}_E = g(z)^{-1}\bar{\partial}_E g(z) =$$

$$= \begin{pmatrix} \bar{\partial}_1 & z \cdot \beta_{12} & z^2 \cdot \beta_{13} \\ 0 & \bar{\partial}_2 & z \cdot \beta_{23} \\ 0 & 0 & \bar{\partial}_3 \end{pmatrix} \longrightarrow \begin{pmatrix} \bar{\partial}_1 & 0 & 0 \\ 0 & \bar{\partial}_2 & 0 \\ 0 & 0 & \bar{\partial}_3 \end{pmatrix} \text{ when } z \to 0.$$

The proof that Φ^0 is holomorphic since Φ is, is exactly the same proof that we presented in 4.2.1. To prove that (E^0, Φ^0) is stable, we must consider the Φ^0-invariant subbundles of E^0: those ones isomorphic to $L_3 := V_2$, and those ones isomorphic to

$L_2 \oplus L_3 = Q \oplus V_2$.
$\mu(L_3) < \mu(E^0)$ trivially, since $V_1 = E_1$ is the maximal destabilizing subbundle of E and $V_2 = E/E_1$, then $\mu(V_2) < \mu(E) < \mu(V_1)$.

Besides, recall that $\mu(N) \geqslant \mu(E) = \mu(E^0)$

$$i.e. \quad \mu(N) = \mu(L_1) \geqslant \frac{1}{3}(\mu(L_1) + \mu(L_2) + \mu(L_3)) = \mu(E^0) \iff$$

$$3\mu(L_1) \geqslant \mu(L_1) + \mu(L_2) + \mu(L_3) \iff 2\mu(L_1) \geqslant \mu(L_2) + \mu(L_3) \iff$$

$$2(\mu(L_1) + \mu(L_2) + \mu(L_3)) \geqslant 3(\mu(L_2) + \mu(L_3)) \iff$$

$$\frac{1}{3}(\mu(L_1) + \mu(L_2) + \mu(L_3)) \geqslant \frac{1}{2}(\mu(L_2) + \mu(L_3))$$

$$\therefore \mu(E^0) \geqslant \mu(L_2 \oplus L_3).$$

Once again, what we have shown is that (E^0, Φ^0) is semistable, but $GCD(3, d) = 1$

$$\therefore (E^0, \Phi^0) = \lim_{z \to 0}(E, z \cdot \Phi) = (L_1 \oplus L_2 \oplus L_3, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix}) \text{ is stable.}$$

♠

Remark 4.2.9. Since E_1 is semi-stable, $\mu(N) \leqslant \mu(E_1)$, and so $\mu(N) \leqslant \mu(Q)$. Then, in this case, the Harder-Narasimhan Type of the limit bundle (E^0, Φ^0) is the vector:

$$HNT(E^0, \Phi^0) : \vec{\lambda} = (\lambda_2, \lambda_1, \lambda_3)$$

where $\lambda_j = \mu(L_j)$. In this subcase, $HNT(E^0, \Phi^0)$ coincides with $HNT(E, \Phi)$ if and only if $\lambda_2 = \lambda_1 = \mu_1 = \mu(V_1)$.

4.2.3 Case (3)

Finally, suppose that (E, Φ) is a Higgs Bundle where E is an unstable vector bundle of $\mathrm{rk}(E) = 3$ with a Harder-Narasimhan Filtration of length 2:

$$HNF(E) : \; 0 = E_0 \subset E_1 \subset E_2 \subset E_3 = E$$

where $\mu(E_1) > \mu(E_2) > \mu(E)$ and $V_1 = E_1,\ V_2 = E_2/E_1,$ and $V_3 = E/E_2$ are semi-stables.

There is a smooth decomposition $E = L_1 \oplus L_2 \oplus L_3 = V_1 \oplus V_2 \oplus V_3$ from the short exact sequences

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow V_2 \longrightarrow 0$$

$$0 \longrightarrow E_2 \longrightarrow E \longrightarrow V_3 \longrightarrow 0.$$

Nevertheless, we can not apply similar proceedings to what we did before, since the Higgs field Φ takes the form

$$\Phi = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{pmatrix}$$

where $\varphi_{31} : L_1 \to L_3 \otimes K$ is not necessarily zero, and the gauge transformation $g \in \mathcal{G}$ given by

$$g(z) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z^2 \end{pmatrix} \in GL_3(\mathbb{C}),$$

will give us

$$g(z) * (z \cdot \Phi) = g(z)^{-1}(z \cdot \Phi)g(z) =$$

$$\begin{pmatrix} z \cdot \varphi_{11} & z^2 \cdot \varphi_{12} & z^3 \cdot \varphi_{13} \\ \varphi_{21} & z \cdot \varphi_{22} & z^2 \cdot \varphi_{23} \\ \frac{1}{z} \cdot \varphi_{31} & \varphi_{32} & z \cdot \varphi_{33} \end{pmatrix}$$

and there is a term of the form $\frac{1}{z}\varphi_{31}$ before we take the limit when $z \to 0$.

We may also think in smooth decompositions of the form

$$E = E_1 \oplus (E/E_1) \text{ or } E = E_2 \oplus (E/E_2)$$

and trying to work the way we did before. However, we are in troubles again, since E_2 and E/E_1 are not semistables:

a. in the first case, $E_1 \subset E_2$ where $\mu(E_1) > \mu(E_2) > \mu(E)$,

b. and in the second case, $E_2/E_1 \subset E/E_1$ where $\mu(E_2/E_1) > \mu(E)$ could also happen.

It seems that these subcases could be worked as above, whereas $\mu(E_2/E_1) < \mu(E)$ or not. Recall, once again, that we have defined $(E^0, \Phi^0) := \lim_{z \to 0}(E, z \cdot \Phi)$, and consider the cases:

(1) $\mu(E_2/E_1) < \mu(E)$

(2) $\mu(E_2/E_1) \geqslant \mu(E)$

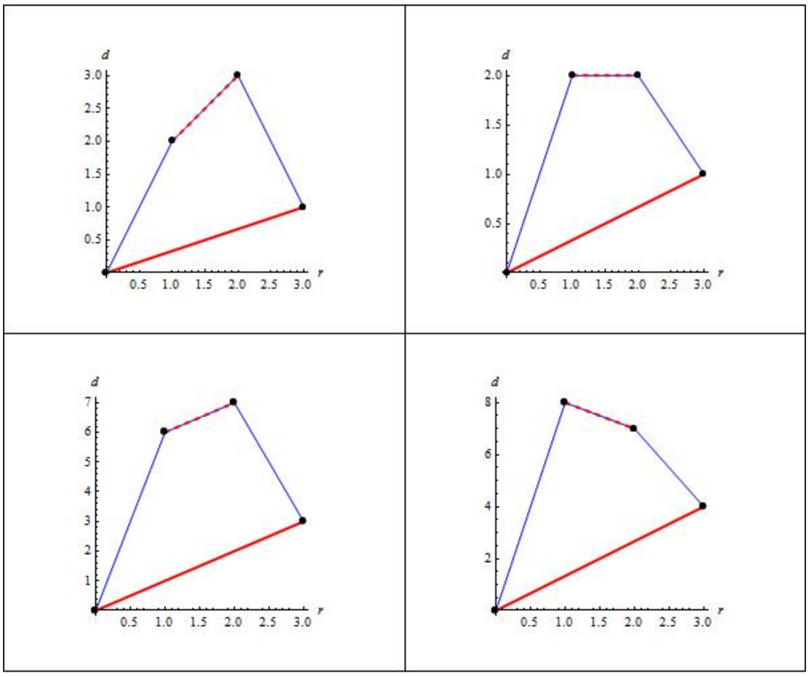


Figure 2: Harder-Narasimhan Polygons with the possible cases mentioned above, the red-dashed line segments represent segments with slope $\mu(V_2) = \mu(E_2/E_1)$ and the red-thick line segments represent segments with slope $\mu(E)$. From left to right, from top to bottom, $\mu(V_2) > \mu(E),\ \mu(V_2) < \mu(E),\ \mu(V_2) = \mu(E),\ \mu(V_2) < \mu(E)$.

Case (3.1)

Suppose that $\mu(E_2/E_1) < \mu(E)$. $E_2/E_1 \subset E/E_1$ is the maximal line bundle such that $\mu(E_2/E_1) > \mu(E/E_1)$. In this case, we will consider the smooth decomposition $E = W_1 \oplus W_2$ from the short exact sequence

$$0 \to W_1 \to E \to W_2 \to 0$$

where $W_1 = E_1$ with $\mathrm{rk}(W_1) = 1$ and $W_2 = E/E_1$ with $\mathrm{rk}(W_2) = 2$; and then, the Higgs field Φ takes the form:

$$\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$$

where $0 \neq \phi_{21} : W_1 \longrightarrow W_2 \otimes K$ is a (1×2)-size block, and every single block $\phi_{ij} \in \Omega^{1,0}(X, Hom(W_j, W_i) \otimes K)$. As well as we did in 4.2.1, we consider $\mathcal{I} \subset V_2$ as the saturated bundle of $\phi_{21}(E_1) \otimes K^{-1}$ where $\mathrm{rk}(\mathcal{I}) = 1$, and we consider also $F = V_1 \oplus \mathcal{I} \subset V_1 \oplus V_2 = E$ where $\mathrm{rk}(F) = 2$.

Recall that we have defined the pair $(E^0, \Phi^0) := \lim_{z \to 0}(E, z \cdot \Phi)$.

Proposition 4.2.10. *With the conditions mentioned above, we have two possibilities:*

i. Suppose that $\mu(F) < \mu(E)$. Then: (E^0, Φ^0) is a $(1, 2)$-VHS.

ii. On the other hand, if $\mu(F) \geqslant \mu(E)$, then: (E^0, Φ^0) is a $(1, 1, 1)$-VHS.

Proof. The proof is essentially the same presented in 4.2.1 and 4.2.1, except for one detail: in $i.$, as we have already mentioned, W_2 is not semistable and, indeed, its maximal destabilizing line bundle is E_2/E_1, but there is nothing to worry about in this case, since we have supposed that $\mu(E_2/E_1) < \mu(E)$, and it gives us stability. ♠

When the limit bundle (E^0, Φ^0) is a $(1, 2)$-VHS, it takes the form:

$$(E^0, \Phi^0) = (W_1 \oplus W_2, \begin{pmatrix} 0 & 0 \\ \phi_{21} & 0 \end{pmatrix})$$

where $W_1 = E_1 = V_1$ and $W_2 = E/E_1 \cong V_2 \oplus V_3$. Then, the Harder-Narasimhan Type of (E^0, Φ^0) is the vector:

$$HNT(E^0, \Phi^0) : \vec{\mu} = (\mu_1, \mu_2, \mu_3)$$

where $\mu_j = \mu(V_j)$, in other words:

$$HNT(E^0, \Phi^0) = HNT(E, \Phi).$$

On the other hand, since here E/E_1 is not semi-stable, we cannot ensure that $\mu(\mathcal{I}) \leqslant \mu(\mathcal{Q})$ as we did in 4.2.1, so the Harder-Narasimhan Type of the limit bundle (E^0, Φ^0) when it is a $(1, 1, 1)$-VHS, is either $(\lambda_1, \lambda_2, \lambda_3)$ or $(\lambda_1, \lambda_3, \lambda_2)$ where $\lambda_j = \mu(L_j)$ and the order of the second and the third entries will depend on who is larger: λ_2 or λ_3. Hence:

$$HNT(E^0,\Phi^0) = HNT(E,\Phi) \Leftrightarrow \begin{cases} (\lambda_1,\lambda_2,\lambda_3) = (\mu_1,\mu_2,\mu_3) & \text{when } \mu(Q) < \mu(\tilde{\mathcal{I}}) \\ \\ (\lambda_1,\lambda_3,\lambda_2) = (\mu_1,\mu_2,\mu_3) & \text{when } \mu(Q) > \mu(\tilde{\mathcal{I}}) \end{cases}$$

Case (3.2)

Suppose now that $\mu(E_2/E_1) \geqslant \mu(E)$. This time, our main concern is that E_2 is not semistable, and actually, $E_1 \subset E_2$ is its maximal destabilizing line subbundle. In this case, we will consider the smooth decomposition $E = W_1 \oplus W_2$ from the short exact sequence

$$0 \to W_1 \to E \to W_2 \to 0$$

where $W_1 = E_2$ with $\operatorname{rk}(W_1) = 2$ and $W_2 = E/E_2$ with $\operatorname{rk}(W_2) = 1$; and then, the Higgs field Φ takes the form:

$$\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$$

but now $0 \neq \phi_{21} : W_1 \longrightarrow W_2 \otimes K$ is a (2×1)-size block, and every single block $\phi_{ij} \in \Omega^{1,0}(X, Hom(W_j, W_i) \otimes K)$. As well as we did in 4.2.2, we consider $N \subset W_1$ as the saturated bundle of $ker(\phi_{21})$, where $\operatorname{rk}(N) = 1$.

Proposition 4.2.11. *With the conditions mentioned above, we have two chances:*

i. If $\mu(N) < \mu(E)$. Then: (E^0,Φ^0) is a $(2,1)$-VHS.

ii. On the other hand, if $\mu(N) \geqslant \mu(E)$, then: (E^0,Φ^0) is a $(1,1,1)$-VHS.

Proof. Basically, the same presented in 4.2.2 and 4.2.2, except for one thing: this time in i. $W_1 = E_2$ is not semistable. Furthermore, $E_1 \subset E_2$ is its maximal destabilizing line subbundle. So, $E_1 \oplus W_2$ could be destabilizing. Nevertheless, we supposed this time that $\mu(E_2/E_1) \geqslant \mu(E)$ or, equivalently:

$$\mu(E) \leqslant 2\mu(E_2) - \mu(E_1) \Longleftrightarrow 3\mu(E) \leqslant 2\mu(E_2) - \mu(E_1) + 2\mu(E) \Longleftrightarrow$$

$$\mu(E_1) + (3\mu(E) - 2\mu(E_2)) \leqslant 2\mu(E) \Longleftrightarrow \mu(E_1) + \mu(E/E_2) \leqslant 2\mu(E)$$

$$\Longleftrightarrow \mu(E_1) + \mu(W_2) \leqslant 2\mu(E) \Longleftrightarrow \mu(E_1 \oplus W_2) \leqslant \mu(E).$$

Since $GCD(3, d) = 1$, we get stability in this subcase, finishing the proof. ♠

When the limit bundle (E^0, Φ^0) is a $(2, 1)$-VHS, it takes the form:

$$(E^0, \Phi^0) = \big(W_1 \oplus W_2, \begin{pmatrix} 0 & 0 \\ \phi_{21} & 0 \end{pmatrix}\big)$$

where $W_1 = E_2 \cong E_1 \oplus E_2/E_1 = V_1 \oplus V_2$ and $W_2 = E/E_2 = V_3$. Then, the Harder-Narasimhan Type of (E^0, Φ^0) is the vector:

$$HNT(E^0, \Phi^0) : \vec{\mu} = (\mu_1, \mu_2, \mu_3)$$

where $\mu_j = \mu(V_j)$, in other words:

$$HNT(E^0, \Phi^0) = HNT(E, \Phi).$$

On the other hand, since here E_2 is not semi-stable, we cannot ensure that $\mu(N) \leqslant \mu(Q)$ as we did in 4.2.2, so the Harder-Narasimhan Type of the limit bundle (E^0, Φ^0) when it is a $(1, 1, 1)$-VHS, is either $(\lambda_1, \lambda_2, \lambda_3)$ or $(\lambda_2, \lambda_1, \lambda_3)$ where $\lambda_j = \mu(L_j)$ and the order of the first and the second entries depend on who is larger: λ_2 or λ_1. Hence:

$$HNT(E^0, \Phi^0) = HNT(E, \Phi) \Leftrightarrow \begin{cases} (\lambda_1, \lambda_2, \lambda_3) = (\mu_1, \mu_2, \mu_3) & \text{when } \mu(Q) > \mu(N) \\ \\ (\lambda_2, \lambda_1, \lambda_3) = (\mu_1, \mu_2, \mu_3) & \text{when } \mu(Q) < \mu(N) \end{cases}$$

4.2.4 The Harder-Narasimhan Type

It would be interesting to ask what happen the other way around: given a limit point $(E^0, \Phi^0) \in F_\lambda$, what is its Harder-Narasimhan Type and, does this $HNT(E^0, \Phi^0)$ coincides with the Harder-Narasimhan Type of (E, Φ) the original bundle, $HNT(E, \Phi)$?

We have already mentioned what the Harder-Narasimhan Type is, but will be very useful if we write it down properly for every single type of critical point. To do that, we will consider the following notation:

Given F_λ a critical submanifold of $\mathcal{M}$, we will denote

$$U_\lambda^+ = \left\{(E,\Phi) \in \mathcal{M} : \lim_{z\to 0}(E, z\cdot\Phi) \in F_\lambda\right\}$$

as the λ-upper-flow Morse subset of $\mathcal{M}$. Recall that these sets will give us a stratification of the moduli space:

$$\mathcal{M} = \bigcup_\lambda U_\lambda^+$$

known as the Morse Stratification and, according to Kirwan [27], equivalent to the Bialynicki-Birula Stratification.

On the other hand, we will denote

$$U_{\vec{\mu}} := \left\{(E,\Phi) \in \mathcal{M} : HNT(E,\Phi)) = \vec{\mu}\right\}$$

as the $\vec{\mu}$-Shatz component of the Shatz Stratification:

$$\mathcal{M} = \bigcup_{\vec{\mu}} U_{\vec{\mu}}.$$

Recall that we denote the pair $(E^0,\Phi^0) := \lim_{z\to 0}(E, z\cdot\Phi)$.

So far, what we know, for the rank three case is that for a given point $(E,\Phi) \in U_{\vec{\mu}}$, there is a particular λ such that $(E,\Phi) \in U_\lambda^+$:

(1) If $(E,\Phi) \in U_{\vec{\mu}}$ with

$$HNF(E,\Phi) : 0 = E_0 \subset E_1 \subset E_2 = E$$

where $\mathrm{rk}(E_1) = 1,\ \mu(E_1) > \mu(E)$ and $V_1 = E_1,\ V_2 = E/E_1$ are semi-stables, then $\vec{\mu} = (\mu_1, \mu_2, \mu_2)$ where $\mu_j = \mu(V_j)$. Then, by the results showed in 4.2.1 and 4.2.1, and considering the sheaf $\mathcal{I} := \phi_{21}(V_1) \otimes K^{-1} \subset V_2$, its saturation $\tilde{\mathcal{I}}$, where

$\mathrm{rk}(\tilde{\mathcal{I}}) = 1$, and also $F := V_1 \oplus \tilde{\mathcal{I}} \subset V_1 \oplus V_2 = E$ where $\mathrm{rk}(F) = 2$, we have two possibilities:
Either

a.
$$(E^0, \Phi^0) = (V_1 \oplus V_2, \begin{pmatrix} 0 & 0 \\ \phi_{21} & 0 \end{pmatrix})$$

is a $(1,2)$-VHS if $\mu(F) < \mu(E)$, and hence, $(E, \Phi) \in F_{d_1}^{(1,2)}$ where $d_1 = \deg(V_1) \in]\frac{d}{3}, \frac{d}{3} + \frac{d_K}{2}[\cap \mathbb{Z}$ (for more details, see Bento [3] or Gothen [14]).
Or

b.
$$(E^0, \Phi^0) = (L_1 \oplus L_2 \oplus L_3, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix})$$

is a $(1,1,1)$-VHS, and so, $(E, \Phi) \in F_{m_1 m_2}^{(1,1,1)}$ where $(m_1, m_2) \in \Omega$ where $M_j := L_j^* L_{j+1} K,\ m_j := \deg(M_j) = d_{j+1} - d_j + d_K$, and

$$\Omega = \left\{ (m_1, m_2) \in \mathbb{N} \times \mathbb{N} \middle| \begin{array}{c} 2m_1 + m_2 < 3d_K \\ m_1 + 2m_2 < 3d_K \\ m_1 + 2m_2 \equiv 0 (mod 3) \end{array} \right\}.$$

For more details of the description of Ω, the reader can see Gothen [14], or Bento [3].
In this case, $L_1 \oplus L_2 \oplus L_3 = V_1 \oplus \tilde{\mathcal{I}} \oplus \mathcal{Q}$.

Hence, summarizing, if $(E, \Phi) \in U_{\vec{\mu}}$ with $\vec{\mu} = (\mu_1, \mu_2, \mu_2)$ then

$$\begin{cases} (E^0, \Phi^0) \in U_{\vec{\mu}} & \text{if} \quad \mu(V_1 \oplus \tilde{\mathcal{I}}) < \mu(E) \\ \\ (E^0, \Phi^0) \in U_{\vec{\lambda}} & \text{if} \quad \mu(V_1 \oplus \tilde{\mathcal{I}}) \geqslant \mu(E) \end{cases}$$

where
$$\vec{\lambda} = (\lambda_1, \lambda_3, \lambda_2)$$

since $\lambda_2 = \mu(\tilde{\mathcal{I}}) \leqslant \mu(\mathcal{Q}) = \lambda_3$. Note that we could have $\lambda_2 = \lambda_3$, and in such a case $\lambda_2 = \lambda_3 = \mu_2$, which implies $\vec{\mu} = \vec{\lambda}$. In other words, if $\lambda_2 = \lambda_3$, then $HNT(E, \Phi) = HNT(E^0, \Phi^0)$.

(2) If $(E, \Phi) \in U_{\vec{\mu}}$ with

$$HNF(E, \Phi) : 0 = E_0 \subset E_1 \subset E_2 = E$$

where $\mathrm{rk}(E_1) = 2,\ \mu(E_1) > \mu(E)$ and $V_1 = E_1,\ V_2 = E/E_1$ are semi-stables, then $\vec{\mu} = (\mu_1, \mu_1, \mu_2)$ where $\mu_j = \mu(V_j)$. Then, by the results showed in 4.2.2 and 4.2.2, and considering the sheaf $N := ker(\phi_{21})$, and its saturation N such that $\mathrm{rk}(N) = 1$ and $N \subset N \subset V_1$, we also have two possibilities:
Either

a.
$$(E^0, \Phi^0) = (V_2 \oplus V_1, \begin{pmatrix} 0 & 0 \\ \phi_{21} & 0 \end{pmatrix})$$

is a $(2,1)$-VHS if $\mu(N) < \mu(E)$, and hence, $(E, \Phi) \in F_{d_2}^{(2,1)}$ where $d_2 = \deg(V_2) \in]\frac{2d}{3}, \frac{2d}{3} + \frac{d_K}{2}[\cap \mathbb{Z}$ (for more details, see Bento [3] or Gothen [14].)
Or

b.
$$(E^0, \Phi^0) = (L_1 \oplus L_2 \oplus L_3, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix})$$

is a $(1,1,1)$-VHS otherwise, and so, $(E, \Phi) \in F_{m_1 m_2}^{(1,1,1)}$ where $(m_1, m_2) \in \Omega$ where $M_j := L_j^* L_{j+1} K,\ m_j := \deg(M_j) = d_{j+1} - d_j + d_K$, and

$$\Omega = \left\{ (m_1, m_2) \in \mathbb{N} \times \mathbb{N} \middle| \begin{array}{c} 2m_1 + m_2 < 3d_K \\ m_1 + 2m_2 < 3d_K \\ m_1 + 2m_2 \equiv 0 (mod 3) \end{array} \right\}.$$

In this case, $L_1 \oplus L_2 \oplus L_3 = N \oplus Q \oplus V_2$.

Then, if $(E,\Phi)\in U_{\vec{\mu}}$ with $\vec{\mu}=(\mu_1,\mu_1,\mu_2)$ then

$$\begin{cases} (E^0,\Phi^0)\in U_{\vec{\mu}} & \text{if} \quad \mu(N)<\mu(E) \\ \\ (E^0,\Phi^0)\in U_{\vec{\lambda}} & \text{if} \quad \mu(N)\geqslant\mu(E) \end{cases}$$

where

$$\vec{\lambda}=(\lambda_2,\lambda_1,\lambda_3)$$

since $\lambda_1=\mu(N)\leqslant\mu(Q)=\lambda_2$. Note that we could have $\lambda_1=\lambda_2$, and in such a case $\lambda_1=\lambda_2=\mu_1$, which implies $\vec{\mu}=\vec{\lambda}$. In other words, if $\lambda_1=\lambda_2$, then $HNT(E,\Phi)=HNT(E^0,\Phi^0)$.

(3) If $(E,\Phi)\in U_{\vec{\mu}}$ with

$$HNF(E,\Phi):0=E_0\subset E_1\subset E_2\subset E_3=E$$

where $\mathrm{rk}(E_1)=1,\ \mathrm{rk}(E_2)=2,\ \mu(E_1)>\mu(E_2)>\mu(E),\ V_1=E_1,\ V_2=E_2/E_1$ and $V_3=E/E_2$ are semi-stables, then $\vec{\mu}=(\mu_1,\mu_2,\mu_3)$ where $\mu_j=\mu(V_j)$. Then, by the results showed in 4.2.3 and 4.2.3, and considering the subbundles $\tilde{\mathcal{I}},\ \mathcal{Q},\ N$ and Q as above, we have four possibilities:

Either $\mu(V_2)<\mu(E)$ and then:

a.
$$(E^0,\Phi^0)=(V_1\oplus V_2,\begin{pmatrix} 0 & 0 \\ \phi_{21} & 0 \end{pmatrix})$$

is a $(1,2)$-VHS if $\mu(F)<\mu(E)$, and hence, $(E,\Phi)\in F^{(1,2)}_{d_1}$ where $d_1=\deg(V_1)\in\,]\frac{d}{3},\frac{d}{3}+\frac{d_K}{2}[\,\cap\mathbb{Z}$, or

b.
$$(E^0,\Phi^0)=(L_1\oplus L_2\oplus L_3,\begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix})$$

is a $(1,1,1)$-VHS if $\mu(F)>\mu(E)$, and so, $(E,\Phi)\in F^{(1,1,1)}_{m_1m_2}$ where $(m_1,m_2)\in$

Ω where $M_j := L_j^* L_{j+1} K,\ m_j := \deg(M_j) = d_{j+1} - d_j + d_K$, and

$$\Omega = \left\{ (m_1, m_2) \in \mathbb{N} \times \mathbb{N} \middle| \begin{array}{c} 2m_1 + m_2 < 3d_K \\ m_1 + 2m_2 < 3d_K \\ m_1 + 2m_2 \equiv 0(mod3) \end{array} \right\}.$$

Or, $\mu(V_2) > \mu(E)$ and so:

c.

$$(E^0, \Phi^0) = (V_2 \oplus V_1, \begin{pmatrix} 0 & 0 \\ \phi_{21} & 0 \end{pmatrix})$$

is a $(2,1)$-VHS if $\mu(N) < \mu(E)$, and hence, $(E, \Phi) \in F_{d_2}^{(2,1)}$ where $d_2 = \deg(V_2) \in \,]\frac{2d}{3}, \frac{2d}{3} + \frac{d_K}{2}[\, \cap \mathbb{Z}$, or

d.

$$(E^0, \Phi^0) = (L_1 \oplus L_2 \oplus L_3, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix})$$

is a $(1,1,1)$-VHS if $\mu(N) > \mu(E)$, and so, $(E, \Phi) \in F_{m_1 m_2}^{(1,1,1)}$ where $(m_1, m_2) \in \Omega$ where $M_j := L_j^* L_{j+1} K,\ m_j := \deg(M_j) = d_{j+1} - d_j + d_K$, and

$$\Omega = \left\{ (m_1, m_2) \in \mathbb{N} \times \mathbb{N} \middle| \begin{array}{c} 2m_1 + m_2 < 3d_K \\ m_1 + 2m_2 < 3d_K \\ m_1 + 2m_2 \equiv 0(mod3) \end{array} \right\}.$$

In this case, $L_1 \oplus L_2 \oplus L_3 = N \oplus Q \oplus V_2$.

Therefore, summarizing, we have:

$$(E,\Phi)\in U_{\vec{\mu}}\Rightarrow\begin{cases}\text{if}\quad \mu(V_2)<\mu(E)\Rightarrow\begin{cases}(E^0,\Phi^0)\in U_{\vec{\mu}} & \text{if}\quad \mu(V_1\oplus\tilde{\mathcal{I}})<\mu(E)\\ \\ (E^0,\Phi^0)\in U_{\vec{\lambda}} & \text{if}\quad \mu(V_1\oplus\tilde{\mathcal{I}})\geqslant\mu(E)\end{cases}\\ \\ \text{if}\quad \mu(V_2)\geqslant\mu(E)\Rightarrow\begin{cases}(E^0,\Phi^0)\in U_{\vec{\mu}} & \text{if}\quad \mu(N)<\mu(E)\\ \\ (E^0,\Phi^0)\in U_{\vec{\rho}} & \text{if}\quad \mu(N)\geqslant\mu(E)\end{cases}\end{cases}$$

where

$$\vec{\lambda}=\begin{cases}(\lambda_1,\lambda_2,\lambda_3) & \text{if}\quad \lambda_2>\lambda_3\\ \\ (\lambda_1,\lambda_3,\lambda_2) & \text{if}\quad \lambda_2\leqslant\lambda_3\end{cases}$$

and

$$\vec{\rho}=\begin{cases}(\rho_1,\rho_2,\rho_3) & \text{if}\quad \rho_1>\rho_2\\ \\ (\rho_2,\rho_1,\rho_3) & \text{if}\quad \rho_1\leqslant\rho_2\end{cases}$$

With the information mentioned above, we can split $U_{\vec{\mu}}$ in terms of its λ-components:

$$U_{\vec{\mu}}=\bigcup_{\lambda}U_{\vec{\mu}\lambda}$$

where we are defining $U_{\vec{\mu}\lambda}:=U_{\vec{\mu}}\cap U_{\lambda}^{+}\quad\forall\lambda$.

Clearly the Shatz Stratification of U_{λ}^{+} will be

$$U_{\lambda}^{+}=\bigcup_{\vec{\mu}}U_{\vec{\mu}\lambda}.$$

We will write down the correspoding decomposition of $F_{\lambda}=\bigcup\limits_{\vec{\mu}}F_{\vec{\mu}\lambda}$ for each VHS, where $F_{\vec{\mu}\lambda}=U_{\vec{\mu}}\cap F_{\lambda}\quad\forall\lambda$.

Variation of Hodge Structure of Type $(1,2)$

Let $F_\lambda = F_{d_1}^{(1,2)}$ be a $(1,2)$-VHS such that $d_1 \in]\frac{d}{3}, \frac{d}{3} + \frac{d_K}{2}[\cap \mathbb{Z}$. In this case there are two components, $i.e.$ for a pair $(E^0, \Phi^0) \in F_{d_1}^{(1,2)}$ we have two possibilities:

1.
$$(E^0, \Phi^0) = (V_1 \oplus V_2, \begin{pmatrix} 0 & 0 \\ \phi_{21} & 0 \end{pmatrix})$$

comes from $(E, \Phi) \in U_{\vec{\delta_1}}$ where

$$HNF(E):\ 0 = E_0 \subset E_1 \subset E_2 = E$$

with $\mathrm{rk}(E_1) = 1,\ V_1 = E_1,\ V_2 = E/E_1$ semi-stables and $\mu(V_1) > \mu(V_2)$, and then

$$\vec{\delta_1} = (\mu_1, \mu_2, \mu_2) \text{ where } \mu_j = \mu(V_j).$$

Here, $HNT(E^0, \Phi^0) = \vec{\delta_1} = HNT(E, \Phi)$ since $E^0 = V_1 \oplus V_2$ where $V_j = V_j$.

2.
$$(E^0, \Phi^0) = (V_1 \oplus V_2, \begin{pmatrix} 0 & 0 \\ \phi_{21} & 0 \end{pmatrix})$$

comes from $(E, \Phi) \in U_{\vec{\rho_1}}$ where

$$HNF(E):\ 0 = E_0 \subset E_1 \subset E_2 \subset E_3 = E$$

with $V_j = E_j/E_{j-1}$ semi-stables and $\mu(V_j) > \mu(V_{j+1})$, and then

$$\vec{\rho_1} = (\mu_1, \mu_2, \mu_3) \text{ where } \mu_j = \mu(V_j).$$

Here, $HNT(E^0, \Phi^0) = \vec{\rho_1} = HNT(E, \Phi)$ since $E^0 = V_1 \oplus V_2$ where $V_1 = E_1$ and $V_2 = E/E_1 \cong E_2/E_1 \oplus E/E_2$.

Briefly, we get then two disjoint components:

$$F_\lambda = F_{d_1}^{(1,2)} = F_{\vec{\delta_1} d_1} \sqcup F_{\vec{\rho_1} d_1}.$$

Variation of Hodge Structure of Type $(2,1)$

Let $F_\lambda = F_{d_2}^{(2,1)}$ be a $(2,1)$-VHS such that $d_2 \in]\frac{2d}{3}, \frac{2d}{3} + \frac{d_K}{2}[\cap\mathbb{Z}$. Similarly, there are two possibilities for a pair $(E^0, \Phi^0) \in F_{d_2}^{(2,1)}$:

1.
$$(E^0, \Phi^0) = (V_2 \oplus V_1, \begin{pmatrix} 0 & 0 \\ \phi_{21} & 0 \end{pmatrix})$$

comes from $(E, \Phi) \in U_{\vec{\delta_2}}$ where

$$HNF(E): \ 0 = E_0 \subset E_1 \subset E_2 = E$$

with $\mathrm{rk}(E_1) = 2, \ V_1 = E_1, \ V_2 = E/E_1$ semi-stables and $\mu(V_1) > \mu(V_2)$, and then

$$\vec{\delta_2} = (\mu_1, \mu_1, \mu_2) \text{ where } \mu_j = \mu(V_j).$$

Here, $HNT(E^0, \Phi^0) = \vec{\delta_2} = HNT(E, \Phi)$ since $E^0 = V_2 \oplus V_1$ where $V_2 = \tilde{E}_1$ and $V_1 = \tilde{E}_2$.

2.
$$(E^0, \Phi^0) = (V_1 \oplus V_2, \begin{pmatrix} 0 & 0 \\ \phi_{21} & 0 \end{pmatrix})$$

comes from $(E, \Phi) \in U_{\vec{\rho_2}}$ where

$$HNF(E): \ 0 = E_0 \subset E_1 \subset E_2 \subset E_3 = E$$

with $V_j = E_j/E_{j-1}$ semi-stables and $\mu(V_j) > \mu(V_{j+1})$, and then

$$\vec{\rho_2} = (\mu_1, \mu_2, \mu_3) \text{ where } \mu_j = \mu(V_j).$$

Here, $HNT(E^0, \Phi^0) = \vec{\rho_2} = HNT(E, \Phi)$ since $E^0 = V_2 \oplus V_1$ where $V_2 = E_2 \cong V_1 \oplus V_2$ and $V_1 = E/E_2 = V_3$.

Therefore, we get then two disjoint components:

$$F_\lambda = F_{d_2}^{(2,1)} = F_{\vec{\delta_2} d_2} \sqcup F_{\vec{\rho_2} d_2}.$$

Variation of Hodge Structure of Type $(1,1,1)$

Let $F_\lambda = F^{(1,1,1)}_{m_1 m_2}$ be a $(1,1,1)$-VHS with $(m_1, m_2) \in \Omega$. Here the situation is quite different: for a pair $(E^0, \Phi^0) \in F^{(1,1,1)}_{m_1 m_2}$ we have three components:

1.
$$(E^0, \Phi^0) = (L_1 \oplus L_2 \oplus L_3, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix})$$

comes from $(E, \Phi) \in U_{\vec{\delta_1}}$ where

$$HNF(E):\ 0 = E_0 \subset E_1 \subset E_2 = E$$

with $\mathrm{rk}(E_1) = 1,\ V_1 = E_1,\ V_2 = E/E_1$ semi-stables and $\mu(V_1) > \mu(V_2)$, and then
$$\vec{\delta_1} = (\mu_1, \mu_2, \mu_2) \text{ where } \mu_j = \mu(V_j).$$

Here, we will denote $\ell_j = \mu(L_j)$ where $L_1 = V_1,\ L_2 = \tilde{\mathcal{I}},\ L_3 = \mathcal{Q}$. Hence:

$$HNT(E^0, \Phi^0) = \begin{cases} (\ell_1, \ell_2, \ell_3) & \text{if} \quad \mu(\tilde{\mathcal{I}}) \geqslant \mu(\mathcal{Q}) \\ \\ (\ell_1, \ell_3, \ell_2) & \text{if} \quad \mu(\tilde{\mathcal{I}}) \leqslant \mu(\mathcal{Q}) \end{cases}$$

Therefore:

$$HNT(E, \Phi) = HNT(E^0, \Phi^0) \Leftrightarrow \ell_2 = \ell_3 = \mu_2 \Leftrightarrow \mu(\tilde{\mathcal{I}}) = \mu(\mathcal{Q}) = \mu(V_2).$$

2.
$$(E^0, \Phi^0) = (L_1 \oplus L_2 \oplus L_3, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix})$$

comes from $(E, \Phi) \in U_{\vec{\delta_2}}$ where

$$HNF(E):\ 0 = E_0 \subset E_1 \subset E_2 = E$$

with $\mathrm{rk}(E_1) = 2,\ V_1 = E_1,\ V_2 = E/E_1$ semi-stables and $\mu(V_1) > \mu(V_2)$, and then

$$\vec{\delta_2} = (\mu_1, \mu_1, \mu_2) \text{ where } \mu_j = \mu(V_j).$$

Here, we will denote $\ell_j = \mu(L_j)$ where $L_1 = N,\ L_2 = Q,$ and $L_3 = V_2$. Hence:

$$HNT(E^0, \Phi^0) = \begin{cases} (\ell_1, \ell_2, \ell_3) & \text{if} \quad \mu(N) \geqslant \mu(Q) \\ \\ (\ell_2, \ell_1, \ell_3) & \text{if} \quad \mu(N) \leqslant \mu(Q) \end{cases}$$

Therefore:

$$HNT(E, \Phi) = HNT(E^0, \Phi^0) \Leftrightarrow \ell_1 = \ell_2 = \mu_1 \Leftrightarrow \mu(N) = \mu(Q) = \mu(V_1).$$

3.

$$(E^0, \Phi^0) = (L_1 \oplus L_2 \oplus L_3, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix})$$

comes from $(E, \Phi) \in U_{\vec{\rho_3}}$ where

$$HNF(E):\ 0 = E_0 \subset E_1 \subset E_2 \subset E_3 = E$$

with $V_j = E_j/E_{j-1}$ semi-stables and $\mu(V_j) > \mu(V_{j+1})$, and then

$$\vec{\rho_3} = (\mu_1, \mu_2, \mu_3) \text{ where } \mu_j = \mu(V_j).$$

Here, once again, we will denote $\ell_j = \mu(L_j)$, but this time the situation is quite different:

$$L_1 \oplus L_2 \oplus L_3 = \begin{cases} V_1 \oplus \tilde{\mathcal{I}} \oplus \mathcal{Q} & \text{if} \quad \mu(V_2) < \mu(E) \quad \text{and} \quad \mu(V_1 \oplus \tilde{\mathcal{I}}) \geqslant \mu(E) \\ \\ N \oplus Q \oplus V_2 & \text{if} \quad \mu(V_2) \geqslant \mu(E) \qquad \text{and} \quad \mu(N) \geqslant \mu(E) \end{cases}$$

Recall that the cases $\mu(V_1 \oplus \tilde{\mathcal{I}}) < \mu(E)$ and $\mu(N) < \mu(E)$, belong to the components of the VHS of type $(1, 2)$ and $(2, 1)$ respectively.

Hence:

$$HNT(E,\Phi) = HNT(E^0,\Phi^0) \Leftrightarrow \begin{cases} \tilde{\mathcal{I}} \cong V_2 \quad \text{and} \quad \mathcal{Q} \cong V_3 \\ \qquad\qquad \text{or} \\ N \cong V_1 \quad \text{and} \quad Q \cong V_2 \end{cases}$$

Therefore, even when we can have two different limit points in the last subcase, we get just three disjoint components:

$$F_\lambda = F^{(1,1,1)}_{m_1 m_2} = F_{\vec{\delta_1}(m_1,m_2)} \sqcup F_{\vec{\delta_2}(m_1,m_2)} \sqcup F_{\vec{\rho_3}(m_1,m_2)}.$$

Chapter 5

Nilpotent Cone

In this chapter, we study the stratification of the Nilpotent Cone given by the Downward Morse Flow, and its relation to the Shatz stratification. The results presented here complement those of Chapter 4. We find a filtration that describes the Nilpotent Cone in terms of the Downward Morse Flow, for rank two and rank three cases.

5.1 The Hitchin Map and The Nilpotent Cone

Recall that we are supposing $\mathrm{GCD}(r,d) = 1$. So, the moduli space of Hitchin pairs, $\mathcal{M}_L(r,d)$, is a non-compact, smooth complex manifold of dimension

$$\dim_{\mathbb{C}}\big(\mathcal{M}_L(r,d)\big) = (r^2-1)\mathrm{deg}(L).$$

$\mathcal{M}(r,d)$ is also a Riemannian manifold with a complete hyperKähler metric, and there is a proper map, the so-called Hitchin map defined by:

$$\begin{aligned} \chi : \mathcal{M}^k(r,d) \quad &\longrightarrow \quad H^0(X,L)\oplus\cdots\oplus H^0(X,L^r) \\ [(E,\Phi)] \quad &\longmapsto \quad \det(\Phi) \end{aligned} \tag{5.1}$$

The Hitchin map is proper, and it is also an algebraically completely integrable Hamiltonian system with respet to the symplectic holomorphic form Ω, with a generic fibre which is a Prym variety corresponding to the espectral cover of X at the image point.

Finally, recall also that the set

$$\chi^{-1}(0) := \left\{[(E,\Phi)] \in \mathcal{M}_L(r,d) : \quad \chi(\Phi) = 0\right\}$$

is known as the Nilpotent Cone, and has been described by Hitchin [24], Hausel [19], among others, as one of the most important fibres of the Hitchin map, and the most singular at the same time.

The Hitchin map is widely studied and descripted by Hausel [19] and [20]. Among his results, the most relevant is the following assertion:

Theorem 5.1.1 (Hausel [20, Theorem 5.2])**.** *The Downward Morse Flow of $\mathcal{M}(r,d)$ coincides with the Nilpotent Cone:*

$$\chi^{-1}(0) \cong \bigcup_{\lambda} D_{\lambda}^{M}.$$

Hence, $[(E,\Phi)] \in \chi^{-1}(0)$ if and only if $\exists \lim_{z\to\infty} [(E,\Phi)] \in \mathcal{M}_L(r,d)$.

5.2 Rank Two Hitchin Pairs in the Nilpotent Cone

From the last theorem, we can conclude our own general results for the Hitchin pairs in the Nilpotent Cone. First, for rank two Hitchin pairs $(E,\Phi) \in \mathcal{M}(2,d)$, we have:

Theorem 5.2.1. *Let $[(E,\Phi)] \in \chi^{-1}(0)$ be a Hitchin pair with $\mathrm{rk}(E) = 2$. Then, there is a filtration*

$$E = E_1 \supset E_2 \supset E_3 = 0$$

such that

$$\Phi(E_j) \subset E_{j+1} \otimes L$$

and

$$(E^{\infty}, \Phi^{\infty}) := \lim_{z\to\infty} (E, z\cdot\Phi) = \left(L_1 \oplus L_2, \begin{pmatrix} 0 & 0 \\ \varphi & 0 \end{pmatrix}\right) \tag{5.2}$$

is a $(1,1)$-VHS where

$$L_j = E_j/E_{j+1} \quad \textit{and} \quad \varphi : L_1 \to L_2 \otimes L.$$

Proof. Consider the kernel subsheaf $N := \ker(\Phi) \subset E$, we know that N is not a subbundle but then, we can consider its saturation $N \subset \tilde{N} \subset E$ which is a line subundle of E. Then, consider the exact sequence:

$$0 \longrightarrow L_2 \longrightarrow E \longrightarrow L_1 \longrightarrow 0$$

where $L_2 = \tilde{N}$ and $L_1 \cong E/\tilde{N}$. Then, there is a smooth splitting: $E \cong_{C^\infty} L_1 \oplus L_2$, and the Higgs field Φ takes the form:

$$\Phi = \begin{pmatrix} 0 & 0 \\ \varphi_{21} & 0 \end{pmatrix}$$

where $\varphi_{21} : L_1 \to L_2 \otimes L$, and the representative holomorphic structure of E, $\bar{\partial}_E = \bar{\partial}_A = \bar{\partial} + A^{0,1} d\bar{z}$ takes the lower triangular form:

$$\bar{\partial}_A = \begin{pmatrix} \bar{\partial}_1 & 0 \\ \beta & \bar{\partial}_2 \end{pmatrix} = \begin{pmatrix} \bar{\partial} + b_{11} d\bar{z} & 0 \\ b_{21} d\bar{z} & \bar{\partial} + b_{22} d\bar{z} \end{pmatrix}$$

where $\varphi_{21} \neq 0$, by the stability of (E, Φ), and $\bar{\partial}_j = \bar{\partial} + b_{jj} d\bar{z}$ is the corresponding holomorphic structure of L_j, and $\beta = b_{21} d\bar{z} \in \Omega^{0,1}(X, Hom(L_1, L_2))$. See Wentworth [39] for more details.

Hence, is enough if we consider the filtration

$$E = E_1 \supset E_2 \supset E_3 = 0$$

where we are taking $E_2 = \tilde{N}$. Trivially: $\Phi(E_2) \subset E_3 \otimes L$, since $E_3 = 0$. Besides, Φ is nilpotent: $\Phi^2 \equiv 0$, and so $\mathrm{im}(\Phi) \subset \ker(\Phi) \otimes L$, and hence $\Phi(E_1) = \Phi(E) \subset E_2 \otimes L$.

All we have to do is to find a gauge transformation $g = g(z) \in GL_2(\mathbb{C})$ such that

$$(E^\infty, \Phi^\infty) = \lim_{z \to \infty} g(z)^{-1}(E, z \cdot \Phi) g(z) \in D_\lambda^M.$$

We may suppose that $g(z)$ is diagonal, so, $g_{12}(z) \equiv 0$ and $g_{21}(z) \equiv 0$. In such a case, we have:

$$g(z) = \begin{pmatrix} g_{11}(z) & g_{12}(z) \\ g_{21}(z) & g_{22}(z) \end{pmatrix} = \begin{pmatrix} g_{11}(z) & 0 \\ 0 & g_{22}(z) \end{pmatrix} \text{ for } z \in \mathbb{C}^*$$

and then:

$$g(z)^{-1} = \frac{1}{det(g)} \begin{pmatrix} g_{22}(z) & 0 \\ 0 & g_{11}(z) \end{pmatrix} = \frac{1}{g_{11}(z)g_{22}(z)} \begin{pmatrix} g_{22}(z) & 0 \\ 0 & g_{11}(z) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{g_{11}(z)} & 0 \\ 0 & \frac{1}{g_{22}(z)} \end{pmatrix} \text{ for } z \in \mathbb{C}^*.$$

Then:

$$g^{-1}(z \cdot \Phi)g = \begin{pmatrix} \frac{1}{g_{11}(z)} & 0 \\ 0 & \frac{1}{g_{22}(z)} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ z \cdot \varphi_{21} & 0 \end{pmatrix} \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{g_{11}}{g_{22}} z \cdot \varphi_{21} & 0 \end{pmatrix}.$$

Similarly:

$$g^{-1}\bar{\partial}_E g = g^{-1}\bar{\partial}_A g = \begin{pmatrix} \frac{1}{g_{11}(z)} & 0 \\ 0 & \frac{1}{g_{22}(z)} \end{pmatrix} \begin{pmatrix} \bar{\partial}_1 & 0 \\ \beta & \bar{\partial}_2 \end{pmatrix} \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix} = \begin{pmatrix} \bar{\partial}_1 & 0 \\ \frac{g_{11}}{g_{22}}\beta & \bar{\partial}_2 \end{pmatrix}.$$

It will be enough if the g_{ij}'s satisfy:

$$\lim_{z\to\infty} \frac{g_{11}(z)}{g_{22}(z)} = 0 \quad \text{and} \quad \lim_{z\to\infty} \frac{g_{11}(z)}{g_{22}(z)} z = 1$$

It seems that we may choose polynomials, or even better, integer powers of z:

$$g_{11}(z) = z^p,\ g_{22}(z) = z^q \text{ for } z \in \mathbb{C}^* :$$

$$g^{-1}(z)(z \cdot \Phi)g(z) = \begin{pmatrix} z^{-p} & 0 \\ 0 & z^{-q} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ z \cdot \varphi_{21} & 0 \end{pmatrix} \begin{pmatrix} z^p & 0 \\ 0 & z^q \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 \\ z^{1-q+p} \cdot \varphi_{21} & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ \varphi_{21} & 0 \end{pmatrix} \text{ when } z \to \infty \Leftrightarrow 1 - q + p = 0 \Leftrightarrow q - p = 1.$$

and also:

$$g^{-1}\bar{\partial}_E g = g^{-1}\bar{\partial}_A g = \begin{pmatrix} \bar{\partial}_1 & 0 \\ z^{p-q}\beta & \bar{\partial}_2 \end{pmatrix}$$

so:

$$g^{-1}\bar{\partial}_E g \to \begin{pmatrix} \bar{\partial}_1 & 0 \\ 0 & \bar{\partial}_2 \end{pmatrix} \text{when } z \to \infty \Leftrightarrow p-q<0 \Leftrightarrow p<q.$$

It is easy to find a pair $(p,q) \in \mathbb{Z}\times\mathbb{Z}$ such that p and q satisfy both conditions, we can consider for instance $p=0$ and $q=1$.

We are almost done. It remains to verify two things: first, that (E^∞, Φ^∞) is stable; and second, that Φ^∞ is holomorphic since Φ is.

Stability follows easily since the original $(E,\Phi) \in \chi^{-1}(0)$ is stable: since $\Phi(L_1) \subset \mathrm{im}(\Phi)\otimes L \subset \ker(\Phi)\otimes L \subset L_2 \otimes L,\ L_1 \cong E/L_2$ is not Φ^∞-invariant, and so, the line subbundles which are Φ^∞-invariant are those that are isomorphic to L_2. But, by the stability of (E,Φ), we know that $\mu(L_2) < \mu(E^\infty)$ trivially, since $\mu(\tilde{N}) < \mu(E) = \mu(E^\infty)$. Hence, (E^∞,Φ^∞) is stable.

$$\bar{\partial}_{End(E)}(\Phi) = 0 \Rightarrow \bar{\partial}_{End(E^\infty)}(\Phi^\infty) = 0 :$$

Recall that

$$0 = \bar{\partial}_{End(E)}(\Phi) = \bar{\partial}_E \circ \Phi - \Phi \circ \bar{\partial}_E$$

Then, in local terms we have:

$$\bar{\partial}_E \circ \Phi - \Phi \circ \bar{\partial}_E =$$

$$\begin{pmatrix} \bar{\partial}_1 & 0 \\ \beta & \bar{\partial}_2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \varphi_{21} & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \varphi_{21} & 0 \end{pmatrix}\begin{pmatrix} \bar{\partial}_1 & 0 \\ \beta & \bar{\partial}_2 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 \\ \bar{\partial}_2\varphi_{21} & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \varphi_{21}\bar{\partial}_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Similarly:

$$\bar{\partial}_{E^\infty} \circ \Phi^\infty - \Phi^\infty \circ \bar{\partial}_{E^\infty} =$$

$$\begin{pmatrix} \bar{\partial}_1 & 0 \\ 0 & \bar{\partial}_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \varphi_{21} & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \varphi_{21} & 0 \end{pmatrix} \begin{pmatrix} \bar{\partial}_1 & 0 \\ 0 & \bar{\partial}_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ \bar{\partial}_2 \varphi_{21} & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \varphi_{21} \bar{\partial}_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

since, by hypothesis

$$(\bar{\partial}_{End(E)}(\Phi))_{21} = \bar{\partial}_2 \circ \varphi_{21} - \varphi_{21} \circ \bar{\partial}_1 \equiv 0.$$

Therefore, Φ^∞ is holomorphic since Φ is. ♠

5.3 Rank Three Hitchin Pairs in the Nilpotent Cone

We would like to say that the result is analogue for rank three Hitchin pairs $(E, \Phi) \in \mathcal{M}_L(3, d)$, but truth is that there is a bizard subcase where we must consider the image subsheaf of the k-Higgs field. So we get the following:

Theorem 5.3.1. *Let* $[(E, \Phi)] \in \chi^{-1}(0)$ *be a Hitchin pair with* $\mathrm{rk}(E) = 3$. *Then:*

(a) either there is a filtration

$$E = E_1 \supset E_2 \supset E_3 \supset E_4 = 0$$

such that

$$\Phi(E_j) \subset E_{j+1} \otimes L$$

and

$$(E^\infty, \Phi^\infty) := \lim_{z \to \infty} (E, z \cdot \Phi) = \left(L_1 \oplus L_2 \oplus L_3, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix} \right) \tag{5.3}$$

is a $(1,1,1)$*-VHS where*

$$L_j = E_j/E_{j+1} \quad \textit{and} \quad \varphi_j : L_{j-1} \to L_j \otimes L$$

(b) or, there is a filtration

$$E = E_1 \supset E_2 \supset E_3 = 0$$

such that

(b.1.) either

$$(E^\infty, \Phi^\infty) := \lim_{z\to\infty}(E, z\cdot\Phi) = \left(V_1 \oplus V_2, \begin{pmatrix} 0 & 0 \\ \varphi_{21} & 0 \end{pmatrix}\right) \tag{5.4}$$

is a $(1,2)$*-VHS where*

$$V_j = E_j/E_{j+1} \quad \textit{and} \quad \varphi : V_1 \to V_2 \otimes L,$$

and where $\Phi(E_j) \subset E_{j+1} \otimes L$*,*

(b.2.) or

$$(E^\infty, \Phi^\infty) := \lim_{z\to\infty}(E, z\cdot\Phi) = \left(W_1 \oplus W_2, \begin{pmatrix} 0 & 0 \\ \varphi_{21} & 0 \end{pmatrix}\right) \tag{5.5}$$

is a $(2,1)$*-VHS, depending on the rank of* E_2*, and depending also on some properties of* Φ*.*

Proof. Since $(E,\Phi) \in \chi^{-1}(0) \subset \mathcal{M}(3,d)$, then $\Phi^3 \equiv 0$. So, either $\Phi^2 \neq 0$ or $\Phi^2 \equiv 0$.

(a) If $\Phi^2 \neq 0$, we may consider the following sequence of subsheaves:

$$N_1 = \ker(\Phi^3) \supset N_2 = \ker(\Phi^2) \supset N_3 = \ker(\Phi) \supset N_4 = 0,$$

and so, we may consider the filtration:

$$E = E_1 \supset E_2 \supset E_3 \supset E_4 = 0$$

where $E_j = \tilde{N}_j$ is the saturated sheaf of N_j. Clearly $\Phi(E_j) \subset E_{j+1} \otimes L$. Then, taking $L_j = E_j/E_{j+1}$, there are morphisms of bundles $\varphi_{ij} : L_j \to L_i \otimes L$ induced by Φ and, since Φ is nilpotent, we may write:

$$\Phi = \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ \varphi_{31} & \varphi_{32} & 0 \end{pmatrix}$$

and then, using

$$g(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z^2 \end{pmatrix}$$

as gauge transformation, we get:

$$g^{-1}(z \cdot \Phi)g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^{-1} & 0 \\ 0 & 0 & z^{-2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ z \cdot \varphi_{21} & 0 & 0 \\ z \cdot \varphi_{31} & z \cdot \varphi_{32} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z^2 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ \frac{1}{z} \cdot \varphi_{31} & \varphi_{32} & 0 \end{pmatrix} \xrightarrow[z \to \infty]{} \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix}$$

and also:

$$g^{-1}\bar{\partial}_E g = g^{-1}\bar{\partial}_A g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^{-1} & 0 \\ 0 & 0 & z^{-2} \end{pmatrix} \begin{pmatrix} \bar{\partial}_1 & 0 & 0 \\ \beta_{21} & \bar{\partial}_2 & 0 \\ \beta_{31} & \beta_{32} & \bar{\partial}_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z^2 \end{pmatrix} =$$

$$\begin{pmatrix} \bar{\partial}_1 & 0 & 0 \\ \frac{1}{z}\beta_{21} & \bar{\partial}_2 & 0 \\ \frac{1}{z^2}\beta_{31} & \frac{1}{z}\beta_{32} & \bar{\partial}_3 \end{pmatrix} \xrightarrow[z \to \infty]{} \begin{pmatrix} \bar{\partial}_1 & 0 & 0 \\ 0 & \bar{\partial}_2 & 0 \\ 0 & 0 & \bar{\partial}_3 \end{pmatrix}.$$

Note that Φ^∞ is holomorphic since Φ is. Recall that:

$$0 = \bar{\partial}_{End(E)}(\Phi) = \bar{\partial}_E \circ \Phi - \Phi \circ \bar{\partial}_E$$

Then, in local terms we have:

$$\bar{\partial}_{End(E^\infty)}(\Phi^\infty) = \bar{\partial}_{E^\infty} \circ \Phi^\infty - \Phi^\infty \circ \bar{\partial}_{E^\infty} =$$

$$\begin{pmatrix} \bar{\partial}_1 & 0 & 0 \\ 0 & \bar{\partial}_2 & 0 \\ 0 & 0 & \bar{\partial}_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix} \begin{pmatrix} \bar{\partial}_1 & 0 & 0 \\ 0 & \bar{\partial}_2 & 0 \\ 0 & 0 & \bar{\partial}_3 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 0 \\ \bar{\partial}_2\varphi_{21} & 0 & 0 \\ 0 & \bar{\partial}_3\varphi_{32} & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21}\bar{\partial}_1 & 0 & 0 \\ 0 & \varphi_{32}\bar{\partial}_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

since $\left(\bar{\partial}_{End(E)}(\Phi)\right)_{21} = \bar{\partial}_2\varphi_{21} - \varphi_{21}\bar{\partial}_1 = 0$ and $\left(\bar{\partial}_{End(E)}(\Phi)\right)_{32} = \bar{\partial}_3\varphi_{32} - \varphi_{32}\bar{\partial}_2 = 0$ by hypothesis, since $\bar{\partial}_{End(E)}(\Phi) = 0$. Hence, Φ^∞ is holomorphic.

To prove stability in this case, is necessary to consider the Φ^∞-invariant subbundles of E^∞, and there are two kinds: those ones isomorphic to L_3, and those ones isomorphic to $L_2 \oplus L_3$. And, by the stability of (E, Φ), we know that $\mu(L_3) < \mu(E^\infty)$ trivially, since E_3 is Φ-invariant and so $\mu(E_3) = \mu(\tilde{N}_3) < \mu(E) = \mu(E^\infty)$. On the other hand, also by the stability of (E, Φ), we have that $\mu(L_2 \oplus L_3) = \mu(E_2) < \mu(E^\infty)$ and E_2 is also Φ-invariant, since $\mu(E_2) = \mu(\tilde{N}_2) < \mu(E) = \mu(E^\infty)$. Hence, (E^∞, Φ^∞) is stable.

(b) On the other hand, suppose that $\Phi^2 \equiv 0$. Then, we may consider:

$$N_1 = \ker(\Phi^3) \supset N_2 = \ker(\Phi^2) \supset N_3 = 0,$$

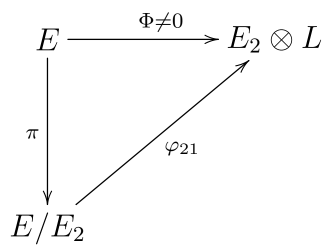
and so, we may consider the filtration:

$$E = E_1 \supset E_2 \supset E_3 = 0$$

where $E_j = \tilde{N}_j$ is the saturated sheaf of N_j. Clearly $\Phi(E_j) \subset E_{j+1} \otimes L$. Then, taking $V_j = E_j/E_{j+1}$, there is a morphism of bundles $\varphi_{21} : V_1 \to V_2 \otimes L$ induced by Φ and so:

$$\Phi = \begin{pmatrix} 0 & 0 \\ \varphi_{21} & 0 \end{pmatrix}$$

The following diagram



factors because $\Phi(E_2) = E_3 = 0$. Now, we must consider two subcases: either $\mathrm{rk}(E_2) = 1$ or $\mathrm{rk}(E_2) = 2$.

When $\mathrm{rk}(E_2) = 1$, we get that

$$(E^\infty, \Phi^\infty) := \lim_{z \to \infty} (E, z \cdot \Phi) = \left(V_1 \oplus V_2, \begin{pmatrix} 0 & 0 \\ \varphi_{21} & 0 \end{pmatrix} \right)$$

is a $(2,1)$-VHS, and the statement of the proof is almost the same of that one presented above for the rank two Hitchin pair, with two main differences: first, $\varphi_{21} : V_1 \to V_2 \otimes L$ is actually a (2×1)-block instead of a (1×1)-block, and so we must take

$$g(z) = \begin{pmatrix} I_2 & 0 \\ 0 & z \end{pmatrix} \in GL_3(\mathbb{C})$$

as our gauge transformation, where $I_2 \in GL_2(\mathbb{C})$ is the identity matrix; and second, stability. In this subcase, the Φ^∞-invariant subbundles are those isomorphic to $V_2 = E_2$, and those isomorphic to the bundle of the form $L' \oplus V_2$, where $L' \subset V_1 = E/E_2$ is any line bundle. But by the stability of (E, Φ) we know that $\mu(E_2) < \mu(E)$ since E_2 is Φ-invariant, so $\mu(V_2) < \mu(E^\infty)$. On the other hand, those bundles of the form $L' \oplus E_2$ also have slope less than E, but the proof is a little bit more sofisticated:

Consider the short exact sequence

$$0 \longrightarrow E_2 \longrightarrow E \stackrel{\pi}{\longrightarrow} E/E_2 \longrightarrow 0.$$

So, define $V := \pi^{-1}(L') \subset E$, and consider the sequence

$$0 \longrightarrow E_2 \longrightarrow V \longrightarrow L'$$

and note that V is Φ-invariant, then $\mu(V) < \mu(E)$, or equivalently, $\mu(L' \oplus E_2) < \mu(E)$. Hence, (E^∞, Φ^∞) is stable.

When $\mathrm{rk}(E_2) = 2$, define $\mathcal{I} := \varphi_{21}(V_1) \otimes K^{-1} \subset V_2$ and its saturation $\tilde{\mathcal{I}}$ such that $\mathcal{I} \subset \tilde{\mathcal{I}} \subset V_2$, and define also $F := V_1 \oplus \tilde{\mathcal{I}}$. If $\mu(F) < \mu(E)$, we get that

$$(E^\infty, \Phi^\infty) := \lim_{z \to \infty} (E, z \cdot \Phi) = \left(V_1 \oplus V_2, \begin{pmatrix} 0 & 0 \\ \varphi_{21} & 0 \end{pmatrix} \right)$$

is a $(1,2)$-VHS, and the statement of the proof follows a similar argument to the one above for the rank two Hitchin pair, also with two main differences: first, $\varphi_{21} : V_1 \to V_2 \otimes L$ is actually a (1×2)-block instead of a (1×1)-block, and so we must take

$$g(z) = \begin{pmatrix} 1 & 0 \\ 0 & I_2 \cdot z \end{pmatrix} \in GL_3(\mathbb{C})$$

as our gauge transformation, where $I_2 \in GL_2(\mathbb{C})$ is again the identity matrix; and second, stability. In this subcase, the Φ^∞-invariant subbundles are those isomorphic to $V_2 = E_2$, those isomorphic to the bundle of the form $L' \subset V_2$, where L' is any line bundle, and those isomorphic to F. By the stability of (E, Φ) we know that $\mu(E_2) < \mu(E)$ since E_2 is Φ-invariant, so $\mu(V_2) < \mu(E^\infty)$. Clearly, those bundles of the form $L' \subset E_2$ also have slope less than E, since $\Phi(L') = 0$ because $L' \subset E_2$, and so it is Φ-invariant, hence $\mu(L') < \mu(E) = \mu(E^\infty)$. On the other hand, we are supposing that $\mu(F) < \mu(E)$, so we are done in this subsubcase.

Finally, when $\mathrm{rk}(E_2) = 2$, if $\mu(F) > \mu(E)$, then we consider the smooth splitting

$$E \equiv \big(E/\tilde{\mathcal{I}}\big) \oplus \tilde{\mathcal{I}}$$

where

$$\begin{array}{ccc} E & \xrightarrow{\Phi \neq 0} & \tilde{\mathcal{I}} \otimes L \\ \tilde{\pi} \downarrow & \nearrow_{\phi_{21}} & \\ E/\tilde{\mathcal{I}} & & \end{array}$$

factors because $\tilde{\mathcal{I}} \subset E_2 = \tilde{N}_2 = \ker(\Phi)$. In such a case, we get that

$$(E^\infty, \Phi^\infty) := \lim_{z \to \infty} (E, z \cdot \Phi) = \left(W_1 \oplus W_2, \begin{pmatrix} 0 & 0 \\ \varphi_{21} & 0 \end{pmatrix} \right)$$

is a $(2,1)$-VHS, where $W_1 = E/\tilde{\mathcal{I}}$ and $W_2 = \tilde{\mathcal{I}}$, and the statement of the proof follows a similar argument to those above. Remains to check stability. The Φ^∞-invariant subbundles of E^∞ are of three kinds: those who are isomorphic to $E_2/\tilde{\mathcal{I}}$, those isomorphic to $L' \oplus \tilde{\mathcal{I}}$ for any line bundle $L' \subset E/\tilde{\mathcal{I}}$, and those isomorphic to $\tilde{\mathcal{I}}$.

$$\mu(E_2/\tilde{\mathcal{I}}) < \mu(E):$$

In this subcase, we are supposing that $\mu(F) > \mu(E)$, which is equivalent to:

$$\mu(V_1 \oplus \tilde{\mathcal{I}}) > \mu(E) \Leftrightarrow 3\big(\deg(V_1) + \deg(\tilde{\mathcal{I}})\big) > 2d \Leftrightarrow$$

$$3\big(d - \deg(V_2) + \deg(\tilde{\mathcal{I}})\big) > 2d \Leftrightarrow d > 3\big(\deg(E_2) - \deg(\tilde{\mathcal{I}})\big) \Leftrightarrow$$

$$\frac{d}{3} > \deg(E_2) - \deg(\tilde{\mathcal{I}}) \Leftrightarrow \mu(E_2/\tilde{\mathcal{I}}) < \mu(E).$$

Note that $\Phi(\tilde{\mathcal{I}}) = 0$ because $\tilde{\mathcal{I}} \subset E_2 = \tilde{N} = \widetilde{\ker(\Phi)}$, and by the stability of (E, Φ) we get $\mu(\tilde{\mathcal{I}}) < \mu(E) = \mu(E^\infty)$.

Finally, to prove that $\mu(L' \oplus \tilde{\mathcal{I}}) < \mu(E^\infty)$, we consider the following short exact sequence

$$0 \longrightarrow \tilde{\mathcal{I}} \longrightarrow E \stackrel{\pi}{\longrightarrow} E/\tilde{\mathcal{I}} \longrightarrow 0.$$

So, define $V := \pi^{-1}(L') \subset E$, and consider the sequence

$$0 \longrightarrow \tilde{\mathcal{I}} \longrightarrow V \longrightarrow L'$$

and note that V is Φ-invariant, then $\mu(V) < \mu(E)$, or equivalently, $\mu(L' \oplus \tilde{\mathcal{I}}) < \mu(E)$. Hence, (E^∞, Φ^∞) is stable.

♠

5.4 Approach for General Rank

Suppose now that $[(E, \Phi)] \in \chi^{-1}(0) \subset \mathcal{M}(r, d)$ is a Hitchin pair of general rank $\mathrm{rk}(E) = r$ and degree $\deg(E) = d$. Let $p \in \mathbb{N}$ be the least positive integer such that $\Phi^p = 0$ and $\Phi^{p-1} \neq 0$, and so consider the subsheaves $K_j := \ker(\Phi^{p+1-j}) \subset E$ and their respective saturations $E_j = \tilde{K}_j$ such that $K_j \subset \tilde{K}_j \subset E$ where $E_j \subset E$ is a subbundle of E $\forall j \in \{1, ..., p\}$. We would like to conclude that there is a filtration

$$E = E_1 \supset E_2 \supset ... ... \supset E_p \supset E_{p+1} = 0$$

such that

$$\Phi(E_j) \subset E_{j+1} \otimes L$$

and that

$$(E^\infty, \Phi^\infty) := \lim_{z \to \infty} (E, z \cdot \Phi) = \left(\bigoplus_{j=1}^{p} V_j, \begin{pmatrix} 0 & ... & ... & ... & 0 \\ \varphi_2 & 0 & ... & ... & 0 \\ 0 & \varphi_3 & 0 & ... & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & ... & 0 & \varphi_p & 0 \end{pmatrix} \right) \tag{5.6}$$

where

$$V_j = E_j / E_{j+1} \quad \text{and} \quad \varphi_j : V_{j-1} \to V_j \otimes L,$$

but this is not always true.

Recall that $[(E,\Phi)]\in\chi^{-1}(0)$ if and only if $\Phi^r=\Phi\circ\Phi\circ...\circ\Phi\equiv 0$ by definition, in general for $rk(E)=r$. So, we know that there is an integer $p\in\mathbb{N},\ p\leqslant r$ such that $\Phi^p=\Phi\circ\Phi\circ...\circ\Phi\equiv 0$ by definition, with equality $p=r$ when the subbundles $E_j=\tilde{K}_j\subset E$ are linear, where $K_j:=\ker(\Phi^{r+1-j})\subset E$ and where $K_j\subset\tilde{K}_j\subset E\ \forall j\in\{1,...,r\}$.

As well as we did for rank two and rank three, we may consider the smooth splitting

$$E\cong_{C^\infty}\bigoplus_{j=1}^{p}V_j$$

where $V_j=E_j/E_{j+1}$, and then, think about the Higgs field taking the triangular form:

$$\Phi=\begin{pmatrix}0&...&...&...&0\\ \varphi_{21}&0&...&...&0\\ \varphi_{31}&\varphi_{32}&0&...&0\\ \vdots&\ddots&\ddots&\ddots&\vdots\\ \varphi_{p1}&...&\varphi_{pp-2}&\varphi_{pp-1}&0\end{pmatrix}$$

where $\varphi_{ij}:V_j\to V_i\otimes L$. In such a case, the holomorphic structure could be of the form:

$$\bar{\partial}_E=\begin{pmatrix}\bar{\partial}_1&0&...&0\\ \beta_{21}&\bar{\partial}_2&\ddots&\vdots\\ \vdots&\ddots&\ddots&0\\ \beta_{p1}&...&\beta_{pp-1}&\bar{\partial}_p\end{pmatrix}$$

where $\bar{\partial}_j$ is the correspoding holomorphic structure of V_j, and $\beta_{ij}\in\Omega^{0,1}(X,Hom(V_j,V_i))$.

We also may consider $g \in \mathcal{G}$ such that:

$$g(z) = \begin{pmatrix} I_{r_1} & 0 & ... & 0 \\ 0 & zI_{r_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & ... & 0 & z^{p-1}I_{r_p} \end{pmatrix} \in GL_r(\mathbb{C})$$

defined by blocks, where $r_j = \mathrm{rk}(V_j)$ is the rank of V_j and $I_{r_j} \in \mathrm{End}(V_j)$ is the identity $\forall j \in \{1, ..., p\}$. Hence:

$$g^{-1}(z)(z \cdot \Phi)g(z) =$$

$$\begin{pmatrix} I_{r_1} & 0 & ... & 0 \\ 0 & z^{-1}I_{r_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & ... & 0 & z^{1-p}I_{r_p} \end{pmatrix} \begin{pmatrix} 0 & ... & ... & 0 \\ z\varphi_{21} & 0 & ... & 0 \\ \vdots & \ddots & \ddots & \vdots \\ z\varphi_{p1} & ... & z\varphi_{pp-1} & 0 \end{pmatrix} \begin{pmatrix} I_{r_1} & 0 & ... & 0 \\ 0 & zI_{r_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & ... & 0 & z^{p-1}I_{r_p} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & ... & ... & ... & 0 \\ \varphi_{21} & 0 & ... & ... & 0 \\ z^{-1}\varphi_{31} & \varphi_{32} & 0 & ... & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ z^{1-p}\varphi_{p1} & ... & z^{-1}\varphi_{pp-2} & \varphi_{pp-1} & 0 \end{pmatrix} \xrightarrow[z \to \infty]{} \begin{pmatrix} 0 & ... & ... & ... & 0 \\ \varphi_{21} & 0 & ... & ... & 0 \\ 0 & \varphi_{32} & 0 & ... & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & ... & 0 & \varphi_{pp-1} & 0 \end{pmatrix},$$

and also:

$$g^{-1}(z)\, \bar{\partial}_E\, g(z) =$$

$$\begin{pmatrix} I_{r_1} & 0 & ... & 0 \\ 0 & z^{-1}I_{r_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & ... & 0 & z^{1-p}I_{r_p} \end{pmatrix} \begin{pmatrix} \bar{\partial}_1 & 0 & ... & 0 \\ \beta_{21} & \bar{\partial}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \beta_{p1} & ... & \beta_{pp-1} & \bar{\partial}_p \end{pmatrix} \begin{pmatrix} I_{r_1} & 0 & ... & 0 \\ 0 & zI_{r_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & ... & 0 & z^{p-1}I_{r_p} \end{pmatrix}$$

$$= \begin{pmatrix} \bar{\partial}_1 & 0 & \dots & 0 \\ z^{-1}\beta_{21} & \bar{\partial}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ z^{-p}\beta_{p1} & \dots & z^{-1}\beta_{pp-1} & \bar{\partial}_p \end{pmatrix} \to \begin{pmatrix} \bar{\partial}_1 & 0 & \dots & 0 \\ 0 & \bar{\partial}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \bar{\partial}_p \end{pmatrix} \text{when } z \to \infty.$$

Φ^∞ is holomorphic since Φ is. To verify that, is enough to do some general calculations similar to those we did for rank two and rank three.

Unfortunately, our main trouble lies in how to prove that

$$\lim_{z\to\infty}(E, z\cdot\Phi) = (E^\infty, \Phi^\infty) = \Big(\bigoplus_{j=1}^{p} V_j, \begin{pmatrix} 0 & \dots & \dots & \dots & 0 \\ \varphi_{21} & 0 & \dots & \dots & 0 \\ 0 & \varphi_{32} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \varphi_{pp-1} & 0 \end{pmatrix}\Big)$$

is stable. This Higgs bundle is not necessarily stable, so we can not conclude a general form of the theorem.

Bibliography

[1] Atiyah, M.F., *K-Theory*, Lecture notes by D.W. Anderson. W.A. Benjamin Inc., New York-Amsterdam, 1967.

[2] Atiyah, M.F. and Bott, R., "The Yang-Mills Equations over Riemann Surfaces", *Phil. Trans. R. Soc. Lond.*, Series A, Vol. **308**, (1982), pp. 523-615.

[3] Bento, S., "Topologia do Espaço Moduli de Fibrados de Higgs Torcidos", Tese de Doutoramento, Universidade do Porto, Porto, Portugal, 2010.

[4] Bradlow, S.B., "Special metrics and stability for holomorphic bundles with global sections", *J. Differential Geom.*, **33**, (1991), pp. 169-213.

[5] Bradlow, S.B. and García-Prada, O., "Stable Triples, Equivariant Bundles and Dimensional Reduction", *Math. Ann.*, **304**, (1996), pp. 225-252.

[6] Bradlow, S.B., García-Prada, O. and Gothen, P.B., "Moduli Spaces of Holomorphic Triples Over Compact Riemann Surfaces", *Math. Ann.*, **328**, (2004), pp. 299-351.

[7] Bradlow, S.B., García-Prada, O. and Gothen, P.B., "Homotopy Groups of Moduli Spaces of Representations", *Topology* **47**, (2008), pp. 203-224.

[8] Corlette, K., "Flat G-Bundles with Canonical Metrics", *J. Differential Geom.*, **28**, (1988), pp. 361-382.

[9] Donaldson, S., "A New Proof of a Theorem of Narasimhan and Seshadri", *J. Differential Geom.* **18**, (1983), no. 2, pp. 269-277.

[10] Frankel, T., “Fixed Points and Torsion on Kähler Manifolds”, *Annals of Mathematics*, Second Series, Vol. **70**, No. 1, (1959), pp. 1-8.

[11] Fulton, W. *Algebraic Topology, A first course*, Springer, New York, U.S.A., 1995.

[12] García-Prada O. and Heinloth, J., “The y-genus of the moduli space of PGL_n-Higgs bundles on a curve (for degree coprime to n)”, *Duke Math. J.* **162**, (2013), no. 14, pp. 2731-2749.

[13] García-Prada O., Heinloth, J. and Schmitt, A. “On the motives of moduli of chains and Higgs bundles”, *Duke Math. J.* **162**, (2013), no. 14, pp. 2731-2749.

[14] Gothen, P.B., “The Betti Numbers of the Moduli Space of Stable Rank 3 Higgs Bundles on a Riemann Surface”, *International Journal of Mathematics*, Vol. **5**, No. 6, (1994), pp. 861-875.

[15] Griffiths, P. and Harris, J., *Principles of Algebraic Geometry*, Wiley, New York, U.S.A., 1978.

[16] Harder,G. and Narasimhan, M.S., “On the Cohomology Groups of Moduli Spaces of Vector Bundles on Curves”, *Math. Ann.*, Vol. **212**, (1975), pp. 215-248.

[17] Hatcher, A., *Algebraic Topology*, Cambridge University Press, Cambridge, U.K. 2002.

[18] Hatcher, A., *Vector Bundles and K-Theory*, unpublished. (http://www.math.cornell.edu/~hatcher/VBKT/VBpage.html)

[19] Hausel, T., “Geometry of Higgs Bundles”, Ph.D. Thesis, Cambridge University Press, Cambridge, United Kingdom, 1998.

[20] Hausel, T., “Compactification of Moduli of Higgs Bundles”, *J. Reine Angew. Math.*, Volume 503, (1998), pp. 169-192.

[21] Hausel, T. Thaddeus, M., “Generators for the Cohomology Ring of the Moduli Space of Rank 2 Higgs Bundles”, *Proc. London Math. Soc.*, (3), **88**, (2004), pp. 632-658.

[22] Hausel, T. and Thaddeus, M. , "Relations in the Cohomology Ring of the Moduli Space of Rank 2 Higgs Bundles", *AMS*, No. 2, Vol. **16**, (2002), pp. 303-329.

[23] Hirsch, M.W., *Differential Topology*, Springer-Verlag, New York, U.S.A. 1994.

[24] Hitchin, N.J., "The Self-Duality Equations on a Riemann Surface", *Proc. London Math. Soc.*, (3), **55**, (1987), pp. 59-126.

[25] Husemoller, D. *Fibre bundles.* Third edition. Graduate Texts in Mathematics, 20. Springer-Verlag, New York, 1994.

[26] James, I.M. (editor) *Handbook of Algebraic Topology*, North-Holland, 1995.

[27] Kirwan, F.C., *Cohomology of Quotients in Symplectic and Algebraic Geometry*, Mathematical Notes 31, Princeton University Press, Princeton, New Jersey, U.S.A., 1984.

[28] Macdonald, I.G., "Symmetric Products of an Algebraic Curve", *Topology*, **1**, (1962), pp. 319-343.

[29] Markman, E., "Generators of the cohomolohy ring of moduli spaces of sheaves on symplectic surfaces", *Journal fur die reine und angewandte Mathematik*, 544, (2002), pp. 61-82.

[30] Markman, E., "Integral generators for the cohomolohy ring of moduli spaces of sheaves over Poisson surfaces", *Adv. in Math.*, 208, (2007), pp. 622-646.

[31] Muñoz, V. Oliveira and A. Sánchez, J., "Motives and the Hodge Conjecture for the Moduli Spaces of Pairs", (2013).

[32] Muñoz, V. Ortega, D. Vázquez-Gallo, M.J., "Hodge Polynomials of the Moduli Spaces of Pairs", *International Journal of Mathematics*, Vol. **18**, No. 6, (2007), pp. 695-721.

[33] Narasimhan, M.S. and Seshadri, C.S., "Stable and Unitary Vector Bundles on a Compact Riemann Surface", *The Annals of Mathematics*, Second Series, Vol. **82**, (1965), pp. 540-567.

[34] Nitsure, N., “Moduli Space of Semistable Pairs on a Curve”, *Proc. London Math. Soc.*, (3), **62**, (1991), pp. 275-300.

[35] Shatz, S.S., “The Decomposition and Specialization of Algebraic Families of Vector Bundles”, *Compositio Mathematica*, Vol. **35**, Fasc. 2., Netherlands, (1977), pp. 163-187.

[36] Simpson, C.T., “Constructing Variations of Hodge Structures Using Yang-Mills Theory and Applications to Uniformization”, *AMS*, No. 4, Vol. **1**, (1988), pp. 867-918.

[37] Simpson, C.T., “Higgs Bundles and Local Systems”, *Inst. Hautes Études Sci. Math. Publ.*, (1992), pp. 5-95.

[38] Thaddeus, M., “Stable Pairs, Integrable Systems and the Verlinde Formula”, *Invent. Math.*, Vol. **117**, (1994), pp. 317-353.

[39] Wentworth, R.A., “Higgs Bundles and Local Systems on Riemann Surfaces”, *Third International School on Geometry and Physics, CRM, Barcelona*, (2012).